On a problem of statistical inference in null recurrent diffusions

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Abstract: We consider a particular example of statistical inference in null recurrent one-dimensional diffusions. In a first parametric model, we prove local asymptotic mixed normality (LAMN) and efficiency of the sequence of maximmum likelihood estimates (MLE): its speed of convergence is $n^{\alpha/2}$ with α ranging over $(0,1)$. In a second semiparametric model (where in addition an unknown nuisance function with known compact support is included in the drift), we prove a local asymptotic minimax bound and specify asymptotically efficient estimates for the unknown parameter.

Key words: diffusions, null recurrence, limit theorems, parametric inference, semiparametric model, LAMN, convolution theorem, local asymptotic minimax bound

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Running head: inference in null recurrent diffusions

Introduction: Problems of statistical inference in one-dimensional ergodic diffusion models have been studied extensively in the last decade, see Prakasa Rao (1999) or Kutoyants (2001) for an overview, whereas only few results seem to exist for models where the observed diffusions are null recurrent. This is due to the fact that existing limit theorems for null recurrent Markov processes are more complicate, with domain of applicability delimited by additional and sometimes really restrictive conditions, and no globally applicable tool (such as the classical martingale convergence theorem in the ergodic cases) exists.

The aim of this note is to consider some examples of statistical inference in null recurrent diffusions. The observed process is a diffusion

$$dX_t \;=\; \left(\vartheta\,\frac{X_t}{1+X_t^2}+g(X_t)\right)dt+\sigma\,dW_t\,,\quad t\geq 0\,,\quad X_0=0$$

where ϑ is an unknown one-dimensional parameter of interest; $\sigma>0$ is known and constant, and

g is some function on $I\!R$ with finite integral $\int |g|(x)dx < \infty$ which is considered as nuisance function. We take $\Theta = \left(-\frac{\sigma^2}{2}, +\frac{\sigma^2}{2}\right)$ as parameter space; this is the maximal open interval on which the above diffusion is recurrent null. ϑ is the parameter which determines speed of convergence of martingales and integrable additive functionals of X, and is linearly related to the tail index (in the sense of regular variation) of the invariant measure of X (note that this invariant measure is of infinite total mass on $I\!R$), or of life cycle length distributions of X between suitably defined successive visits of the diffusion to 0.

In section 1, we state the basic limit theorem (theorem A) from which asymptotic properties of statistical models and of estimates of ϑ in this null recurrent setting will be deduced. This limit theorem puts together results due to Khasminskii (1980) and to Touati (1988); for a proof (general Harris recurrent Markov processes) see Höpfner and Löcherbach (2000).

In section 2, we consider the case of the parametric model – without nuisance function in the drift – for the diffusion X, i.e. $g \equiv 0$. We determine the limit distribution of the maximum likelihood estimate $\widehat{\vartheta}_n$ for ϑ based on continuous observation of X up to time n: for $\vartheta \in \Theta$, we have convergence in law

$$(*) \qquad n^{\alpha(\vartheta)/2}\left(\widehat{\vartheta}_n - \vartheta\right), \quad \alpha(\vartheta) := \frac{1}{2} - \frac{\vartheta}{\sigma^2} \in (0,1), \quad \vartheta \in \Theta$$

to a variance mixture of normals

$$(**) \qquad \int \mathcal{N}(0, j^{-1}) P(J \in dj), \quad J = J(\vartheta)$$

where the mixing variable $J(\vartheta)$ is stricly positive and has – up to constants specified in theorem 1 below – a Mittag Leffler law of index $\alpha(\vartheta)$, i.e. $J(\vartheta)$ is a level crossing time for the stable increasing process of index $\alpha(\vartheta)$ (see Bingham, Goldie and Teugels, 1987, p. 391). Then we discuss the local structure of the parametric model at fixed reference points ϑ: we have local asymptotic mixed normality (LAMN) with local scale $n^{-\alpha(\vartheta)/2}$ at ϑ. For the statistical background on LAMN, we refer to LeCam and Yang (1990), Jeganathan (1988), and Davies (1985). For our problem, we can apply both the convolution theorem and the local asymptotic minimax theorem to see that the sequence of estimates $(\widehat{\vartheta}_n)_{n\geq 1}$ is asymptotically efficient for $\vartheta \in \Theta$. (We mention also a minor but handsome variant of the convergence result, in form of random norming

$$\sqrt{\int_0^n \left(\frac{X_t}{1+X_t^2}\right)^2 dt}\ \left(\widehat{\vartheta}_n - \vartheta\right) \quad \to \quad \mathcal{N}(0, \sigma^2)$$

thanks to the fact that the information process in our model does not depend on the parameter).

