Introductory Particle Cosmology

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Lecture 4

Introduction

Mathematical Tools

Generic Vector Bases
Basis Change
Vectors in Curved Geometries
Tensors, Metric Tensor
(Riemann) Manifolds
Connection
Geodesics and Curvature

General Relativity

Equivalence Principle
Einstein Equations

Cosmology

Cosmological Principle
FLRW Metric
Friedmann Equations
Cosmic Distances
Cosmological Models

Observation

CMB
Structure Formation
Red-shift/Distance

Standard Model

Brief History
Particle Content
Gauge Principle, CPV, Strong CP
EW Symmetry Breaking
Beyond the SM

Particle Cosmology

Dark Matter (Models + Exp.)
Dark Energy (Models + Exp)
Inflationary Models
Gravitational Waves
Density Perturbations

Plan

	Datum	Von	Bis	Raum
1	Di, 17. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
2	Do, 19. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
3	Di, 24. Apr. 2018	10:00	12:00	05 119 Minkowski-Raum
4	Do, 26. Apr. 2018	08:00	10:00	05 119 Minkowski-Raum
5	Do, 3. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
6	Di, 8. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
7	Di, 15. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
8	Do, 17. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
9	Di, 22. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
10	Do, 24. Mai 2018	08:00	10:00	05 119 Minkowski-Raum
11	Di, 29. Mai 2018	10:00	12:00	05 119 Minkowski-Raum
12	Di, 5. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
13	Do, 7. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
14	Di, 12. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
15	Do, 14. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
16	Di, 19. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
17	Do, 21. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
18	Di, 26. Jun. 2018	10:00	12:00	05 119 Minkowski-Raum
19	Do, 28. Jun. 2018	08:00	10:00	05 119 Minkowski-Raum
20	Di, 3. Jul. 2018	10:00	12:00	05 119 Minkowski-Raum
21	Do, 5. Jul. 2018	08:00	10:00	05 119 Minkowski-Raum

H.Minkowski
(1864-1909)

Physics Dept. Building, 5th Floor

Cosmological Models

The Cosmological Principle
The FLRW metric
Friedmann Equations
Cosmological Models and Parameters
LambdaCDM

