Übungsblatt 7

Exercise 1

A compelling model for dark matter (DM) is the WIMP one, where DM is described as a cold thermal relic from the early Universe.

1. A cold relic has $x \gg 1$: why? Try to be quantitative using e.g. special relativity.
2. The number density for a non-relativistic ("cold") particle is
$$n \sim (m_\chi T)^{3/2} e^{-m_\chi/T} \quad . \tag{1}$$
Remembering the freeze-out condition $n\sigma \sim H$ and the Hubble parameter in a radiation-dominated Universe ($H \sim T^2/M_P$), show that
$$\sqrt{x} e^{-x} = \frac{1}{M_P \cdot \sigma \cdot m_\chi} \quad . \tag{2}$$

Exercise 2

Euler's equation of fluid dynamics is
$$\rho \frac{dv}{dt} = -\nabla P - \rho \nabla \phi \quad . \tag{3}$$
Applying it to a cluster of galaxies in equilibrium, (dv/dt=0) with only the gravitational force acting we have
$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \quad , \tag{4}$$
where $M(r)$ is the mass within a radius r. Using the ideal gas law $P = \rho k_B T/m_p$, where m_p is the proton mass, show that
$$M(r) = \frac{k_B T r}{G m_p} \left(-\frac{d \ln \rho}{d \ln r} - \frac{d \ln T}{d \ln r} \right) \tag{5}$$
which is the basis of mass distribution reconstruction with X-ray surveys.

Exercise 3

Weak gravitational lensing is based on the bending of light rays by large masses. In order to calculate the deflection angle (see notes), we start from the space-time metric generated by a point mass (in the weak-field approximation)
$$ds^2 = c^2 \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \left(1 - \frac{2\phi}{c^2}\right) dr^2 \quad , \tag{6}$$
where ϕ is the gravitational potential.
1) Show that for a light ray ($ds^2 = 0$) the effective velocity *at first order* ($\phi \ll c^2$) is
$$c' = |dr/dt| = c\left(1 + \frac{2\phi}{c^2}\right) \quad . \tag{7}$$

2) Show that at first order the "refraction index" of a gravitational lens is $n = c/c' \approx 1 - 2\phi/c^2$.