From an inference point of view, the parametric model for our time-continuously observed diffusion X (with $g = 0$) shares all its essential statistical features with a certain one-parametric model for null recurrent birth-and-death processes considered earlier in Höpfner (1990 a): first, for every value of the parameter ϑ, the process has i.i.d life cycles between suitably defined successive visits to some recurrent point (e.g. state 0), and the expected amount of information collected over a full life cycle of the process is finite; second, tails of the life cycle distribution vary regularly at ∞ with some tail index $-\alpha$, $0 < \alpha < 1$, which depends on ϑ and ranges over the full interval $(0, 1)$ with varying ϑ. See also Höpfner (1990 b, 1993). The first property can be reformulated as follows: all information processes are integrable (in the sense of invariant measure) additive functionals of X. This is the reason why we call the parametric model under consideration (here in its diffusion process version) a reference model for null recurrent Markov processes.

In section 3, we turn to the general semiparametric model where an unknown nuisance function g is present in the drift of the diffusion X. We suppose that g ranges over the class $\mathcal{H}_c$ of bounded measurable functions supported by a known compact $[-c, +c]$, and consider estimates $\widehat{\vartheta}_n^c$ for ϑ which retain – out of the trajectory of X observed up to time n – only segments corresponding to time intervals where $|X| > c$. Models of this type occur naturally when we work with measurement devices unable to record 'small' values of the process, or unreliable below some known threshold c characterizing the measurement device. We prove convergence of $\widehat{\vartheta}_n^c$ in the semiparametric model: the speed of convergence is again $n^{\alpha(\vartheta)/2}$, and the limit law coincides with $(**)$ up to a factor of spread $\gamma(\vartheta, c) > 1$:

$$(***) \qquad n^{\alpha(\vartheta)/2}\left(\widehat{\vartheta}_n^c - \vartheta\right) \quad \to \quad Z(\vartheta) \sim \gamma(\vartheta, c) \int \mathcal{N}(0, j^{-1}) P(J(\vartheta) \in dj)\,.$$

The remarkable point is that $\gamma(\vartheta, c)$ does not depend on the nuisance function g from class $\mathcal{H}_c$. Then we investigate local properties of the semiparametric model. For $\vartheta_0 \in \Theta$ and for nuisance functions g_0 with full support $\mathrm{supp}(g_0) = [-c, +c]$, we consider one-parametric submodels in $\Theta \times \mathcal{H}_c$ parametrized by ϑ, passing through (ϑ_0, g_0) in direction $(1, \widetilde{g})$, where $\widetilde{g}$ ranges over $\mathcal{H}_c$. In all these, LAMN with local scale $n^{-\alpha(\vartheta_0)/2}$ holds at ϑ_0, and $\widehat{\vartheta}_n^c$ is regular at ϑ_0. We spell out one particular direction which is least informative for ϑ, and in which $\widehat{\vartheta}_n^c$ is efficient at ϑ_0: in this least informative submodel, $\widehat{\vartheta}_n^c$ is a maximum likelihood estimate for ϑ. As a consequence, we have a local asymptotic minimax theorem at ϑ_0. Writing for short $Q_{(\vartheta_0, g_0), n, h, \widetilde{g}}$ for the law of the solution of $dX_t = \left(\vartheta \frac{X_t}{1+X_t^2} + g(X_t)\right) dt + \sigma dW_t$ when $\vartheta = \vartheta_0 + n^{-\alpha(\vartheta_0)/2} h$ and $g = g_0 + n^{-\alpha(\vartheta_0)/2} h \widetilde{g}$,

$$\sup_{\widetilde{g} \in \mathcal{H}_c} \left[\sup_{d<\infty} \liminf_{n\to\infty} \sup_{|h|<d} E_{(\vartheta_0, g_0),\, n,\, h,\, \widetilde{g}} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right) \right] \geq E\left(l\left(Z(\vartheta_0) \right) \right)$$

for arbitrary sequences of $\mathcal{G}_n$-measurable estimates $\widetilde{\vartheta}_n$ for ϑ, with $Z(\vartheta_0)$ of $(***)$; the bound is attained for $\widetilde{\vartheta}_n := \widehat{\vartheta}_n^c$. This is - with exact constants etc specified there – theorem 2 in section 3.