The Cosmological Principle

The Universe is spatially homogeneous and isotropic

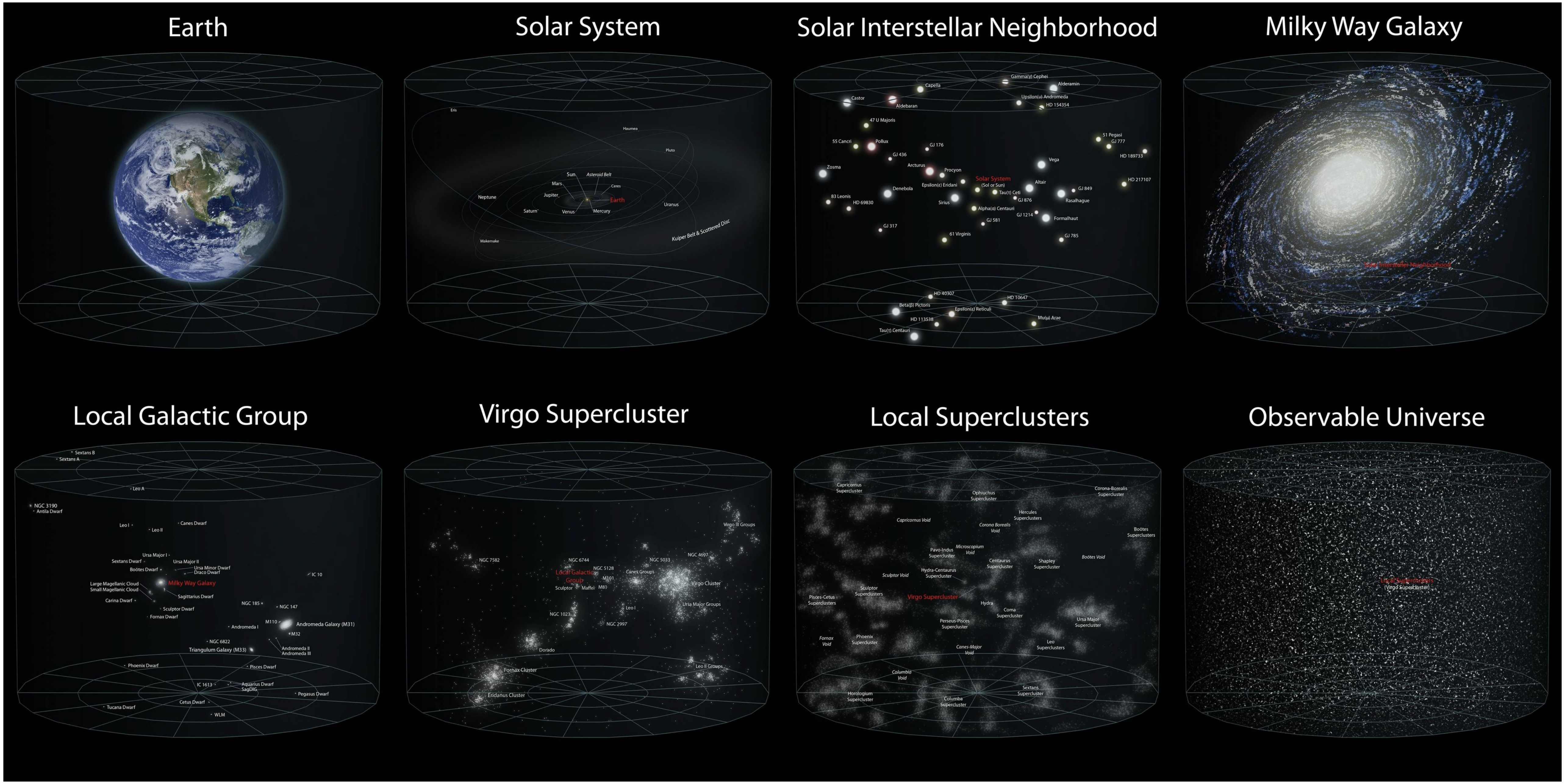


Image from Cryhavoc

The FLRW Metric

From the Cosmological Principle

we obtain the Friedmann-Lemaitre-Robertson-Walker metric:

$$\boxed{ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]}$$

Noting the rescaling invariance

$$R \to \frac{R}{\lambda}$$
$$r \to \lambda r$$
$$k \to \frac{k}{\lambda^2}$$

we can always choose k=+1,0,-1.
R can be scaled to e.g. 1 for t=today. Usually R(t) is called a(t): the scale factor.

