NOTES ON MACROECONOMIC MODELS

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0.1 Von Neumann Growth Model

In the 1937 John von Neumann published (originally in German [1]) a simple linear model to describe optimal economic growth. Von Neumann introduced many simplifications to make equilibrium possible: constant returns to scale, pure and perfect competition, unlimited quantities of goods available through the productive process, no savings from workers who are depicted as very simple agents, and no consumption from producers who save the totality of their income. Production is considered a temporal process of transforming one set of goods into another. For simplicity, von Neumann also assumed that the cost of production of one good depends on the value of the goods necessary for its production, plus the interest rate.
He introduced a system in which N "technologies" can operate by combining M "commodities". Said in other words, N possible production processes act on M possible goods for producing other goods.
We define a matrix B_{cp} where $p = 1..N$ and $c = 1..M$ and represents, for a given process p, how much of the compound c it uses. Assuming B represents the "inputs" to a production process, an analog matrix A_{cp} will represent the "outputs", i.e. A describes how much of the compound j, the process i outputs. Another assumption of the model is linearity, in the following sense: if a process is run at scale s, then sB inputs will produce sA outputs. We further assume that alla the matrix elements and the scales s_p are zero or positive.
In a perfect supply-demand equilibrium (and forgetting for now about the scales s), considering a specific compound c we have the proportionality

$$\sum_p A_{cp} = \rho \sum_p B_{cp} \quad . \tag{1}$$

Considering compound prices p_c, we can now calculate the cost of running a certain process p: $\sum_p p_p B_{pc}$, which generates a revenue $\sum_p p_p A_{pc}$ (the vector p contains the prices). In equilibrium we expect

$$\sum_p p_p A_{cp} = \sigma \sum_p p_p X_{cp} \quad . \tag{2}$$

While the parameter ρ represents the grow factor, $\sigma = 1 + r$ is connected to the interest rate r one pays for borrowing the money for realizing the

production process (which has to be returned after each step).
The von Neumann growth problem is a relaxation of the latter equilibrium formulas and it is given by the dual optimization problem

$$max_{\rho>0}(\rho) \quad : \quad \sum_{p}^{N}(A_{pc}-\rho B_{pc})s_p \geq 0 \quad \forall c \tag{3}$$

$$min_{\sigma>0}(\sigma) \quad : \quad \sum_{c}^{M}p_c(A_{pc}-\sigma B_{pc}) \leq 0 \quad \forall p \tag{4}$$

with $p \geq 0$, $s \geq 0$.
Intuitively, the first problem (the "technological expansion problem") says that we would like to maximize the output of the economy and we have to find the maximum ρ and intensity of the processes s (the "optimal intensity vector") such that the output As exceeds the input Bs. The solution does not trivially correspond to choose a large ρ (the "technological expansion rate"), otherwise $A-\rho B$ will be negative.
The second (dual) problem requires that what we gain from the output is at most equal to what we spend for the input ,finding the optimal σ (the "optimal economic expansion rate") and p (the "optimal price vector").
It can be showed that generally the solutions $\bar{\rho}$ and $\bar{\sigma}$ respect $\bar{\rho} \geq \bar{\sigma}$ and with some more restrictive conditions $\bar{\rho} = \bar{\sigma}$ [2].
Another way to see the two dual problems is that the first is a resource allocation problem, while the second is a resource valuation problem.

0.2 Production Functions

A production function (PF) is a function $F : R_{+}^{N} \rightarrow R^{+}$ which connects the output of an activity to its production factors. The "activity" could be also the whole economy of one or more countries. A common example is the production function connecting the outout Y to capital K and labor L : $Y = F(K, L)$. More in general, $Y = F(x_1, x_2, x_3, ..)$.
A property of the PF which is generally assumed is *constant return to scale*

$$\lambda Y = F(\lambda x_1, \lambda x_2, ..) \quad , \tag{5}$$

so that if all the production factors are scaled by a quantity λ, also the output is scaled by the same quantity. Constant return to scale can be

applied only to some of the production factors: the typical example is the PF $Y = F(K, L, A)$, where A is an "efficiency" applied to one of the factors. Constant return to scale is applied only to K and L.
Relevant quantities characterizing a PF are the derivatives with respect to the production factors