1 A limit theorem under null recurrence

A one-dimensional diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

(with drift $b : I\!\!R \to I\!\!R$ and diffusion coefficient $\sigma : I\!\!R \to (0, \infty)$ satisfying Lipschitz and linear growth conditions) is recurrent if and only if the following function S

$$S(x) := \int_0^x s(y)dy\,, \quad s(y) := \exp\left(-\int_0^y \frac{2b}{\sigma^2}(v)dv \right)$$

is a space transformation on $I\!\!R$, i.e.

$$\lim_{x\to-\infty} S(x) = -\infty\,, \quad \lim_{x\to+\infty} S(x) = +\infty$$

(Khasminskii, 1980, example 2 in section 3.8). In this case, there is a unique (up to multiplication with a constant) invariant measure μ for X

$$\mu(dx) := \frac{2}{\sigma^2(x)} \exp\left(\int_0^x \frac{2b}{\sigma^2}(v)dv \right) dx\,, \quad x \in I\!\!R$$

and the process $\widetilde{X} := (S(X_t))_{t\geq 0}$ is a diffusion without drift, with diffusion coefficient $\widetilde{\sigma}$ and invariant measure $\widetilde{\mu}$ given by

$$\widetilde{\sigma} = (s \cdot \sigma) \circ S^{-1}\,, \quad \widetilde{\mu}(dx) = \frac{2}{\widetilde{\sigma}^2(x)} dx\,, \quad x \in I\!\!R$$

where S^{-1} is the inverse of the space transformation S (with $S(0) = S^{-1}(0) = 0$). A recurrent process X is termed positive (ergodic) if μ is a finite measure, and null else.

In ergodic cases, it well known that for measurable functions F with $0 < \mu(F^2) < \infty$, pairs of type (martingale, angle brackett)

$$\left(\frac{1}{n^{1/2}} \int_0^{tn} F(X_s)dW_s\,,\; \frac{1}{n} \int_0^{tn} F^2(X_s)ds \right)_{t\geq 0}$$

under linear change of time $t \to tn$ and $\sqrt{n}$-renormalization converge weakly as $n \to \infty$ in the Skorohod space $D(\mathbb{R}^+, \mathbb{R}^2)$ to the limit process

$$\left(\sqrt{\mu(F^2)}\, B_t \,,\, \mu(F^2)\, t \right)_{t \geq 0}$$

where B is standard Brownian motion. Limit theorems for martingales and integrable additive functionals of X in null recurrent cases require an additional condition: tails $P(R > \cdot)$ of suitably defined life cycle length distributions of X have to vary regularly at infinity with some index $-\alpha$, $0 < \alpha < 1$, or integrated tails $t \to \int_0^t P(R > r)dr$ have to vary slowly at ∞ (case $\alpha = 1$). See Khasminskii (1980, 2001) for (one-dimensional marginals of) integrable additive functionals of X, see Touati (1988) for weak convergence of martingales, see also Höpfner and Löcherbach (2000, section 3). (In case $\alpha = 0$, limit theorems of different type arise, see Kasahara (1986). The case of non-integrable additive functionals - known only partially up to now, see Khasminskii (2000) - also leads to different types of limit results. Index $\alpha = 1$ arises under ergodicity and in some null recurrent cases on the frontier to ergodicity.) We are interested in limit theorems if $0 < \alpha < 1$. The following theorem is a combination of results of Khasminskii (1980) and Touati (1988), see Höpfner and Löcherbach (2000, Cor. 3.2 and Ex. 3.5).

The following processes appear in the limit theorem. For $0 < \alpha < 1$, the stable increasing process S^α of index α is the PIIS (process with stationary independent increments) with

$$E\left(e^{-\lambda S_t^\alpha}\right) = e^{-t\,\lambda^\alpha} \;, \quad \lambda \geq 0 \,, t \geq 0 \,, \quad S_0^\alpha \equiv 0$$

and with càdlàg nondecreasing paths; the Mittag Leffler process W^α of index α

$$W_t^\alpha := \inf\{s > 0 : S_s^\alpha > t\} \;, \quad t \geq 0$$

is the process inverse of S^α, with continuous nondecreasing paths; for standard Brownian motion B which is *independent* of W^α, we write

$$B(W^\alpha) = (B\,(W_t^\alpha))_{t \geq 0}$$

for B time changed by $t \to W_t^\alpha$.