FLRW

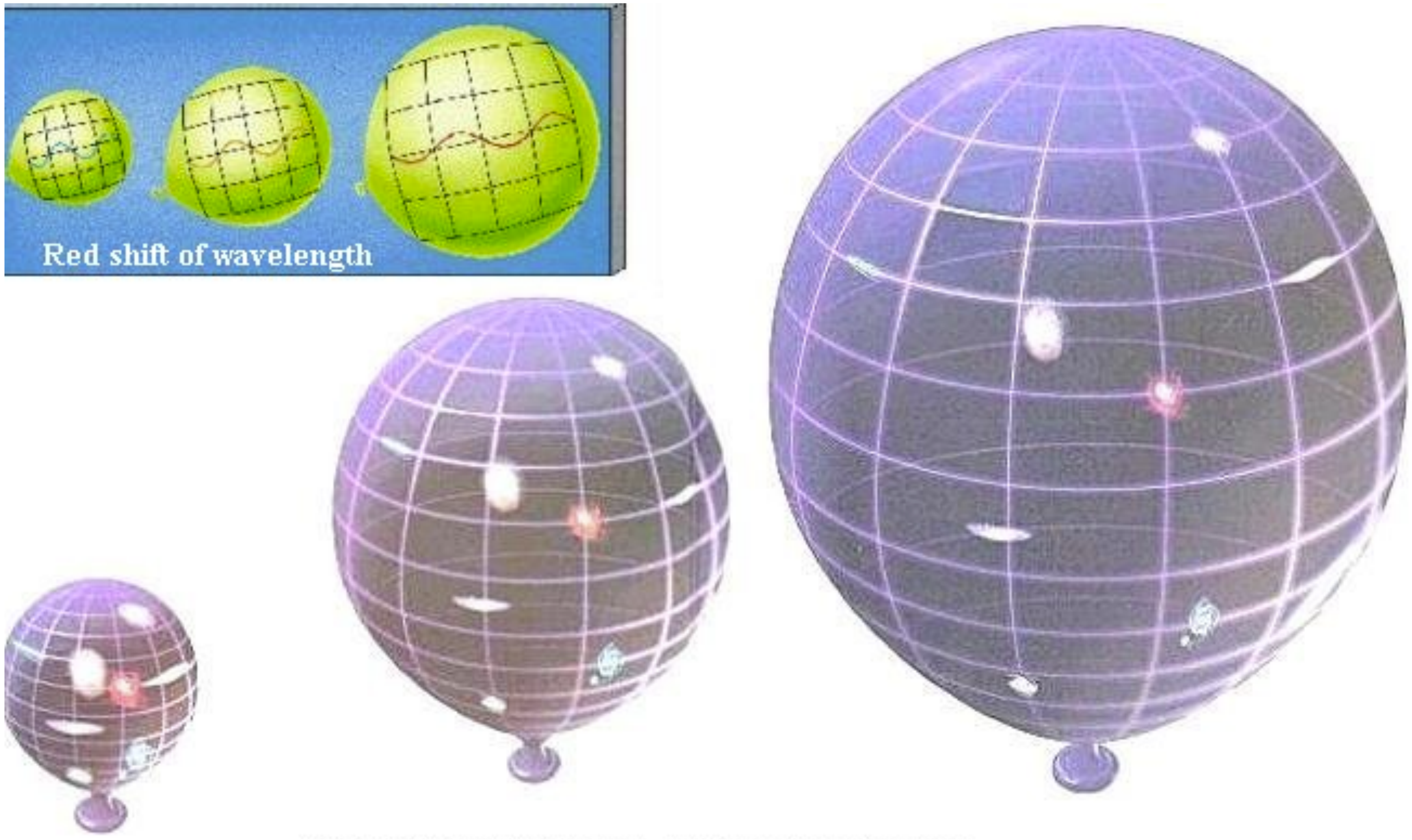
Alexander Friedmann (1888-1925)
Russian Physicist and Mathematician

Georges Lemaître (1894-1966)
Belgian Catholic Priest and Astronomer

Howard P. Robertson (1903-1961)
USA physicist and mathematician

Arthur G. Walker (1909-2001)
UK Mathematician

Comoving Coordinates



Einstein/Ricci Tensor for FLRW metric

Non-zero components of the FLRW metric tensor

$$g_{00} = 1$$
$$g_{11} = -\frac{a^2(t)r^2}{1-kr^2}$$
$$g_{22} = -a^2(t)r^2$$
$$g_{33} = -a^2(t)r^2\sin^2\theta$$

Non-zero components of the FLRW Christoffer symbols

$$\Gamma^0_{11} = \frac{\dot{a}a}{1-kr^2} \quad ; \quad \Gamma^0_{22} = a\dot{a}r^2 \quad ; \quad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta$$
$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}$$
$$\Gamma^1_{22} = -r(1-kr^2) \quad ; \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta \quad ; \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \quad ,$$

Non-zero components of the Ricci tensor

$$R_{00} = -3\frac{\ddot{a}}{a}$$
$$R_{11} = \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2}$$
$$R_{22} = r^2(a\ddot{a}+2\dot{a}^2+2kr)$$
$$R_{33} = r^2(a\ddot{a}+2\dot{a}^2+2kr)\sin^2\theta$$

Ricci curvature scalar

$$R = \frac{6}{a^2}(a\ddot{a}+2\dot{a}^2+k)$$

Energy-Momentum Tensor

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}} = \boxed{8\pi T_{\mu\nu}}$$

Calculated in the previous slide

Still missing…but we can use again the cosmological principle:

E/m tensor for an isotropic homogeneous "fluid":

$$T_{\mu\nu} = (\rho + P) v_\mu v_\nu - P g_{\mu\nu}$$

Density

Pressure

4-velocity

Note this!

