Introductory Particle Cosmology

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Lecture 2

Introduction

Mathematical Tools

Generic Vector Bases
Basis Change
Vectors in Curved Geometries
Tensors, Metric Tensor
(Riemann) Manifolds
Connection
Geodesics and Curvature

General Relativity

Equivalence Principle
Einstein Equations

Cosmology

Cosmological Principle
FLRW Metric
Friedmann Equations
Cosmic Distances
Cosmological Models

Observation

CMB
Structure Formation
Red-shift/Distance

Standard Model

Brief History
Particle Content
Gauge Principle, CPV, Strong CP
EW Symmetry Breaking
Beyond the SM

Particle Cosmology

Dark Matter (Models + Exp.)
Dark Energy (Models + Exp)
Inflationary Models
Gravitational Waves
Density Perturbations

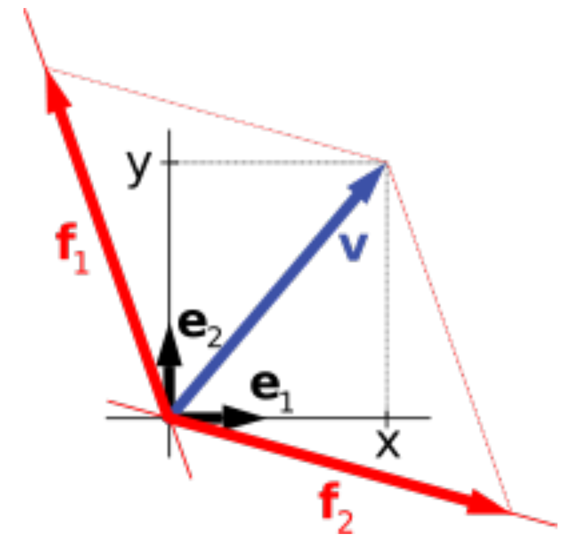
Plan

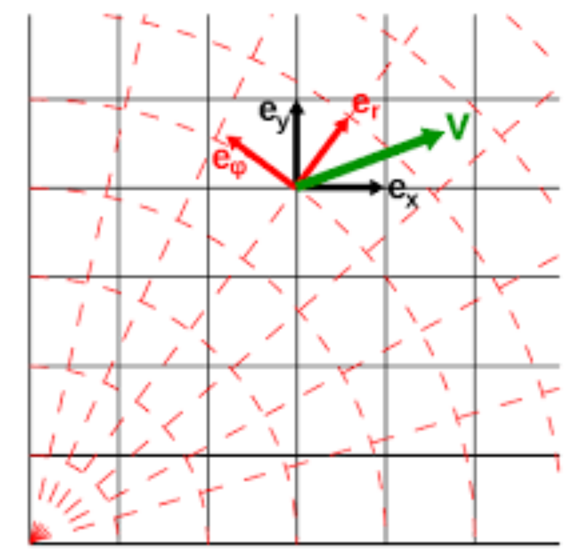
	Datum	Von	Bis	Raum
1	Di, 17. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
2	Do, 19. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
3	Di, 24. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
4	Do, 26. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
5	Do, 3. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
6	Di, 8. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
7	Di, 15. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
8	Do, 17. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
9	Di, 22. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
10	Do, 24. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
11	Di, 29. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
12	Di, 5. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
13	Do, 7. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
14	Di, 12. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
15	Do, 14. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
16	Di, 19. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
17	Do, 21. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
18	Di, 26. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
19	Do, 28. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
20	Di, 3. Jul. 2018	10:00	12:00	05 119 Minkowski-Raum
21	Do, 5. Jul. 2018	08:00	10:00	05 119 Minkowski-Raum

