Introductory Particle Cosmology

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Lecture 3

JG|U

Introduction

Mathematical Tools

Generic Vector Bases
Basis Change
Vectors in Curved Geometries
Tensors, Metric Tensor
(Riemann) Manifolds
Connection
Geodesics and Curvature

General Relativity

Equivalence Principle
Einstein Equations

Cosmology

Cosmological Principle
FLRW Metric
Friedmann Equations
Cosmic Distances
Cosmological Models

Observation

CMB
Structure Formation
Red-shift/Distance

Standard Model

Brief History
Particle Content
Gauge Principle, CPV, Strong CP
EW Symmetry Breaking
Beyond the SM

Particle Cosmology

Dark Matter (Models + Exp.)
Dark Energy (Models + Exp)
Inflationary Models
Gravitational Waves
Density Perturbations

Plan

	Datum	Von	Bis	Raum
1	Di, 17. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
2	Do, 19. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
3	Di, 24. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
4	Do, 26. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
5	Do, 3. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
6	Di, 8. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
7	Di, 15. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
8	Do, 17. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
9	Di, 22. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
10	Do, 24. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
11	Di, 29. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
12	Di, 5. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
13	Do, 7. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
14	Di, 12. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
15	Do, 14. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
16	Di, 19. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
17	Do, 21. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
18	Di, 26. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
19	Do, 28. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
20	Di, 3. Jul. 2018	10:00	12:00	05 119 Minkowski-Raum
21	Do, 5. Jul. 2018	08:00	10:00	05 119 Minkowski-Raum


H.Minkowski
(1864-1909)

Physics Dept. Building, 5th Floor

Part 3: General Relativity

The Equivalence Principle
Free-Falling Bodies and Geodesics
Non-relativistic Limit
Energy-Momentum Tensor
Einstein Equations

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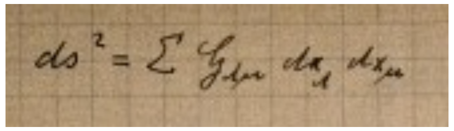
Brief History of GR

1907: While working at the Bern’s Patent Office, A. Einstein realized how to start generalizing special relativity to generic reference frames.

1908: First paper about acceleration and relativity by A. Einstein. In this paper he states the Equivalence Principle and derives time dilation caused by gravitational fields.

1911: Second paper by A. Einstein on time dilation by gravitational fields where also light deviation by massive bodies was approximately derived.

1912: Einstein consults with M. Grossman about non-euclidean geometry.



October 1915: First guess: $R_{ij} = T_{ij}$

November 1915: Einstein publishes the General Theory of Relativity as we know it today. D. Hilbert obtained the same equations almost at the same time.

1919: Eddington confirms the deviation of light formula from GR using a solar eclipse in Brazil.

1959: Pound-Rebka Experiment (gravitational red shift)

1971: Hafele-Keating Experiment (time dilation)

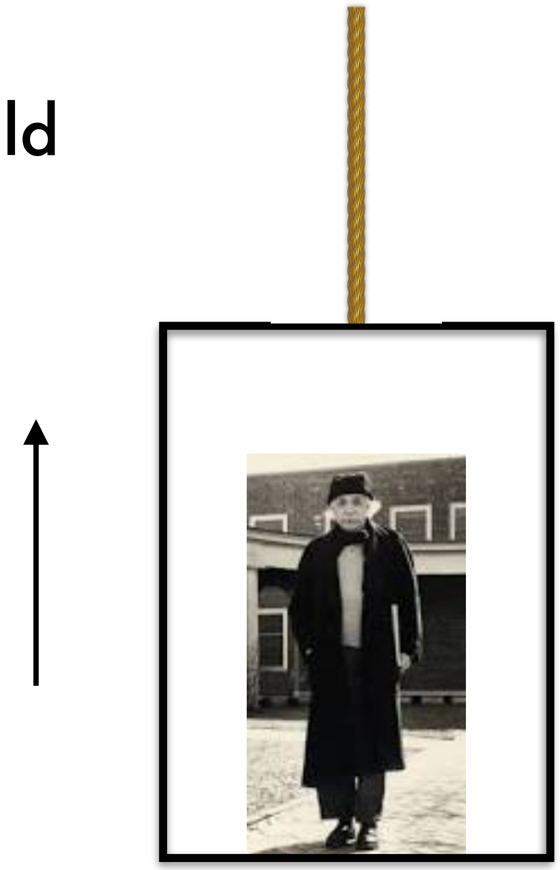
1974: Hulse-Taylor binary pulsar.