Theorem A : Assume that μ has infinite total mass, and that for $\widetilde{\sigma}$ as above

(1) $$\frac{2}{\widetilde{\sigma}^2(x)} \;\sim\; A_\pm |x|^\beta \,, \quad x \to \pm\infty$$

for some $0 < \alpha < 1$, with $\beta := -2 + \frac{1}{\alpha}$ and nonnegative constants $A_\pm$ such that $A_+ + A_- > 0$. Then for measurable functions F with $0 < \mu(F^2) < \infty$, pairs

$$\left(\frac{1}{n^{\alpha/2}} \int_0^{tn} F(X_s) dW_s \, , \, \frac{1}{n^\alpha} \int_0^{tn} F^2(X_s) ds \right)_{t \geq 0}$$

converge weakly as $n \to \infty$ in the Skorohod space $D(I\!R^+, I\!R^2)$ to the limit

$$\left(K(\alpha, A_\pm, F)^{1/2} \, B(W^\alpha) \, , \, K(\alpha, A_\pm, F) \, W^\alpha \right)$$

where standard Brownian motion B and Mittag Leffler process W^α are independent, and

$$K(\alpha, A_\pm, F) = \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)\, \alpha^{2\alpha}} \, \frac{1}{A_+^\alpha + A_-^\alpha} \, \mu(F^2) \, .$$

Proof : We define life cycles jointly for X and $\widetilde{X} = S(X)$ by

$$R_n = \inf\{ t > r_n : X_t = 0 \} \, , \quad r_n = \inf\{ t > R_{n-1} : X_t = S^{-1}(1) \} \, , \quad R_0 = 0 \, .$$

Khasminskii considered the process $\widetilde{X} = S(X)$ and proved (Khasminskii, 1980, theorem 11.2, corollary, remark 3, theorem 11.3, see also Khasminskii, 2001, theorem 1.1) that condition (1) implies

$$P(R_2 - R_1 > t) \; \sim \; \frac{\alpha^{2\alpha} \left(A_+^\alpha + A_-^\alpha \right)}{\Gamma(1+\alpha)} \, t^{-\alpha} \, , \quad t \to \infty \, . \tag{2}$$

Note also that for $f \in L^1(\mu)$

$$E\left(\int_{R_1}^{R_2} f(X_s) ds \right) = E\left(\int_{R_1}^{R_2} (f \circ S^{-1})(\widetilde{X}_s) ds \right) = \widetilde{\mu}(f \circ S^{-1}) = \mu(f) \tag{3}$$

by (Khasminskii, 1980, lemma 10.5). Combining (2) – viewed as tail condition for the process X – with (3), theorem A is a direct application of Höpfner and Löcherbach (2000, Cor. 3.2), a result obtained first by Touati (1988). □

Remark : Theorem A together with the ratio limit theorem

$$h, g \in L^1(\mu) \, , \mu(g) > 0 \; : \quad \lim_{t \to \infty} \frac{\int_0^t h(X_s) ds}{\int_0^t g(X_s) ds} = \frac{\mu(h)}{\mu(g)} \quad P\text{-a.s.}$$

yields joint convergence of martingales and arbitrary integrable additive functionals of X.

2 A reference model for null recurrent diffusions

We consider a statistical model

$$(4)\qquad dX_t \;=\; \vartheta\,\frac{X_t}{1+X_t^2}\,dt + \sigma\,dW_t\,,\quad t\ge 0\,,\quad X_0=0$$

where ϑ is a one-dimensional parameter of interest; $\sigma>0$ is known and constant. Let Q_ϑ denote the law of X under ϑ: Q_ϑ is a probability measure on the canonical path space $C:=C(I\!R^+,I\!R)$ endowed with σ-field and filtration generated by the canonical process $\eta=(\eta_t)_{t\ge 0}$:

$$I\!\!G=(\mathcal{G}_t)_{t\ge 0}\,,\quad \mathcal{G}_t=\bigcap_{r>t}\sigma\{\eta_s:0\le s\le r\}\,,\quad \mathcal{C}=\sigma\{\eta_s:s\ge 0\}\,.$$