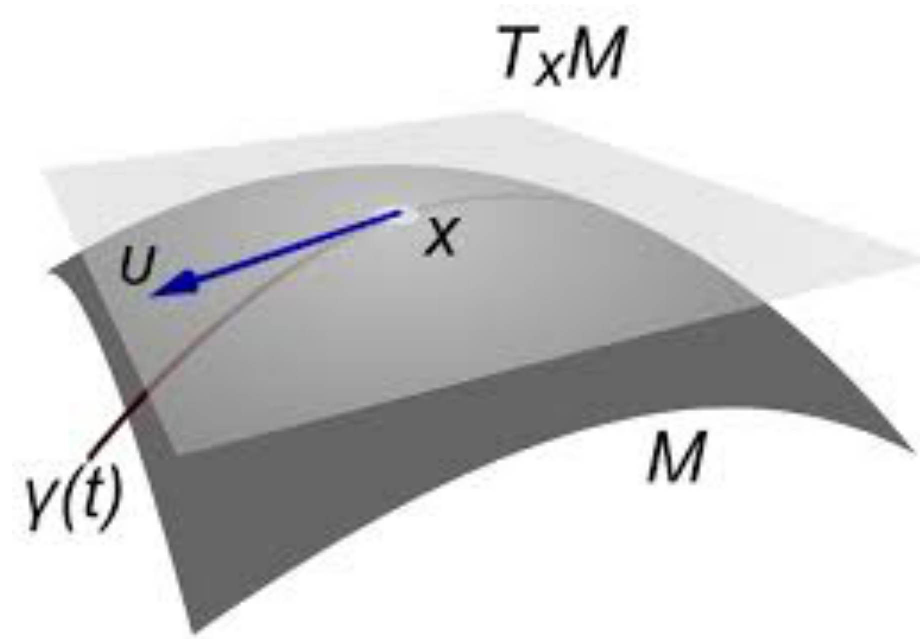
H.Minkowski
(1864-1909)

Physics Dept. Building, 5th Floor

Vector Spaces
(Ortho-normal) bases
Change of Basis
Non-orthonormal bases
Covariant and contravariant vectors
Tensors
Curved Spaces
Vectors on curved spaces
Covariant Differentiation
Geodetics
Curvature