GR is in general "hard" to solve: the metric shows up on BOTH sides of the Einstein Equations..

The "fluid" is at rest wrt the co-moving coordinates: v=(1,0,0,0)

The Friedmann Equations

Putting together the previous calculations, we can obtain the Friedmann equations:

Acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Hubble par. equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

The Friedmann equations are the Einstein equations for the FLRW metric and an isotropic homogenous fluid.

How to solve them for the scale factor a(t)? We need to know pressure and density of the "fluid", or at least a relation between then: an equation of state.

Equation of State

Energy-momentum conservation: $\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\beta} T^\beta_\nu - \Gamma^\beta_{\mu\nu} T^\mu_\beta = 0$

For the 0-th component: $\partial_0 \rho(t) + 3\frac{\dot{a}(t)}{a(t)}(\rho(t) + P(t)) = 0$

Choose the generic equation of state: $P = w\rho$

$$\boxed{\begin{aligned} \frac{\dot{\rho}}{\rho} &= -3(1+w)\frac{\dot{a}}{a} \\ \rho(t) &\propto a(t)^{-3(1+w)} \end{aligned}}$$

Cosmological Parameters

$$\boxed{\begin{aligned} H(t) &= \frac{1}{a}\frac{da}{dt} \quad \text{Hubble Parameter} \\ q(t) &= -\frac{1}{a}\frac{d^2a}{dt^2} \Big/ \left(\frac{1}{a}\frac{da}{dt}\right)^2 \quad \text{Deceleration} \end{aligned}}$$

$$j(t) = \frac{1}{a}\frac{d^3a}{dt^3} \Big/ \left(\frac{1}{a}\frac{da}{dt}\right)^3 \quad \text{"Jerk"}$$

$$s(t) = \frac{1}{a}\frac{d^4a}{dt^4} \Big/ \left(\frac{1}{a}\frac{da}{dt}\right)^4 \quad \text{"Snap" (or Jounce)}$$

$$c(t) = \frac{1}{a}\frac{d^5a}{dt^5} \Big/ \left(\frac{1}{a}\frac{da}{dt}\right)^5 \quad \text{"Crackle"}$$

$$p(t) = \frac{1}{a}\frac{d^6a}{dt^6} \Big/ \left(\frac{1}{a}\frac{da}{dt}\right)^6 \quad \text{"Pop"}$$

...