Then the likelihood ratio process $L^{\xi/\vartheta}$ of Q_ξ with respect to Q_ϑ relative to $I\!\!G$ is

$$L_t^{\xi/\vartheta}=\exp\left(\frac{\xi-\vartheta}{\sigma^2}\int_0^t\frac{\eta_s}{1+\eta_s^2}\,dM_s^\vartheta\;-\;\frac{1}{2}\frac{(\xi-\vartheta)^2}{\sigma^2}\int_0^t\left(\frac{\eta_s}{1+\eta_s^2}\right)^2ds\right)\,,\quad t\ge 0$$

where M^ϑ is the Q^ϑ-martingale part of η (see Liptser and Shiryaev, 1978, Jacod and Shiryaev, 1987, Kutoyants, 1994). Obviously

$$L_t^{\xi/\vartheta}=\exp\left(\frac{\xi-\vartheta}{\sigma^2}\int_0^t\frac{\eta_s}{1+\eta_s^2}\,d\eta_s\;-\;\frac{1}{2}\frac{\xi^2-\vartheta^2}{\sigma^2}\int_0^t\left(\frac{\eta_s}{1+\eta_s^2}\right)^2ds\right)\,,\quad t\ge 0\,,$$

so the score function martingale at ϑ is

$$\Delta_t(\vartheta):=\frac{1}{\sigma^2}\int_0^t\frac{\eta_s}{1+\eta_s^2}\,dM_s^\vartheta$$

and the information process is

$$I_t:=\frac{1}{\sigma^2}\int_0^t\left(\frac{\eta_s}{1+\eta_s^2}\right)^2ds\,,\quad t\ge 0\,.$$

Note that I does not depend on ϑ. The process of maximum likelihood estimates (MLE)

$$\widehat{\vartheta}_t=\int_0^t\frac{\eta_s}{1+\eta_s^2}\,d\eta_s\;\Big/\;\int_0^t\left(\frac{\eta_s}{1+\eta_s^2}\right)^2ds\,,\quad t\ge 0$$

has the representation

$$\widehat{\vartheta}_t-\vartheta=\Delta_t(\vartheta)\,/\,I_t\,,\quad t\ge 0\,.$$

Calculating s^ϑ, S^ϑ as in the beginning of section 1, we see that the canonical process η under Q^ϑ is recurrent if and only if $\frac{2\vartheta}{\sigma^2}\le 1$, with invariant measure given by

$$(5)\qquad \mu^\vartheta(dx)=\frac{2}{\sigma^2}\,\sqrt{1+x^2}^{\,\frac{2\vartheta}{\sigma^2}}\,dx\,,\quad \frac{2\vartheta}{\sigma^2}\le 1\,.$$

Here μ^ϑ has infinite total mass if $|\vartheta| \le \frac{2}{\sigma^2}$. We restrict attention to the maximal open null recurrent submodel

$$\Theta := (-\frac{\sigma^2}{2}, +\frac{\sigma^2}{2}) .$$

Proposition 1 : Define

$$\alpha(\vartheta) := \frac{1}{2} - \frac{\vartheta}{\sigma^2} \in (0,1) , \quad \vartheta \in \Theta .$$

With notation $f(x) := \dfrac{x}{1+x^2}$, we have for every $\vartheta \in \Theta$ weak convergence of

$$\left(\frac{1}{n^{\alpha(\vartheta)/2}} \int_0^{tn} f(\eta_s) dM_s^\vartheta \,,\, \frac{1}{n^{\alpha(\vartheta)}} \int_0^{tn} f^2(\eta_s) ds \right)_{t \ge 0}$$

under Q^ϑ as $n \to \infty$ in $D(I\!R^+, I\!R^2)$ to

$$\left(\sigma\, \widetilde{K}(\vartheta, f)^{1/2}\, B(W^{\alpha(\vartheta)}) \,,\, \widetilde{K}(\vartheta, f)\, W^{\alpha(\vartheta)} \right)$$

where the constants $\widetilde{K}(\vartheta, f)$ are given by

$$\widetilde{K}(\vartheta, f) = \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)\,\alpha^{2\alpha}}\; \frac{1}{A_+^\alpha + A_-^\alpha}\; \mu^\vartheta(f^2) \,, \quad \alpha = \alpha(\vartheta) \tag{6}$$

with

$$A_+ = A_- = A(\vartheta) := \frac{2}{\sigma^2} \left(1 - \frac{2\vartheta}{\sigma^2} \right)^{\frac{4\vartheta}{\sigma^2 - 2\vartheta}} . \tag{7}$$