Covariance and Contravariance

See notes

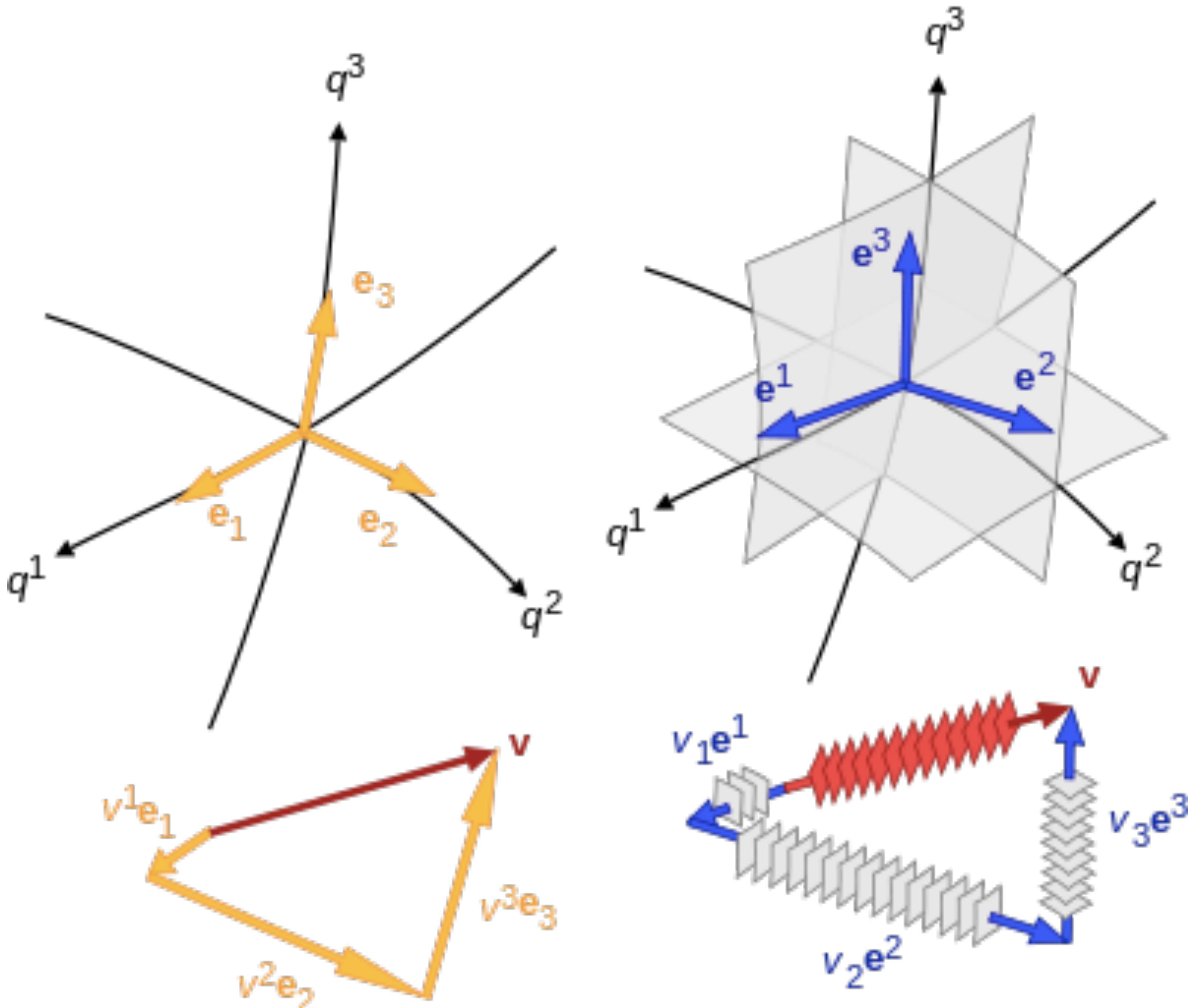


Figure from wikipedia

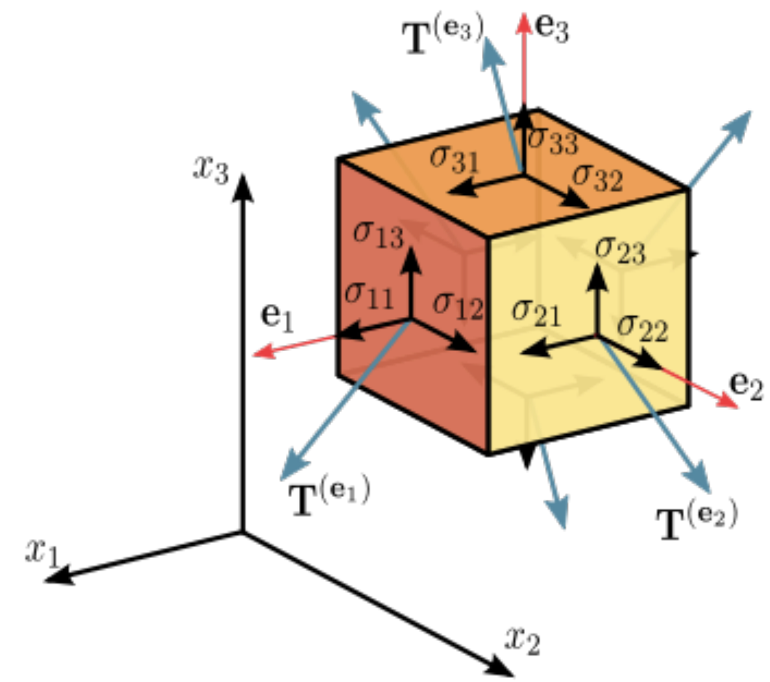
Tensors

Stress Tensor

$$T_i = \sigma_{ij} n_j$$

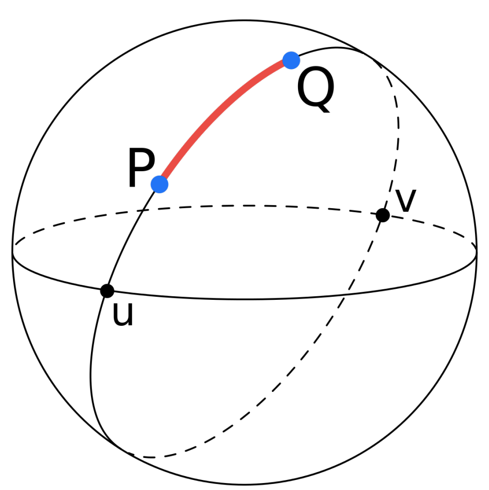
T= stress vector

n= normal to the plane to decompose on e_i



Inertial momentum tensor

$$L_i = I_{ij} \omega_j$$



Metric tensor

$$dl^2 = g_{ij} dx^i dx^j$$

Transformations

Contravariant $$v'^{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}} v^{\mu}$$

Covariant $$v'_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} v_{\mu}$$

Tensor with two covariant indices $$X'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} X_{\gamma\delta}$$