2004: Gravity Probe-B and frame dragging (published in 2011)

2016: Direct detection of gravitational waves by LIGO

The Equivalence Principle

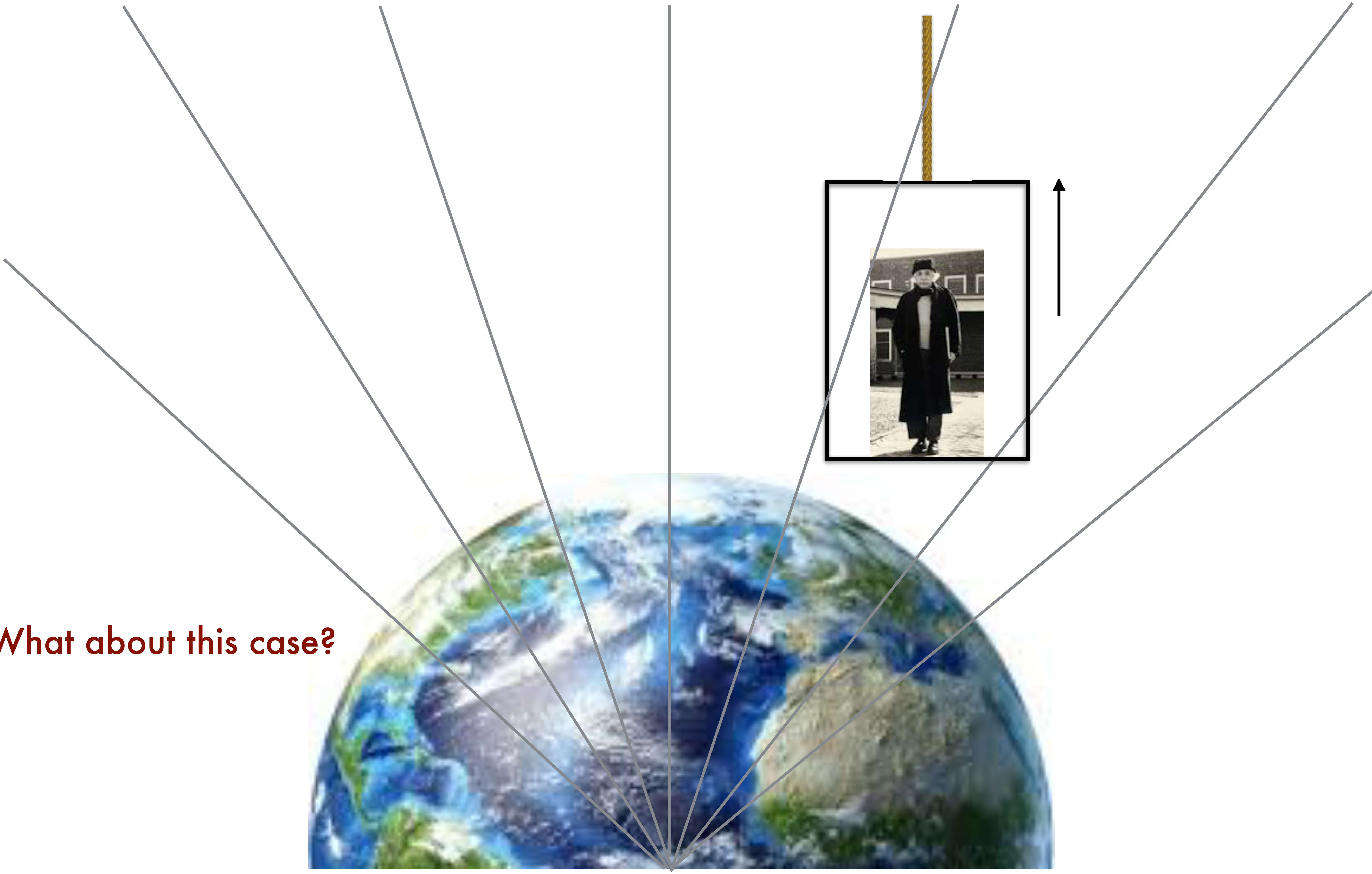
Observer in an
uniform gravitational field



Accelerated Observer

How do you tell the difference?