Hubble's Law

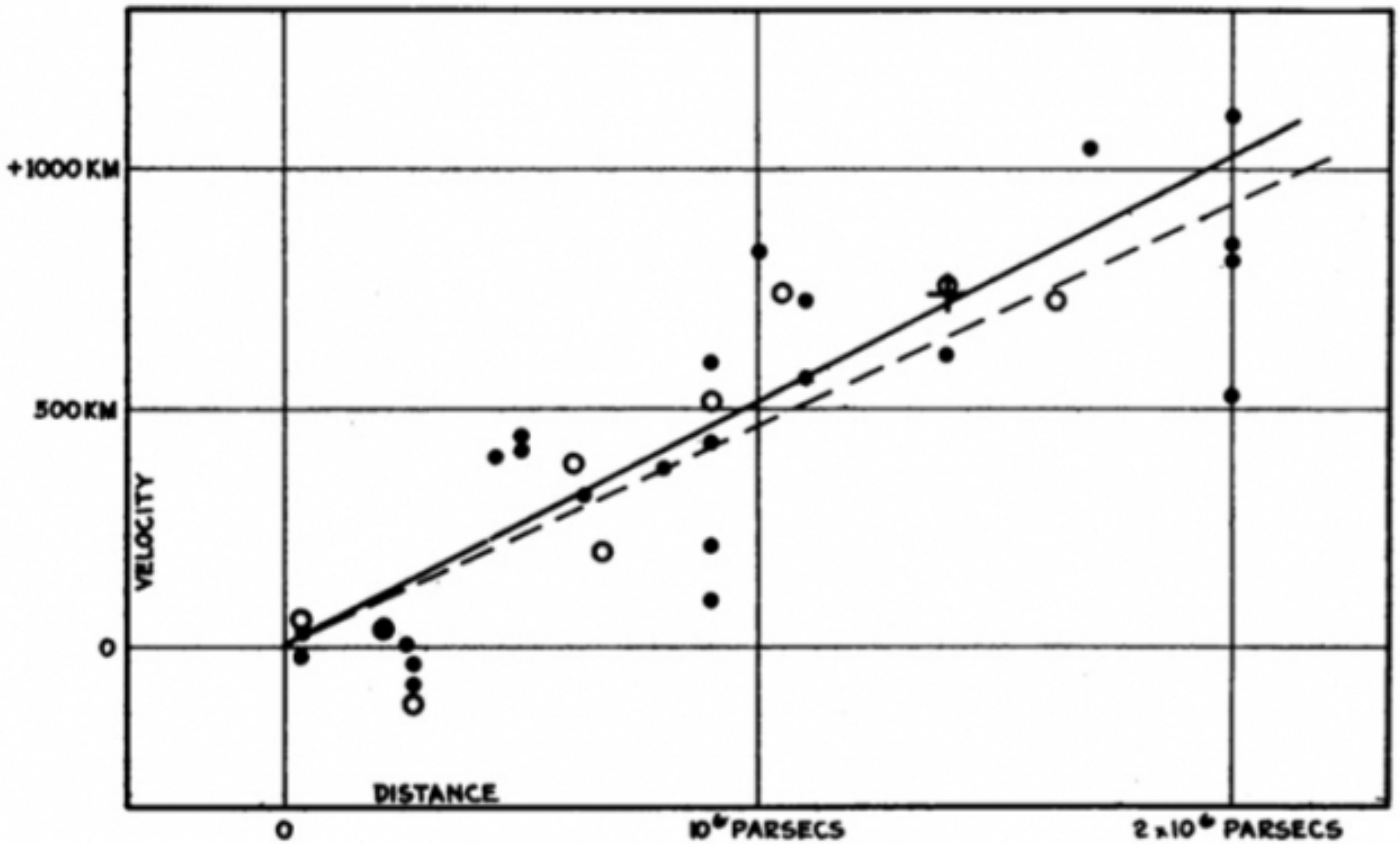
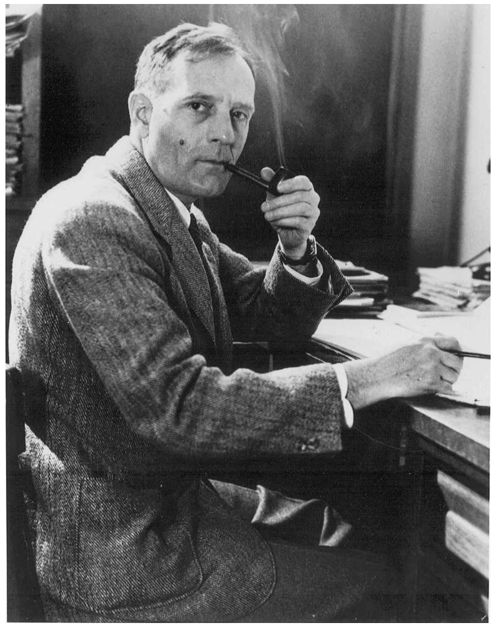


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.