Mixed tensor $$X'^{\alpha}_{\beta\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}} \frac{\partial x^{\varepsilon}}{\partial x'^{\beta}} \frac{\partial x^{\mu}}{\partial x'^{\gamma}} X^{\delta}_{\varepsilon\mu}$$

Covariant Differentiation

How does a basis vector change in the direction of another basis vector?

$$\nabla_{e_j} e_i = \Gamma^k_{ij} e_k$$

For general vectors and directions

$$\nabla_v w = \nabla_{v^j e_j}(w^i e_i) = v^j \nabla_{e_j}(w^i e_i) = v^j w^i \nabla_{e_j} e_i + v^j e_i \nabla_{e_j} w^i$$
$$= v^j w^i \Gamma^k_{ij} e_k + v^j \frac{\partial w^i}{\partial x^j} e_i = \left(v^j w^i \Gamma^k_{ij} + v^j \frac{\partial w^k}{\partial x^j} \right) e_k$$

And for a tensor with generic indices

$$(\nabla_{e_c} T)^{a_1..a_r}_{b_1..b_r} = \frac{\partial \nabla_{e_c} T^{a_1..a_r}_{b_1..b_s}}{\partial x^c}$$
$$+\Gamma^{a_1}_{dc} T^{d a_2..a_r}_{b_1..b_s} + ... + \Gamma^{a_r}_{dc} T^{a_1..a_{r-1} d}_{b_1..b_s}$$
$$-\Gamma^{d}_{b_1 c} T^{a_1..a_r}_{d b_2..b_s} - ... - \Gamma^{d}_{b_s c} T^{a_1..a_r}_{b_1..b_{s-1} d}$$

Christoffel Symbols

Metric compatibility condition: $\nabla_\rho g_{\mu\nu} = 0$

implies cycling the indices:

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\nu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0$$
$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0$$
$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0$$

and summing to zero: $\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu} = 0$

we have $\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} = 0$

and solving for Gamma multiplying by the metric tensor

$$\boxed{\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)}$$

Geodesics

Characterization of a geodesic: the change while transported along its own direction must be zero:

$$\nabla_{T^\mu} T^\nu = T^\mu \nabla_\mu T^\nu = 0$$

Writing the covariant derivative:

$$\frac{dT^\mu}{dt} + \Gamma^\mu_{\alpha\beta} T^\alpha T^\beta = 0$$

Introducing coordinates and a parameterized curve:

$$\boxed{\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0}$$

Geodesics as extremal curves

Squared curve length $$L = \int_{t_1}^{t_2} g_{ij}\dot{x}^i\dot{x}^j dt$$

Variation: $$\delta L = \int_{t_1}^{t_2} \left(\delta g_{ij}\dot{x}^i\dot{x}^j + 2g_{ij}\delta(\dot{x}^i)\dot{x}^j\right) dt = 0$$

$$\delta L = \int_{t_1}^{t_2} \left(\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j\delta x^k + 2g_{ij}\dot{x}^i\frac{d}{dt}\delta x^j\right) dt =$$

$$\int_{t_1}^{t_2} \left(\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j - 2\frac{d}{dt}\left[g_{ij}\dot{x}^i\right]\right)\delta x^k dt + g_{ij}\frac{dx^i}{dt}\delta x^j|_{t_1}^{t_2} = 0$$

$$\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j - 2\frac{d}{dt}\left[g_{ij}\dot{x}^i\right]$$

$$= \left(\frac{\partial g_{ij}}{\partial x^k} - 2\frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j - 2g_{ij}\frac{d^2x^i}{dt^2}$$

$$= \frac{d^2x^k}{dt^2} + \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)\dot{x}^i\dot{x}^j$$