$$F_{x_i} = \frac{\partial F}{\partial x_i} \quad . \tag{6}$$

Economically relevant properties of a PF are

$$\frac{\partial F}{\partial x_i} > 0 \tag{7}$$

$$H_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j} \quad \text{negative semidefinite} \tag{8}$$

which mean that F is a (monothonically) growing function of its inputs and that the growth rate decreases as the inputs increase ("law" of decreasing returns).
The latter properties describe F as a concave, growing function of the production factors.
Often other assumed properties are the so-called *Inada conditions*

$$f(0) = 0 \tag{9}$$

$$\lim_{x \to 0} F_{x_i} = +\infty \tag{10}$$

$$\lim_{x \to +\infty} F_{x_i} = 0 \quad . \tag{11}$$

Intuitively, the PF rises sharply as soon as we "switch on" a production factor x_i and then, as the factor increases, the PF levels off towads a zero growth regime.

0.3 Elasticity of Substitution

Elasticity of substitution (EoS) for a PF with two production factors is defined as

$$\sigma = \frac{d(x_2/x_1)/(x_2/x_1)}{d(F_{x_1}/F_{x_2})/(F_{x_1}/F_{x_2})} \quad . \tag{12}$$

EoS is the ratio of percentage change of a production factor ratio to the percentage change of the output changes. The most common production

factors are capital ($x_1 = K$) and labor ($x_2 = L$) and in the following we will stick to this specific choice.
Dividing the PF by L, $F(K, L) \to F(K/L, 1)$ and using the constant return tu scale property, $Y/L = F(K/L, 1)$. Introducing the *capital intensity* k=K/L (the capital per unit labor) and renaming $y = Y/L$, the PF reduces to

$$\frac{Y}{L} = F\left(\frac{K}{L}, 1\right) \Rightarrow y = f(k) \quad . \tag{13}$$

We would like now to relate the partial derivatives of F to the derivative of f. To this aim, we introduce first *Euler's theorem*. We start from the definition of an homogeneous function of degree θ, which has the defining property

$$\lambda^{\theta} Y = F(\lambda K, \lambda L) \quad . \tag{14}$$

If $\theta = 1$ we are clearly in the constant returns to scale case.
Theorem (Euler): if F(K,L) is homogeneous of degree θ, then

$$\theta F(K, L) = F_K K + F_L L \quad . \tag{15}$$

Proof: Differentiating the definition of homogeneous function with respect to λ

$$\frac{\partial F}{\partial(\lambda K)} \frac{\partial \lambda K}{\partial \lambda} + \frac{\partial F}{\partial(\lambda L)} \frac{\partial \lambda L}{\partial \lambda} = \theta \lambda^{\theta - 1} F(K, L) \tag{16}$$

$$\frac{\partial F}{\partial(\lambda K)} K + \frac{\partial F}{\partial(\lambda L)} L = \theta \lambda^{\theta - 1} F(K, L) \tag{17}$$

and fixing $\lambda = 1$ concludes the proof.
We can now go back to the problem of finding the relationship between derivatives of F and f.
Differentiating $Y/L = F(K, L)/L = f(k)$ with respect to K:

$$F_K = \frac{\partial F}{\partial K} = L \cdot f'(k) \cdot \frac{1}{L} = f'(k) \quad , \tag{18}$$

where $f'(k) = df/dk$.
Using Euler's theorem together with Eq. 18

$$y = Y/L = F_K k + F_L = f \Rightarrow \frac{\partial F}{\partial L} = F_L = f(k) - k f'(k) \tag{19}$$

Applying the definition of EoS (Eq. 12), we can obtain a differential equation for the elasticity:

$$\sigma = \frac{d(K/L)/(K/L)}{d(F_L/F_K)/(F_L/F_K)} = \frac{dk/k}{d\left(\frac{f-kf'}{f'}\right)} / \left(\frac{f-kf'}{f'}\right) \quad , \tag{20}$$

which can be rearranged as

$$\sigma = \frac{1}{k}\frac{f-kf'}{f'\frac{d}{dk}\left(\frac{f-kf'}{f'}\right)} \quad . \tag{21}$$

Calculating the derivative in the denominator we arrive at the form

$$\sigma = \frac{f'(f-kf')}{-kff''} \quad . \tag{22}$$

A relevant case considered in the economic literature is the one where σ is a constant (*constant elasticity of substiution* case, CES): Eq. 22 represents a second order differential equation which solutions will be discussed in the next section.