Edwin Hubble (1889 – 1953)

$$d \propto v \quad \Rightarrow \quad \frac{\dot{a}}{a} = H \quad \Rightarrow \quad \dot{a} = Ha$$

Simplest Cosmological Models

Friedmann Equations
with Cosmological Constant Term

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P) + \boxed{\frac{\Lambda}{3}}$$
$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \boxed{\frac{\Lambda}{3}} \, .$$

Equations of state
$$P = w\rho$$

$$w = 0 \Rightarrow \rho \sim \frac{1}{a^3} \quad \text{"Dust"}$$
$$w = 1/3 \Rightarrow \rho \sim \frac{1}{a^4} \quad \text{"Radiation"}$$
$$w = -1 \Rightarrow \rho \sim const. \quad \text{"Vacuum"}$$

Matter-Dominated Universe

"Dust": $w = 0 \quad \Rightarrow \quad \rho \cdot a(t)^3 = A$

Define the Conformal Time $\dfrac{d\eta}{dt} = \dfrac{1}{a}$

The 2nd Friedmann equation becomes $a'^2 = \dfrac{8\pi G}{3} A a - k a^2$

With solutions

$$k = 1 \Rightarrow a = \frac{4\pi G A}{3}(1 - \cos\eta) \quad ; \quad t = \frac{4\pi G A}{3}(\eta - \sin\eta)$$

$$k = 0 \Rightarrow a = \frac{2\pi G A}{3}\eta^2 \quad ; \quad t = \frac{2\pi G A}{9}\eta^3$$

$$k = -1 \Rightarrow a = \frac{4\pi G A}{3}(\cosh\eta - 1) \quad ; \quad t = \frac{4\pi G A}{3}(\sinh\eta - \eta)$$

Radiation-Dominated Universe

Radiation: $w = 1/3 \Rightarrow \quad \dot{a}^2 = \frac{8\pi G}{3}\frac{A}{a^2} - k$