The Equivalence Principle

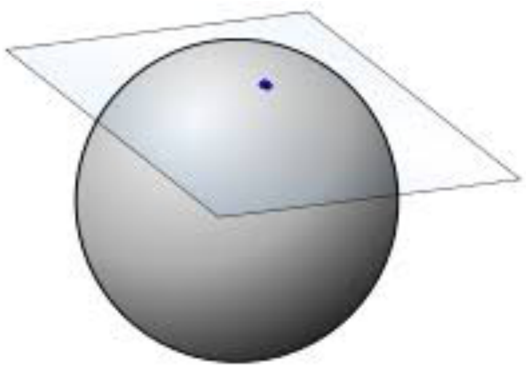


What about this case?

The Equivalence Principle

Strong Equivalence Principle:

At every space-time point in a gravitational field, it is possible to choose a locally inertial coordinate system such that in a sufficiently small neighbourhood of that point, the laws of nature can be expressed in the same form as in an unaccelerated coordinate system.



Weak Equivalence Principle:

Change “laws of nature” with “laws of motion of free-falling bodies” (gravity).

Free-Falling Bodies

$$\frac{d^2\xi^\mu}{d\tau^2} = 0$$

Equation of motion for a free-falling body

Changing to a generic coordinate system:

$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

Equation of motion:

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$

Non-Relativistic Limit

Quasi-flat metric:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \text{ with } |h_{\alpha\beta}| \ll 1$$

"slow" particles:

$$\frac{d\bar{x}}{dt} \ll \frac{dt}{d\tau}$$

The geodesic equation becomes $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$

Corresponding Christoffel symbol (time derivatives vanish):

$$\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial g_{00}}{\partial x_\beta}$$

Substituting into the geodesic equation:

$$\left\{\begin{array}{ll} \frac{d^2t}{d\tau^2} = 0 & (1) \\ \frac{d^2\mathbf{x}}{d\tau^2} - \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla h_{00} = 0 & (2) \end{array}\right.$$

From (1) $\frac{dt}{d\tau}$ is a constant and dividing (2) by it ...

Non-Relativistic Limit

...we obtain

$$\frac{d^2\bar{x}}{dt^2} = \frac{1}{2}\nabla h_{00}$$

Comparing with $d^2x/dt^2 = -\nabla\phi$

we have $h_{00} = -2\phi + C$ with C=0 at infinity

and thus $\boxed{g_{00} = -(1+2\phi)}$

–>

1) The metric tensor plays the role of gravitational potential.
2) The geodesic equation can have the correct Newtonian limit