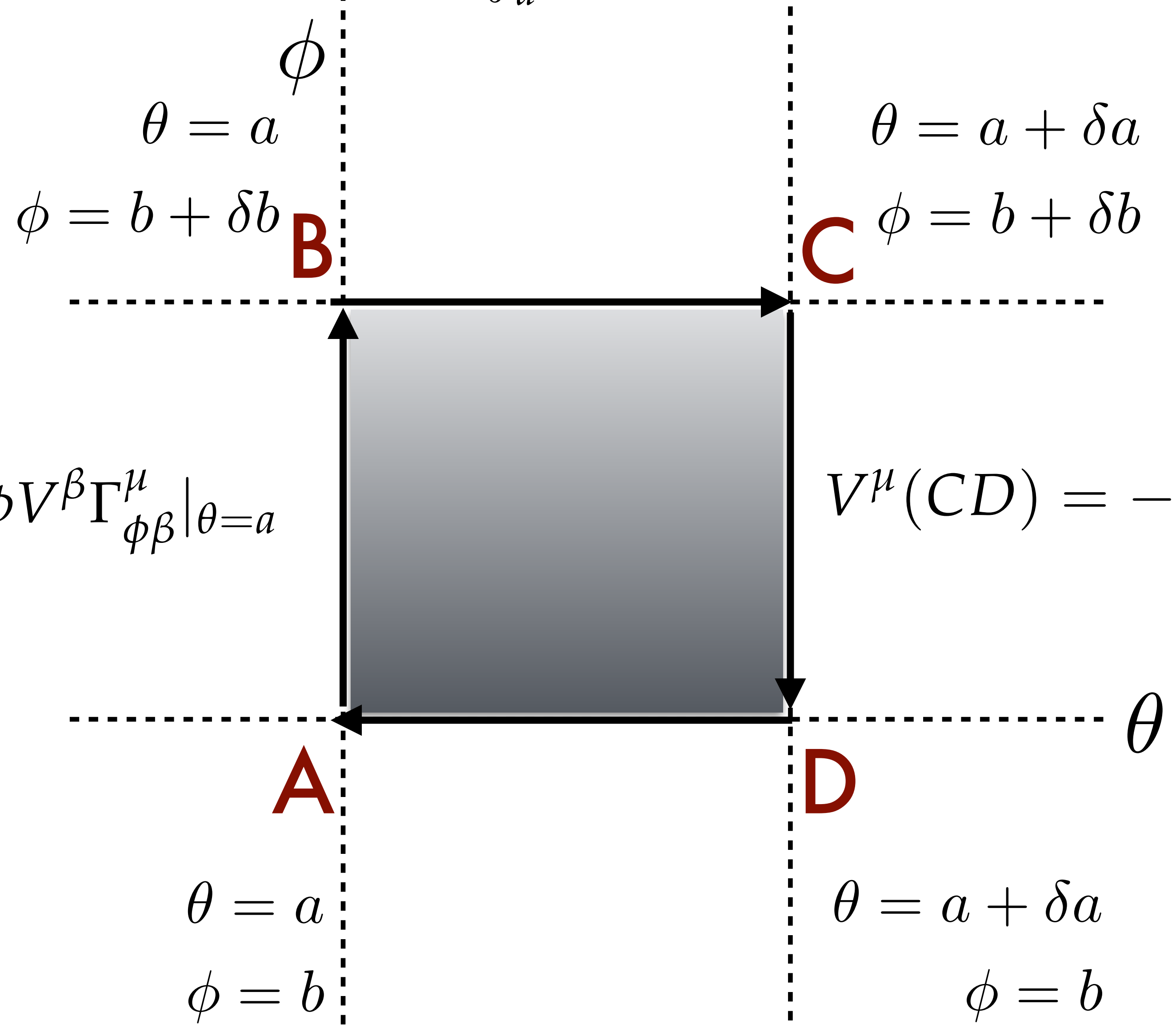
$$= \boxed{\frac{d^2x^k}{dt^2} + \frac{1}{2}g^{kl}\Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0}$$

Curvature

Parallel Transport along a closed path

$$V^\mu(BC) = -\int_a^{a+\delta a} d\theta V^\beta \Gamma^\mu_{\theta\beta}|_{\phi=b+\delta b}$$

$$V^\mu(AB) = -\int_b^{b+\delta b} d\phi V^\beta \Gamma^\mu_{\phi\beta}|_{\theta=a}$$



$$V^\mu(CD) = -\int_{b+\delta b}^{b} d\phi V^\beta \Gamma^\mu_{\phi\beta}|_{\theta=a+\delta a}$$

$$V^\mu(DA) = -\int_{a+\delta a}^{a} d\theta V^\beta \Gamma^\mu_{\theta\beta}|_{\phi=b}$$

