Übungsblatt 1

Exercise 1

As a warm-up with cosmological quantities, show that the term $\ddot{a}/a$ contained in one of the Friedmann equations can be written as $\dot{H} + H^2$. These kind of manipulations will be useful in many calculations. As a hint, try to calculate what is the time derivative of the Hubble parameter H.

Exercise 2

The measurement of wavelengths of astronomical objects as a function of their distance shows the phaenomenon of *red-shift*, which proves the expansion of the Universe. The red-shift z is defined as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \quad , \tag{1}$$

where λ_{obs} and λ_{em} are the observed (today) and emitted (in the past) wavelengths. The stretching of the wavelengths is directly connected to the change of the scale factor:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(now)}{a} = \frac{1}{a} \quad , \tag{2}$$

where we choose the current scale factor a(now)=1.
Show that the following relation between the Hubble parameter and redshift holds:

$$H = -\frac{1}{1+z}\dot{z} \quad . \tag{3}$$

A "dot" on a quantity denotes its total time derivative.

Exercise 3

The Einstein Universe is a static solution of the Friedmann equations. At that time, Einstein (like almost everybody) did not consider an evolving Universe and therefore he was looking for static solutions of his equations. Such a solution is possible only if $\Lambda \neq 0$. It turns out that this solution is not stable against small perturbations and therefore it is not really leading to a static Universe. This is the famous Einstein's "blunder" which made him reject the expanding solution required by the discoveries of Hubble. Later, another "blunder" was made, setting $\Lambda = 0$, which nowadays looks disfavoured by current observations, although the scenario is still evolving with more precise measurements.
To see that the Einstein's solutions is unstable, consider the Friedmann equation for the acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \quad . \tag{4}$$

Einstein imagined a Universe filled with just ordinary matter. Treating galaxies and clusters as non-interacting "dust", we can set $P = 0$. Let's consider now small perturbations of the scale factor $a(t) \sim 1 + \delta a(t)$. Considering only matter, we have $\rho = \rho_0 a^{-3}$.
1) Prove that the perturbation of the density at first order (*i.e.* neglecting δa^2 or higher order terms) is

$$\rho = \rho_0(1 - 3\delta a) \quad . \tag{5}$$

2) Substitute the small perturbations for the scale factor and density in Eq. 4
3) Neglect terms of $O(\delta a^2)$
4) Substitute the static version of Eq. 4 ($\ddot{a} = 0$) to obtain

$$\delta\ddot{a} = \Lambda\delta a \tag{6}$$

Discuss how small perturbations evolve in time and why the Einstein Universe is not stable.