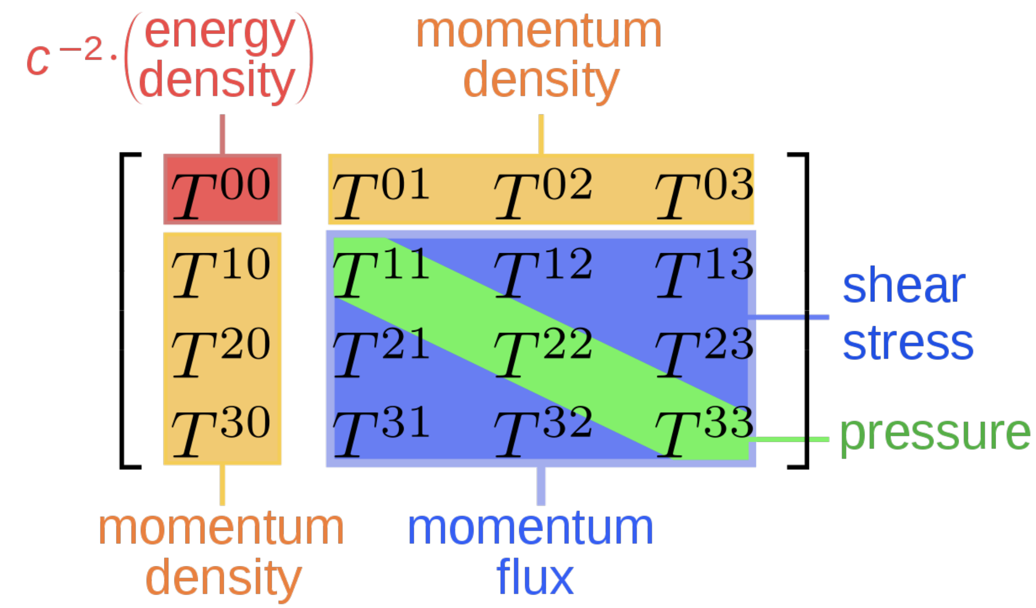
Energy-Momentum Tensor

$$T^{\alpha 0} = \sum_n p_n^\alpha \delta^3(x - x_n)$$

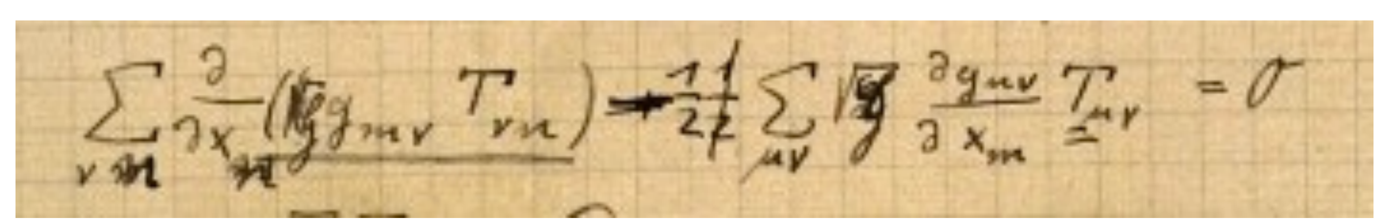
$$T^{\alpha i} = \sum_n p_n^\alpha \frac{dx_n^i}{dt} \delta^3(x - x_n)$$

$$T^{\alpha\beta} = \sum_n p_n^\alpha \frac{dx_n^\beta}{dt} \delta^3(x - x_n) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(x - x_n)$$

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = F^\alpha$$



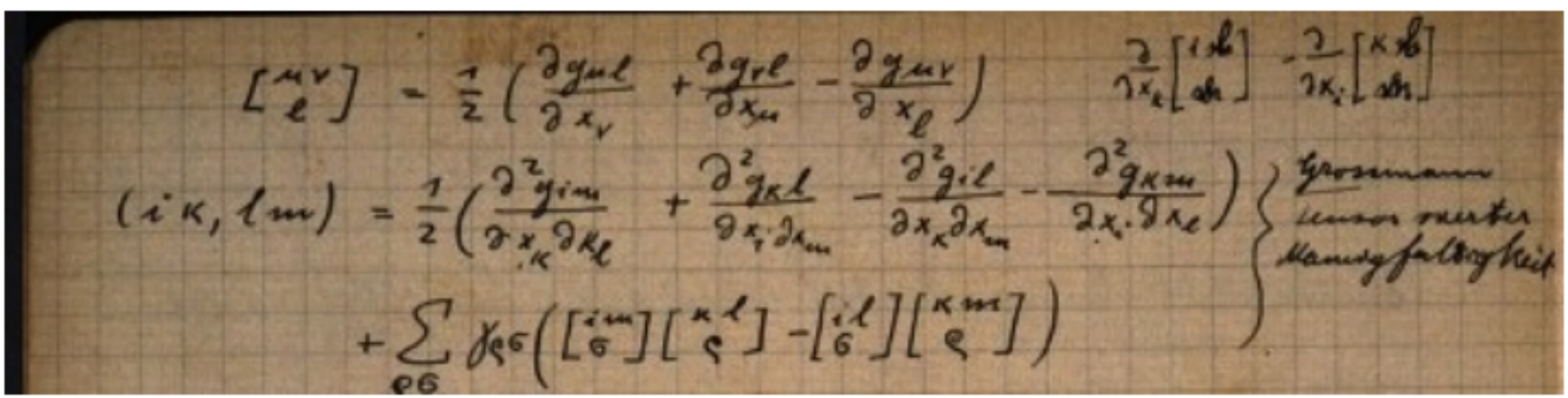
Einstein's notes on the covariant derivative of T



Einstein Equations

Requirements:

1) It should be a tensor equation (the same in any coordinate system).
2) In analogy to other situations known in physics, it should be of second order at most in the relevant variable (the gravitational potential, or in this case the metric tensor).
3) The equation must reduce to the Poisson equation in the non-relativistic limit.
4) The source of the gravitational field should be the energy-momentum tensor T.
5) If T=0, space-time must be flat



First time the Riemann tensor appears in Einstein’s notebooks.