Curvature

$$\Delta V^\mu = \int_b^{b+\delta b} d\phi \left[V^\beta \Gamma^\mu_{\phi\beta}|_{\theta=a+\delta a} - V^\beta \Gamma^\mu_{\phi\beta}|_{\theta=a} \right]$$
$$- \int_a^{a+\delta a} d\theta \left[V^\beta \Gamma^\mu_{\theta\beta}|_{\phi=b+\delta b} - V^\beta \Gamma^\mu_{\theta\beta}|_{\phi=b} \right]$$
$$= - \left[\int_a^{a+\delta a} d\theta \delta b \frac{\partial}{\partial\phi} \left(V^\beta \Gamma^\mu_{\theta\beta} \right) \int_b^{b+\delta b} d\phi \delta a \frac{\partial}{\partial\theta} \left(V^\beta \Gamma^\mu_{\phi\beta} \right) \right]$$
$$= -\delta a \delta b \left[\frac{\partial}{\partial\phi} V^\beta \Gamma^\mu_{\theta\beta} - \frac{\partial}{\partial\theta} V^\beta \Gamma^\mu_{\phi\beta} \right]$$

The vector is parallel along one side:

$$\nabla_i V^\mu = \frac{\partial V^\mu}{\partial x^i} + \Gamma^\mu_{i\beta} = 0$$

Using the definition of partial derivative

$$\Delta V^\mu = -\delta A \left(\frac{\partial V^\beta}{\partial\phi} \Gamma^\mu_{\theta\beta} + V^\beta \frac{\partial \Gamma^\mu_{\theta\beta}}{\partial\phi} - \frac{\partial V^\beta}{\partial\theta} \Gamma^\mu_{\phi\beta} - V^\beta \frac{\partial \Gamma^\mu_{\phi\beta}}{\partial\theta} \right)$$
$$= -\delta A \left(-V^\nu \Gamma^\beta_{\phi\nu} \Gamma^\mu_{\theta\beta} + V^\beta \frac{\partial \Gamma^\mu_{\theta\beta}}{\partial\phi} + V^\nu \Gamma^\beta_{\theta\nu} \Gamma^\mu_{\phi\beta} - V^\beta \frac{\partial \Gamma^\mu_{\phi\beta}}{\partial\theta} \right)$$
$$= \delta A V^\beta \left(\Gamma^\nu_{\phi\beta} \Gamma^\mu_{\theta\nu} - \frac{\partial \Gamma^\mu_{\theta\beta}}{\partial\phi} - \Gamma^\nu_{\theta\beta} \Gamma^\mu_{\phi\nu} + \frac{\partial \Gamma^\mu_{\phi\beta}}{\partial\theta} \right)$$

$$\boxed{R^\mu_{\beta\gamma\lambda} = \Gamma^\nu_{\lambda\beta} \Gamma^\mu_{\gamma\nu} - \frac{\partial \Gamma^\mu_{\gamma\beta}}{\partial x^\lambda} - \Gamma^\nu_{\gamma\beta} \Gamma^\mu_{\lambda\nu} + \frac{\partial \Gamma^\mu_{\lambda\beta}}{\partial x^\gamma}}$$

Properties of the Riemann Tensor

Another characterization of the Riemann tensor (cov. deriv. commutator):

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_c = [\nabla_c, \nabla_b] V_c = R^d_{abc} V_d$$

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{badc} \quad \text{(symmetry)}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad \text{(antisymmetry)}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \quad \text{(cyclicity)}$$

Bianchi Identities:

