Übungsblatt 3

Exercise 1

The Einstein Universe is a static solution of the Friedmann equations. It can be obtained only if $\Lambda \neq 0$ in order to balance the tendency of the Universe to either contract or expand. It turns out that this solution is not stable against even very small perturbations. To see this, consider the Friedmann equation for the acceleration

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \quad . \tag{1}$$

Einstein imagined a Universe filled with just ordinary matter. Treating galaxies and clusters as non-interacting "dust", we can set $P = 0$. Let's consider now small perturbations of the scale factor $a(t) \sim 1+\delta a(t)$. Considering only matter, $\rho = \rho_0 a^{-3}$.
1) Prove that the perturbation of the density at first order is

$$\rho = \rho_0(1 - 3\delta a) \quad . \tag{2}$$

2) Substitute the small perturbations for the scale factor and density in Eq. 1
3) Neglect terms of $O(\delta a^2)$
4) Substitute the static version of Eq. 1 ($\ddot{a} = 0$) to obtain

$$\delta \ddot{a} = \Lambda \delta a \tag{3}$$

Discuss how small perturbations evolve in time and why the Einstein Universe is not stable.

Exercise 2

Show explicitly that the following relation between Hubble parameter and redshift holds:

$$H = -\frac{1}{1+z}\dot{z} \tag{4}$$

Exercise 3

Show that in the inflationary regime ($\dot{H} \sim 0 \Rightarrow H \sim const.$) the following equation for the density parameter holds:

$$\frac{d\Omega}{d\ln a} = (1 + 3w)\Omega(1 - \Omega) \quad . \tag{5}$$

Hints:
1) Consider the Friedmann equations with $\ddot{a}/a = \dot{H} + H^2$ and $\dot{a}/a = H$.
2) Eliminate the pressure with the equation of state $P = w\rho$.

3) Introduce the density parameter $\Omega = \rho/\rho_c$ and the critical density.
4) Eliminate by substitution H^2
5) Remember that $\frac{d}{da}(\frac{1}{a^2}) = -\frac{2}{a^3}$.

Exercise 4

Perform a stability analysis of Eq. 5:
- what are the equilibrium points?
- what about the stability of the equilibria?
- are there attractors?