Einstein Equations

From 1) + 4) we postulate $G^{\mu\nu} \propto T^{\mu\nu}$ with some tensor G.

Since $\nabla_\mu T^{\mu\nu} = 0$, then $\nabla_\mu G^{\mu\nu} = 0$

and since T is symmetric, also G should be.
The only tensor we know with this properties and also respecting condition 2) and 5) is the Einstein tensor, so we can set

$$\boxed{R_{\mu\nu} - g_{\mu\nu} R = k \cdot T_{\mu\nu}}$$

The constant k can be fixed thanks to the requirement 3) and a direct calculation (see notes) gives k = 8π

Einstein Equations

Die Feldgleichungen der Gravitation.

Von A. Einstein.

In zwei vor kurzem erschienenen Mitteilungen[1] habe ich gezeigt, wie man zu Feldgleichungen der Gravitation gelangen kann, die dem Postulat allgemeiner Relativität entsprechen, d. h. die in ihrer allgemeinen Fassung beliebigen Substitutionen der Raumzeitvariabeln gegenüber kovariant sind.

Der Entwicklungsgang war dabei folgender. Zunächst fand ich Gleichungen, welche die Newtonsche Theorie als Näherung enthalten und beliebigen Substitutionen von der Determinante 1 gegenüber kovariant waren. Hierauf fand ich, daß diesen Gleichungen allgemein kovariante entsprechen, falls der Skalar des Energietensors der »Materie« verschwindet. Das Koordinatensystem war dann nach der einfachen Regel zu spezialisieren, daß $\sqrt{-g}$ zu 1 gemacht wird, wodurch die Gleichungen der Theorie eine eminente Vereinfachung erfahren. Dabei mußte aber, wie erwähnt, die Hypothese eingeführt werden, daß der Skalar des Energietensors der Materie verschwinde.

Neuerdings finde ich nun, daß man ohne Hypothese über den Energietensor der Materie auskommen kann, wenn man den Energietensor der Materie in etwas anderer Weise in die Feldgleichungen einsetzt, als dies in meinen beiden früheren Mitteilungen geschehen ist. Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation unberührt. Ich gebe hier nochmals die ganze Betrachtung, damit der Leser nicht genötigt ist, die früheren Mitteilungen unausgesetzt heranzuziehen.

Aus der bekannten Riemannschen Kovariante vierten Ranges leitet man folgende Kovariante zweiten Ranges ab:

$$G_{im} = R_{im} + S_{im} \tag{1}$$

$$R_{im} = -\sum_{l} \frac{\partial \begin{Bmatrix} im \\ l \end{Bmatrix}}{\partial x_l} + \sum_{l\rho} \begin{Bmatrix} il \\ \rho \end{Bmatrix} \begin{Bmatrix} m\rho \\ l \end{Bmatrix} \tag{1a}$$

$$S_{im} = \sum_{l} \frac{\partial \begin{Bmatrix} il \\ l \end{Bmatrix}}{\partial x_m} - \sum_{l\rho} \begin{Bmatrix} im \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho l \\ l \end{Bmatrix} \tag{1b}$$

[1] Sitzungsber. XLIV, S. 778 und XLVI, S. 799, 1915.

“Trace-Inverted” Einstein Equations

Taking the trace of the Einstein equation: $-R + 4\Lambda = 8\pi T$

Substituting back in the equation:

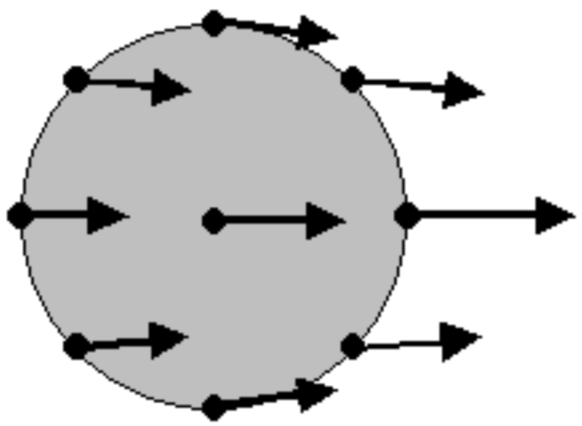
$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})$$

Observations:

- Gravitational part condensed in only the Ricci tensor
- Energy/Matter side explicitly dependent from the metric
- The cosmological constant part can be suggestively moved to the RHS:

$$R_{\mu\nu} = 8\pi \left[T_{\mu\nu} - \left(\frac{T}{2} + \frac{\Lambda}{8\pi} \right) g_{\mu\nu} \right]$$

Tidal Forces



Newton's equation for nearby points

$$\frac{d^2 x^i}{dt^2} = -\delta^{ik} \frac{\partial \Phi(x^j)}{\partial x^k},$$

$$\frac{d^2 (x^i + \xi^i)}{dt^2} = -\delta^{ik} \frac{\partial \Phi(x^j + \xi^j)}{\partial x^k}$$

Expanding in series:

$$\frac{d^2 (x^i + \xi^i)}{dt^2} = -\delta^{ik} \left(\frac{\partial \Phi(x^i)}{\partial x^k} + \frac{\partial}{\partial x^j} \left(\frac{\partial \Phi(x^i)}{\partial x^k} \right) \xi^j + \dots \right)$$

Newton's equation for tidal forces

$$\frac{d^2 \xi^i}{dt^2} = -\delta^{ik} \boxed{\left(\frac{\partial^2 \Phi}{\partial x^k \partial x^j} \right)} \xi^j$$

Tidal Tensor

Geodesic Deviation and Gravitation

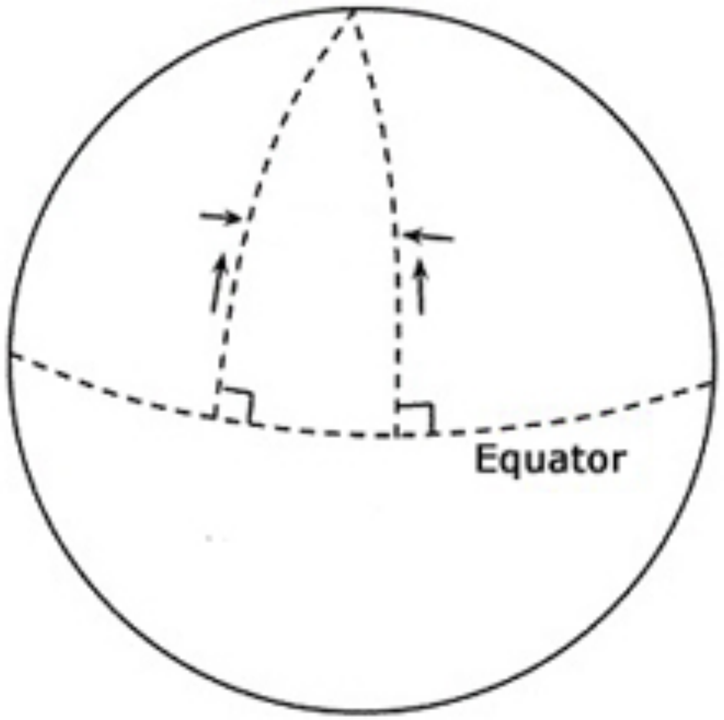
Trajectories of two nearby free-falling particles:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\frac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu_{\nu\lambda}(x^\mu + \delta x^\mu)\frac{d}{d\tau}(x^\nu + \delta x^\nu)\frac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0$$

Subtracting and keeping only 1st order terms:

$$\frac{d^2\delta x^\mu}{d\tau^2} + \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\rho}\delta x^\rho\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} + 2\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$



Introducing the covariant derivation along the curve:

$$\frac{D^2}{D\tau^2}\delta x^\lambda = R^\lambda_{\nu\mu\rho}\delta x^\mu\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

The Riemann tensor quantifies the amount of geodesic deviation.

Geodesic Deviation and Gravitation

If $R_{ijkl}=0$, there is no gravity
If $R_{ijkl}=0$, there are no tidal forces.
–> Tidal forces are the true manifestation of gravity.

If $R_{ijkl}=0$, we can always find a coordinate system which makes the metric equivalent to the one of a flat space (Minkowski metric).

Non-uniform gravitational fields are observable through geodesic deviation (only locally equivalent to accelerated frames: equivalence principle).

Uniform gravitational fields can be eliminated by a coordinate transformation and are non distinguishable from an uniformly accelerated frame.