With solutions

$$k = 1 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t - t^2}$$

$$k = 0 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t}$$

$$k = -1 \Rightarrow a = \sqrt{2\sqrt{\frac{8\pi GA}{3a^2}}t + t^2}$$

Vacuum-Dominated Universe

Vacuum: $$P = \rho = 0 \Rightarrow \dot{a}^2 = \frac{\Lambda a^2}{3} - k$$

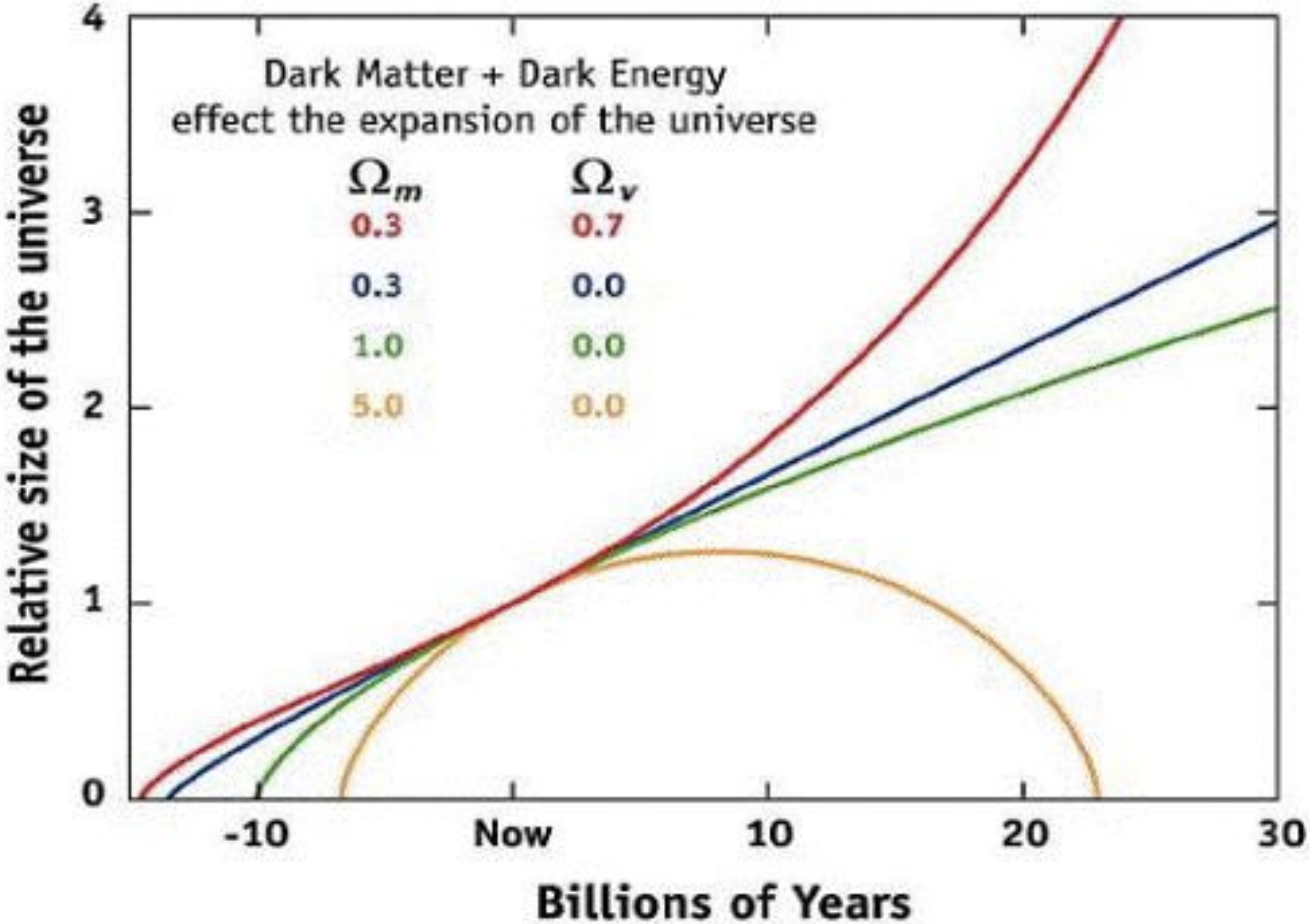
With solutions

$$k = 1 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right)$$

$$k = 0 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \exp\left(\sqrt{\frac{\Lambda}{3}} t\right)$$ ⟵ Note this result

$$k = -1 \Rightarrow a = \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}} t\right)$$

Summary (1)


Dark Matter + Dark Energy
effect the expansion of the universe
Ω_m
Ω_v
0.3
0.7
0.3
0.0
1.0
0.0
5.0
0.0
Relative size of the universe
0
1
2
3
4
-10
Now
10
20
30
Billions of Years

The Density Parameter

Remember the Hubble parameter: $$H(t) = \frac{\dot{a}(t)}{a(t)}$$

The 2nd Friedmann Equation becomes $$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \Rightarrow \frac{8\pi G}{3H^2}\rho - 1 = \frac{k}{H^2 a^2}$$

Introducing the Critical Density $$\rho_c = \frac{3H^2}{8\pi G}$$

And the Density Parameter $\Omega = \rho/\rho_c$ we have: $$\boxed{\Omega - 1 = \frac{k}{H^2 a^2}}$$

and therefore
$$\begin{aligned}
\rho < \rho_c &\Longleftrightarrow \Omega < 1 \Longleftrightarrow k = -1 \qquad \text{(Open)} \\
\rho = \rho_c &\Longleftrightarrow \Omega = 1 \Longleftrightarrow k = 0 \qquad \text{(Flat)} \\
\rho > \rho_c &\Longleftrightarrow \Omega > 1 \Longleftrightarrow k = 1 \qquad \text{(Closed)}
\end{aligned}$$