0.4 CES Production Functions

A production function is a relation between the output of an economy and its aggregate factors, like capital, labour, land, etc.. . In general, the output Y can be written as a function of the production factors X_i as

$$Y = F(X_1, X_2, X_3, ..) \quad . \tag{23}$$

A class of production functions can have specific properties connected with the *elasticity of substitution* (ES). ES of a variable X with respect to another variable Y is the ratio of their relative change:

$$ES(X,Y) = \frac{dX/X}{dY/Y} = \frac{dX}{dY}\frac{Y}{X} = \frac{d(\ln X)}{d(\ln Y)} \quad . \tag{24}$$

ES is a convenient definition since it is the ratio of dimensionless quantities.

Another useful case is when we would like to calculate the ES of a function f with respect of one of its variables x:

$$ES(f(x,y,z,..),x) = \frac{df/f}{dx/x} = \frac{xdf/fx}{f} = \frac{xf'}{f} \quad . \tag{25}$$

An interesting class of production functions is the one where ES is constant (usually the class is called CES: constant ES):

$$Y(X_1, X_2, X_3, ..) = A\left(\sum_i a_i X_i^{\rho}\right)^{k/\rho} \quad . \tag{26}$$

with $A > 0$, $k > 0$, $a_i \geq 0$, and $\sum_i a_i = 1$. The constants a are the *factor shares* and determine how much weight the single production factors X have in the total output Y.
We can show that the ES is $\sigma = 1/(1-\rho)$:

0.5 The Cobb-Douglas Production Function

A particularly simple (and convenient in the calculations) production function is the Cobb-Douglas form (CD). The CD function in its simplest form has only two production factors, for example capital K and labor L:

$$Y = K^{\alpha} L^{\beta} \quad . \tag{27}$$

By direct calculation we can show that CD has constant elasticity of substitution with respect to both the factors:

$$E(Y,K) = \frac{\partial Y}{\partial K}\frac{K}{Y} = \alpha \tag{28}$$

$$E(Y,L) = \frac{\partial Y}{\partial L}\frac{L}{Y} = \beta \tag{29}$$

The returns to scale condition $aY = f(aK, aL)$ (the production increases by an amount a if the factors increase by the same amount) forces the identification $\beta = 1 - \alpha$.
It is easy also to show that the CD respects also the Inada conditions.

Since the CD has the CES property, it must be contained in the family of functions defined by Eq. 26. Indeed, the CD results in the limit $\rho \to 0$:

$$\lim_{\rho \to 0} (\alpha K^{\rho} + (1-\alpha) L^{\rho})^{1/\rho} = K^{\alpha} L^{1-\alpha} \tag{30}$$

This can be proven taking the logarithm of the limiting function and then apply de L'Hopital's theorem. By comparison between the CD and the general CES function, α and $1-\alpha$ are the factor shares.
More in general, a CD production function can be written as

$$Y = TK^{\alpha} L^{1-\alpha} \quad . \tag{31}$$

where the constant T is the "total factor productivity".

0.6 The Harrow-Domar Model

The production function output depends only from the capital K:

$$Y = F(K) \quad , \tag{32}$$

and Y(0)=0 (no output with no capital). The marginal product of capital is assumed to be constant:

$$\frac{dY}{dK} = c \quad , \tag{33}$$

and if constant returns to scale is assumed

$$\frac{dY}{dK} = \frac{Y}{K} \quad . \tag{34}$$

If s is the fraction of the output which goes into savings S, it is assumed

$$sY = S = I \quad , \tag{35}$$

where I are the investments. It is therfore assumed that all the savings go into investments. If δ is the depreciation of the capital K, the variation in capital is

$$\Delta K = I - \delta K \quad . \tag{36}$$

From Eq. **??**, and Eq. 34, $Y = cK \quad \Rightarrow \quad \log Y = \log c + \log K$. Taking the time derivative of the last "log" equation,