Properties of the Einstein Equations

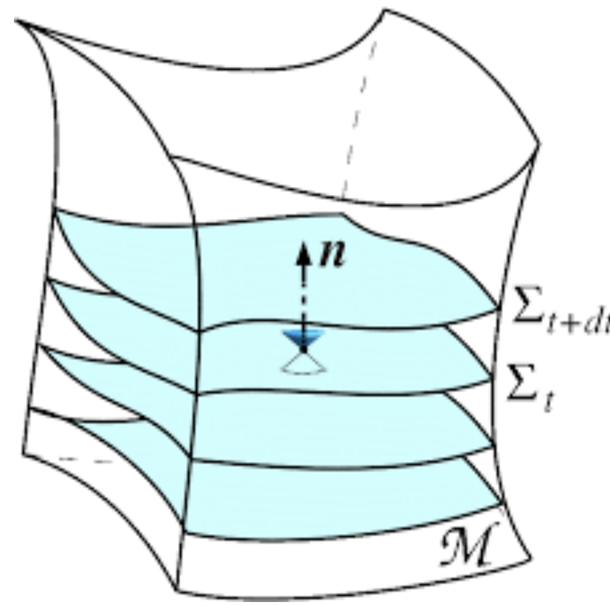
- Partial differential equations:
    - 10 Equations
    - non linear
    - hyperbolic
    - coupled

- Analog to the Maxwell equations for the 4-potential A.
    - Fundamental difference:
    - - for EM, given the 4-current, the fields can be calculated.
    - for GE, given T, we cannot calculate the metric tensor g, since it appears also in T.

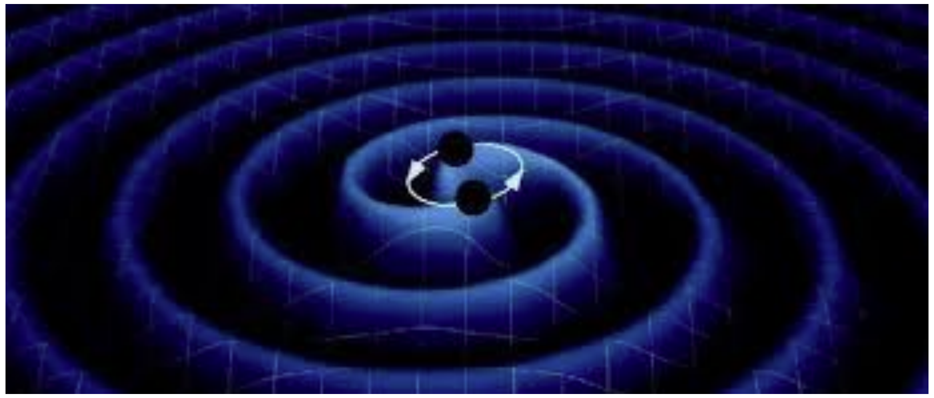
Solution of the Einstein Equations

Cauchy initial condition problem

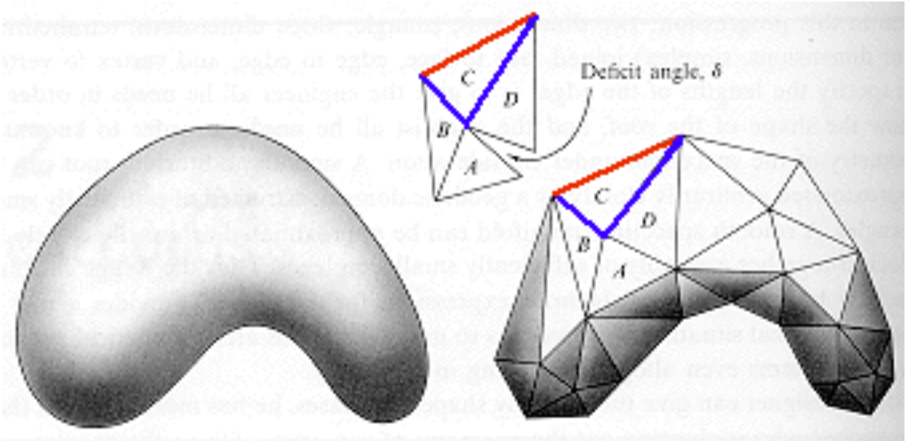
ADM Decomposition



Numerical Methods
(main recent dev.: adaptive meshes)



Regge Calculus



$$I_{\text{Regge}} = \frac{1}{8\pi} \sum_{\substack{\text{hinges,}\\ h}} A_h \varepsilon_h$$