Mixed Models

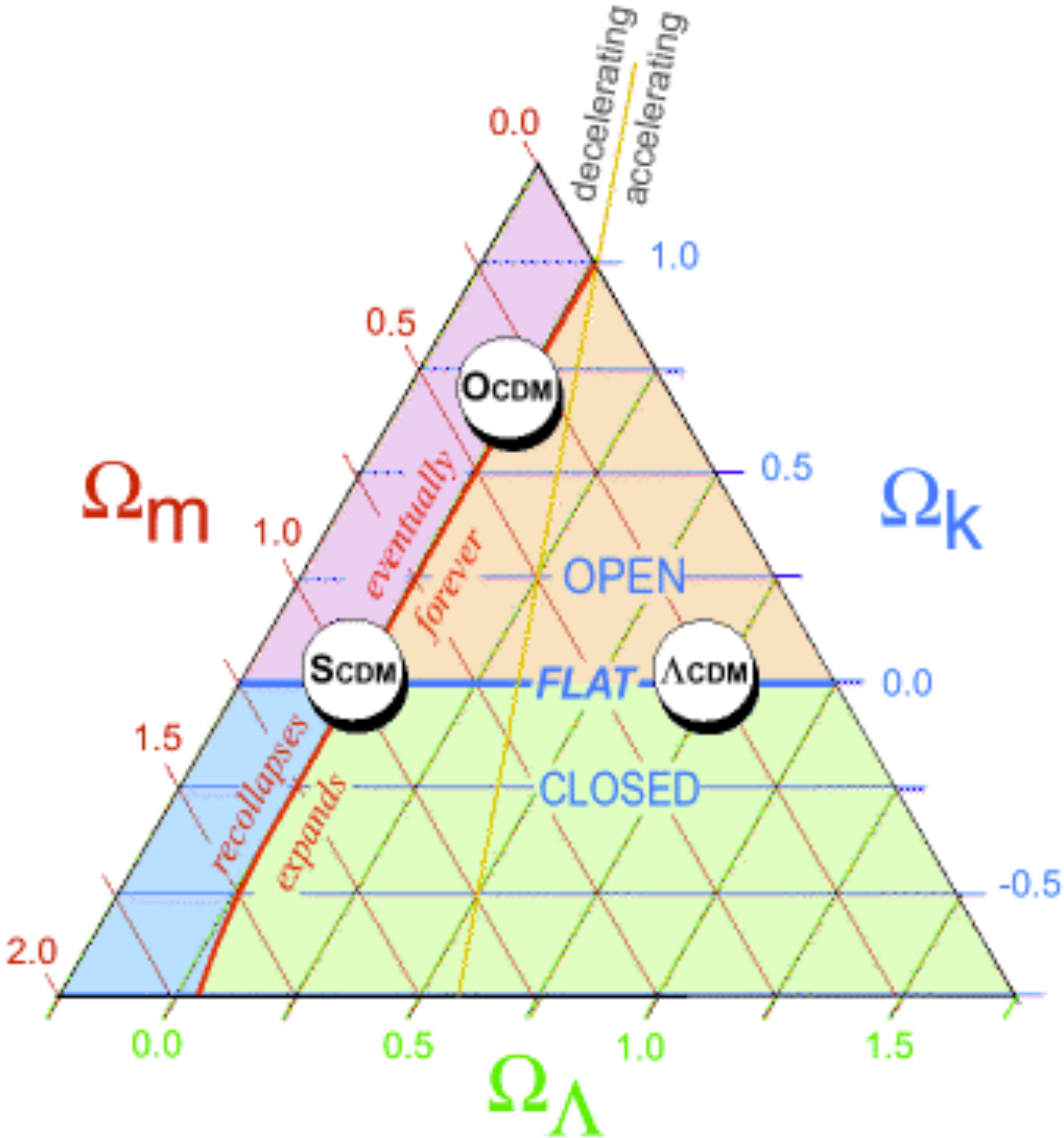
If more than a component is present $\rho_{TOT}(a) = \sum_i \rho_i(a) = \rho_C \sum_i \Omega_i a^{-3(1+w_1)}$

The Friedmann equation is then $\frac{k}{a^2} = H^2(\Omega_{TOT} - 1)$

And introducing the today's observed density parameters

$$\frac{k}{a_0^2} = H_0(\Omega_m + \Omega_r + \Omega_\Lambda - 1)$$

Summary 2



decelerating
accelerating
0.0
1.0
0.5
0.5
Ω_m
1.0
Ω_k
OCDM
eventually
forever
OPEN
SCDM
FLAT
ΛCDM
0.0
1.5
recollapses
expands
CLOSED
-0.5
2.0
0.0
0.5
1.0
1.5
Ω_Λ