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} \quad . \tag{37}$$

Using now Eq 35 and Eq. 36

$$\frac{\dot{K}}{K} = \frac{I}{K} - \delta = s\frac{Y}{K} - \delta \quad , \tag{38}$$

which yields the final dynamical equation of the model for the output rate as a function of the parameters

$$\frac{\dot{Y}}{Y} = sc - \delta \quad . \tag{39}$$

0.7 The Solow-Swan Model

The Solow model builds on the Harrow-Domar model including labor L as an additional variable. Moreover, the labor "intensity" A is also considered and measures the efficiency with which labor is employed.
The production function is thus

$$Y = F(K, L, A) = F(K, AL) \quad , \tag{40}$$

where the factor A enters only multiplicatively with the labor. In general, the production factors are functions of time (e.g. K=K(t)).
Assuming constant returns to scale,

$$\frac{Y}{AL} = F\left(\frac{L}{AL}, 1\right) = \frac{1}{AL}F(K, AL) \quad . \tag{41}$$

Defining y=Y/AL and k=K/AL,

$$y = f(k) \quad . \tag{42}$$

Assuming a Cobb-Douglas form for F (which respects constant returns to scale and the Inada conditions)

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \quad , \tag{43}$$

we can now to specify the time evolution of A and L. A common choice is a constant growth rate

$$\dot{L} = nL \Rightarrow L(t) = L(0)e^{nt} \quad , \tag{44}$$

$$\dot{A} = gA \Rightarrow A(t) = A(0)e^{gt} \quad , \tag{45}$$

where n and g are given ("exogenous") constants.
The capital evolution can be written as

$$\dot{K} = sY - \delta K \quad , \tag{46}$$

where s is the fraction of output saved for investments, and δ is the existing capital depreciacion.
For deriving the time evolution, we can derive the "capital intensity" k = K/AL with respect to time

$$\dot{k} = \frac{d}{dt}\left(\frac{K}{AL}\right) = \frac{\dot{K}}{AL} - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A} = \frac{sY - \delta K}{AL} - kn - kg \quad , \tag{47}$$

where we used Eq. 44, 45, 46. Using again the capital intensity definition and that $Y/AL = f(k) = k^\alpha$ we obtain the dynamical equation for the Solow model (for a Cobb-Douglas production function)

$$\dot{k} = sk^\alpha - (n + g + \delta)k \quad . \tag{48}$$

The last equation says that the change in capital per effective labor is equal to the difference between investments per effective labor and the "breakeven investment". The breakeven investment is the minimal investment needed for keeping k at the existing level.
The solution of the Solow equation converges to an asymptotic capital intensity $k = k^*$, which can be calculated considering when k does not change anymore, or $\dot{k} = 0$:

$$k^* = \left(\frac{s}{n + g + \delta}\right)^{1/(1-\alpha)} \quad . \tag{49}$$

Considering the output intensity at k^*

$$y^* = \left(\frac{s}{n + g + \delta}\right)^{\alpha/(1-\alpha)} \quad , \tag{50}$$

and taking the ratio eliminating AL

$$\frac{K}{Y} = \frac{s}{n + g + \delta} \quad , \tag{51}$$

which states that the capital/output ratio depends only from the growth constants of the model.
When $k = k^*$, since $K = ALk$ and AL grows at rate $(n+g)$, so does K. From the constant returns assumption, since A, L, and K grow at constant rate, so does the output Y. One of the main conclusions of the model is that on the "balanced growth path" (i.e. when $k = k^*$), all the variables grow at a constant rate, and this will happen regardless from the starting point.

0.8 Analytic solution of the Solow Model

In the case of a Cobb-Douglas production function $Y = AK^\alpha L^{1-\alpha}$ (with $0 < \alpha < 1$), the Solow model

$$\dot{k} = sAk^\alpha - (n+g+\delta)k \quad . \tag{52}$$

admits an analytic solution.
Noting that the previous equation is a Bernouilli-type differential equation, it can be solved with the standard substitution $z = Ak^{1-\alpha}$ obtaining a non-homogeneous linear differential equation for $z(t)$:

$$\dot{z} = (1-\alpha)A^2 s - (1-\alpha)(n+g+\delta)z = (1-\alpha)(A^2 s - \omega z) \quad . \tag{53}$$

where $\omega = n+g+\delta$. This equation has an economic interpretation since $z(t)$ is the capital-output ratio. Therefore, the capital-output ratio in the Solow model with Cobb-Douglas production obeys a simple linear differential equation.
Eq. 53 can be solved by separation of variables:

$$\frac{dz}{(A^2 s - \omega z)} = (1-\alpha)dt \quad , \tag{54}$$

and integrating both sides

$$-\frac{1}{\omega}\ln(A^2 s - \omega z) = (1-\alpha)t + C \quad , \tag{55}$$

where C is an integration constant. After some algebra we obtain the solution

$$z(t) = \frac{sA^2}{\omega} + Ce^{-\omega(1-\alpha)t} \quad . \tag{56}$$

Restoring the original capital intensity variable $k = z^{\alpha-1}/A$ and fixing the initial condition $k(0) = k_0$ we find for the integration constant

$$C = Ak_0^{1\alpha} - \frac{sA^2}{\omega} \quad . \tag{57}$$

The analytic solution of the model is finally

$$k(t) = \left[\frac{sA^2}{\omega} + \left(Ak_0^{1-\alpha} - \frac{sA^2}{\omega}\right) e^{-\omega(1-\alpha)t}\right]^{\frac{1}{1-\alpha}} \quad . \tag{58}$$

For large times, the exponential term goes to zero and the solution converges to a constant k^* which is exactly what we calculated in the previous section (Eq. 49) with the condition $\dot{k} = 0$

0.9 The Mankiw-Romer-Weil Model

Along the lines of the Solow model, one can consider a third production factor, for example the human capital H, and the (Cobb-Douglas) production function becomes

$$Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta} \quad . \tag{59}$$

Assuming that H depreciates with the same rate as K and a similar dynamical law as for the capital (see the Solow model), the resulting dynamics is given by

$$\dot{k} = s_K k^\alpha h^\beta - (n + g + \delta)k \tag{60}$$

$$\dot{h} = s_H k^\alpha h^\beta - (n + g + \delta)h \tag{61}$$

0.10 A model with four production factors

The production function can in principle contain even more factors. One example is a model where we consider capital, labor, available land T, and natural resources R

$$Y = K^\alpha R^\beta T^\gamma (AL)^{1-\alpha-\beta-\gamma} \quad . \tag{62}$$

Since land cannot grow, $\dot{T} = 0$ and since resources should diminish with time we assume $\dot{R} = -bR$ with $b > 0$.
The model is now quite more complicated and there is no convergence to a fixed capital intensity k^*. The analysis strategy for this model is to look for a balanced growt path where K and Y grow at a constant rate. From the capital change equation

$$\dot{K} = sY - \delta K \Rightarrow \frac{\dot{K}}{K} = s\frac{Y}{K} - \delta \quad , \tag{63}$$

thus the growth rate for K is constant only if Y/K is constant, i.e. the growth rate of Y and K must be the same. We can use the production function to see when this happens. Taking the logarithm of the production function we have

$$\ln Y = \alpha \ln K + \beta \ln R + \gamma \ln T + (1 - \alpha - \beta - \gamma)(\ln A + \ln L) \quad , \tag{64}$$

and differentiating with respect to time we convert logs into growth rates

$$g_Y = \alpha g_K + \beta g_R + \gamma g_T + (1 - \alpha - \beta - \gamma)(g_A + g_L) \quad . \tag{65}$$

Substituting the grow rates we decided for our model

$$g_Y = \alpha g_K + \beta b + \gamma g_T + (1 - \alpha - \beta - \gamma)(n + g) \quad . \tag{66}$$

Since on the balanced growth path Y and K grow at the same rate, $g_Y = g_K$ we have

$$g_Y^* = \frac{(1 - \alpha - \beta - \gamma)(n + g) - \beta b}{1 - \alpha} \tag{67}$$

From the last equation,

$$g_{Y/L}^* = g_Y^* - g_L^* = \frac{(1 - \alpha - \beta - \gamma)(n + g) - \beta b}{1 - \alpha} - n = \frac{(1 - \alpha - \beta - \gamma)g - \beta b - (\beta + \gamma)n}{1 - \alpha} \quad . \tag{68}$$

The last equation indicates that the income per worker (Y/L) can have positive or negative values. The limitations in land and resources can limit or invert the income/worker growth rate.