$$\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = 0$$

$$\nabla_e R_{bd} + \nabla_d R_{be} + \nabla_c R^c_{bde} = 0$$

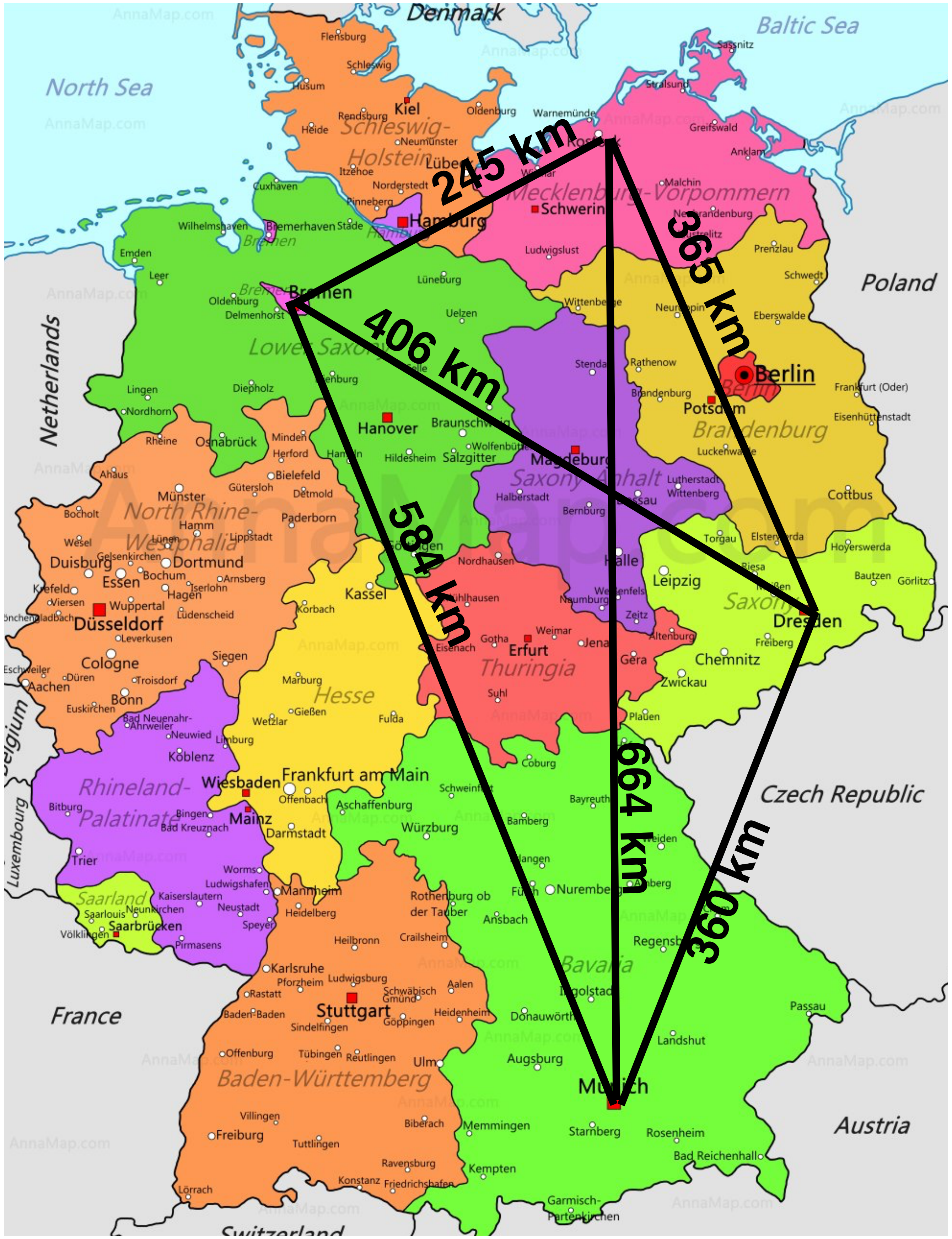
Einstein Tensor:
$$\boxed{\nabla_b (R^b_e - \frac{1}{2} \delta^b_e R) = \nabla_b G^b_e = 0}$$

The Riemann tensor is unique in the following sense:
it is the only tensor which can be constructed using only the metric tensor and its first and second derivatives (ant it is linear in the second derivatives).

Ist Deutschland flach ?

A= Muenchen
B= Bremen
C = Rostock
D= Dresden

AB = 584 km
AC = 664 km
AD = 360 km
BC = 245 km
BD = 406 km
CD = 365 km



Exercise: Try to calculate AC with the available data and confront with AC=664 km

Ist Deutschland flach ?

A more elegant way:
calculate the volume of the tetrahedron in 3D space

$$V^2 = \frac{1}{288} det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \bar{AB}^2 & \bar{AC}^2 & \bar{AD}^2 \\ 1 & \bar{AB}^2 & 0 & \bar{BC}^2 & \bar{BD}^2 \\ 1 & \bar{AC}^2 & \bar{BC}^2 & 0 & \bar{CD}^2 \\ 1 & \bar{AD}^2 & \bar{BD}^2 & \bar{CD}^2 & 0 \end{pmatrix}$$

with the Cayley-Menger determinant.
If the determinant vanishes, the tetrahedron "collapses" on a plane
–> condition for the 4 points to lie on a plane (can you explain why?).

If Germany is not flat, how could you estimate the curvature radius?

For a crude estimate, you can gather some ideas from
https://en.wikipedia.org/wiki/Horizon

For a better calculation, use spherical trigonometry.