0.11 The Utility Function (simplified treatment)

In the following we describe a simple ("neoclassical") consumption model based on a rational consumer who maximizes its utility.
For simplifying the analysis, we consider only two "times": the present and the future, assuming that the past does not influence the consumer's choices.
We define the financial wealth of an individual at time t with f_t. For example, the present wealth is f_{today} and the future one f_{future}. Analogously, the labor income is y_{today} ($y_{tomorrow}$).
If the consumption of the individual is c, then we have the two budgetary constraints:

$$c_{today} = y_{today} - (f_{future} - f_{today}) \tag{69}$$

$$c_{future} = y_{future} - (1+R)f_{future} \tag{70}$$

The two latter equations refer to different point in time but have the same form as:

Consumption equals income - savings

with R the interest rate. Substituting f_{future} in the second equation into the first we obtain the **intertemporal budget constraint**

$$c_{today} + \frac{c_{future}}{1+R} = f_{today} + y_{today} + \frac{y_{future}}{1+R} \tag{71}$$

The equation simply states that consumption must equal the total wealth [1]. Future incomes and consumptions are discounted by the interest rate.
We define now the "utility" which the consumer obtains from consumption as a function $u(c)$

$$U = u(c_{today}) + \beta u(c_{future}) \tag{72}$$

[1]A caveat to this equation is that y should be interpreted as income after taxes. The **Ricardian Equivalence** assumes that a change in timing of the taxes does not affect consumption. In the present model, the Ricardian equivalence is respected, since the total wealth $W = f_{today} + y_{today} + \frac{y_{future}}{1+R}$ does not change if taxes are subtractred today and added in the future or the other way around.

where we distinguished present and future consumption and a possible different weight between the two indicated by β. Following the *deminishing marginal utility* assumption, we expect u(c) to have a decreasing first derivative with respect to c. We assume that the target of the consumer is to maximize its utility under the budgetary constraint, so the problem is

$$\max_{c_{today}, c_{future}} u(c_{today}) + \beta u(c_{future}) \tag{73}$$

$$c_{today} + \frac{c_{future}}{1+R} = W \tag{74}$$

where W is the total (lifetime) wealth (see Eq. 71). Solving the constrained optimization problem we obtain

$$u'(c_{today}) = \beta(1+R)u'(c_{future}) \quad , \tag{75}$$

which is called the **Euler Equation** for the consumption.
The equation says that the consumer is indifferent in consuming one "unit" today or saving it for the future or the other way around.
A possible (respecting the deminishing marginal utility) very simple for for the utility function is $u(c) = \log(c)$. With this form, the Euler equation is

$$\frac{c_{future}}{c_{today}} = \beta(1+R) \quad . \tag{76}$$

The left-hand side is the growth rate of consumption and the equation says that it is equal to the interest rate (modified by his propensity to consume more or less in the future through the parameter β). Thus the less weight on future utility ($\beta < 1$), the lower the consumption grows.
If the growth rate is the one of the whole economy, as for example the one described in a Solow-like model, then the Euler equation establishes its connection with the interest rate for the consumers.
The general consideration here is that growth rates and interest rates are tightly related.

0.12 The Utility Function

The "instantaneous" utility function $u(c)$ is a function of the consumption c. Its absolute value has no significance, so u is in general defined

modulo an affine transformation.
What characterizes the utility function the most is its concavity (convexity) which differentiates among risk-averse or risk-affine agents. The higher the curvature, the high the risk-aversion (affinity), thus the second derivative of u contains the aversion information. Since the absolute value of u does not matter and a quantity describing risk should not be dependent from affine transformations and the unit of measure, we define the **Arrow-Pratt** relative risk aversion as

$$RRA = -c\frac{u''(c)}{u'(c)} \quad . \tag{77}$$

Renaming for now RRA=k, the previous definition is equivalent to the differental equation

$$u'' + \frac{k}{c}u' = 0 \quad . \tag{78}$$

Defining $v(c) = u'(c)$ the equation becomes $v' + (k/c)v = 0$, which can be resolved by separation of variables. Restoring the original utility function $u(c)$ by integration, the solution for a constant RRA is

$$u(c) = \frac{c(t)^{1-\theta} - 1}{1-\theta} \quad , \tag{79}$$

By direct calculation from the definition in Eq. 77, for the CRRA function we have $RRA = \theta$.
Another interesting property is the **elasticity of intertemporal substitution (EIS)**. As defined before, the elasticity is the ratio of two relative changes. In the consumption case, the EIS is the relative change in consumtion rate for a relative change of the interest rate (or the change in utility rate)

$$EIS = -\frac{\partial(\dot{c}/c)}{\partial(\dot{u'}/u')} = -\frac{\partial(\dot{c}/c)}{\partial u'' \cdot c/u'} = -\frac{\partial(\dot{c}/c)}{\partial(RRA \cdot \dot{c}/c)} = \frac{1}{RRA} = -\frac{u'}{u''c} \quad , \tag{80}$$

where the dots are time derivatives and primes are derivatives with respect to the consumption. For the CRRA function therefore,

$$EIS = \frac{1}{RRA} = \frac{1}{\theta} \quad . \tag{81}$$

Another observation is that for $\theta = 1$, the CRRA function reduces to $u(c) = \log c$.

0.13 A basic DSGE Model

Dynamical Stochastic General Equilibrium models are an evolution of the RBC models including a stochastic dynamics for some of their variables. Here we take a look at one of their simplest forms. In general, the problem consists in maximizing the utility function subject to a budget constraint. The utility u is a function of the consumption at time t c_t, which is a stochastic variable:

$$\max_{c(t)} \sum_{t=0}^{\infty} \beta^t E[u(c_t)] \quad . \tag{82}$$

The problem is formulated in discrete time and the utility is maximized over the lifetime $[0, \infty]$, while β is the discount factor. The operator $E[]$ is the expectation value.
The budget constraint is

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \quad , \tag{83}$$

where f is a production function depending on the capital k. For example, we can take $f = A_t k^\alpha$, where the productivity A_t is a stochastic variable, i.e. A_t is known, but A_{t+1} is not. The parameter δ is the capital depreciacion. The budget constraint means that one can choose between consumption today, or more capital tomorrow (t+1) and their sum must be equal to the production plus the previous capital level $(1-\delta)k_t$.
The constrained optimization problem to solve is thus

$$L = E\left[\sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \lambda_t(c_t + k_{t+1} - f(k_t) - (1-\delta)k_t)\right]\right] \quad . \tag{84}$$

Before solving the maximization problem, we notice first that the expectation value operator is linear ($E[ax+by] = aE[x] + bE[y]$ if x,y are stochastic and a,b numbers). This means that the operator acts on all the terms in the sum over times and β^t, being a number can be factored out.
Second, when taking the derivatives with respect to a variable at a certain time, all the terms at different times vanish. The first order conditions are the derivatives of the Lagrangian Eq. 84 with respect to the variables c_t and k_{t+1}, plus the derivative with respect to the constraint λ_t. The

most involved calculation involves k_{t+1} and we will give here some more details about that:

$$\frac{dL}{dk_{t+1}} = \frac{d}{dk_{t+1}} E\left[\sum_{t=0}^{\infty} -\lambda_t(c_t + k_{t+1} - f(k_t) - (1-\delta)k_t)\right] = \tag{85}$$

$$= E\left[-\beta^t \lambda_t \frac{dk_{t+1}}{dk_{t+1}} - \frac{d}{dk_{t+1}} \sum_{t=0}^{t=\infty} \left(\beta^t \lambda_t(-f(k_t) - (1-\delta)k_t)\right)\right] = \tag{86}$$

$$-\lambda_t \beta^t + ... + \frac{d}{dk_{t+1}} E\left[\beta^{t+1}(\lambda_{t+1} f(k_{t+1}) + 1 - \delta) + ...\right] = \tag{87}$$

$$= -\lambda_t \beta^t + \beta^{t+1} E\left[\lambda_{t+1} f'(k_{t+1}) + 1 - \delta\right] \quad . \tag{88}$$

where we dropped the $u(c_t)$ and c_t terms since they do not depend on k and we used the linearity of E. The terms of the sum marked with "..." are zero under the action of the derivative.
We can write now the first order conditions:

$$\frac{dL}{dc_t} = u'(c_t) - \lambda_t = 0 \quad , \tag{89}$$

$$\frac{dL}{dk_{t+1}} = -\lambda_t + \beta E\left[\lambda_{t+1}[f'(k_{t+1}) + 1 - \delta]\right] = 0 \quad , \tag{90}$$

$$\frac{dL}{d\lambda_t} = c_t + k_{t+1} - f(k_{t+1}) - (1-\delta)k_t = 0 \quad . \tag{91}$$

Combining the first two equations, we obtain the so-called Euler equation (compare with Eq. 75)

$$\frac{u'(c_t)}{\beta E[u'(c_{t+1})]} = E[f'(k_{t+1}) + 1 - \delta] \quad . \tag{92}$$