Proof : Proceeding as in section 1, the functions S and s as there are given by

$$S^\vartheta(x) = \int_0^x s^\vartheta(y) dy \,, \quad s^\vartheta(y) = \exp\left(-\frac{2\vartheta}{\sigma^2} \int_0^y \frac{v}{1+v^2} dv \right) = \sqrt{1+y^2}^{\;-\frac{2\vartheta}{\sigma^2}} ;$$

For $\vartheta \in \Theta$, write

$$\gamma(\vartheta) := \frac{2\vartheta}{\sigma^2} \,, \quad \alpha(\vartheta) = \frac{1-\gamma(\vartheta)}{2} \,, \quad \beta(\vartheta) := -2 + \frac{1}{\alpha(\vartheta)} = \frac{2\gamma(\vartheta)}{1-\gamma(\vartheta)} = \frac{4\vartheta}{\sigma^2 - 2\vartheta} .$$

We have to calculate the diffusion coefficient $\widetilde{\sigma}^\vartheta$ of $\widetilde{\eta} = S^\vartheta(\eta)$ under $\vartheta \in \Theta$. Obviously

$$S^\vartheta(x) \;\sim\; \mathrm{sgn(x)}\, \frac{1}{1-\gamma(\vartheta)}\, |\mathrm{x}|^{(1-\gamma(\vartheta))} \,, \quad \left(\mathrm{S}^\vartheta\right)^{-1}(\mathrm{x}) \;\sim\; \mathrm{sgn(x)}\; \left((1-\gamma(\vartheta))\, |\mathrm{x}|\right)^{(1-\gamma(\vartheta))^{-1}}$$

for $x \to \pm\infty$; thus the diffusion coefficient $\widetilde{\sigma}^\vartheta = (\sigma \cdot s^\vartheta) \circ \left(S^\vartheta\right)^{-1}$ of $\widetilde{\eta}$ behaves as

$$\widetilde{\sigma}^\vartheta(x) \;\sim\; \sigma \left| \left(S^\vartheta\right)^{-1}(x) \right|^{-\gamma(\vartheta)} \;\sim\; \sigma\, \left((1-\gamma(\vartheta))\, |x|\right)^{\frac{-\gamma(\vartheta)}{1-\gamma(\vartheta)}}$$

as $x \to \pm\infty$. We arrive at

$$\frac{2}{\left(\widetilde{\sigma}^\vartheta\right)^2(x)} \;\sim\; \frac{2}{\sigma^2}\, \left((1-\gamma(\vartheta))\, |x|\right)^{\frac{2\gamma(\vartheta)}{1-\gamma(\vartheta)}} \;=\; A_\pm(\vartheta) |x|^{\beta(\vartheta)}$$

with notation

$$A_+(\vartheta) = A_-(\vartheta) := \frac{2}{\sigma^2}\,(1-\gamma(\vartheta))^{\frac{2\gamma(\vartheta)}{1-\gamma(\vartheta)}} = \frac{2}{\sigma^2}\,\left(1-\frac{2\vartheta}{\sigma^2}\right)^{\frac{4\vartheta}{\sigma^2-2\vartheta}} .$$

Finally, we note that the function $f(x) = \frac{x}{1+x^2}$ belongs to $L^2(\mu^\vartheta)$ for all $\vartheta \in \Theta$ (this is not a harmless condition in general, and is violated in our setting for $\vartheta_0 := +\frac{2}{\sigma^2} \in \overline{\Theta}$, $\overline{\Theta}$ the closure of Θ, on the frontier to transience). We have checked all the conditions of theorem A in section 1, from which proposition 1 now follows. □

Theorem 1 : For every $\vartheta \in \Theta$, we have the following:

a) the MLE under Q^ϑ converges to a 'mixed normal' limit

$$n^{\alpha(\vartheta)/2}\left(\widehat{\vartheta}_n - \vartheta\right) \quad \to \quad \sigma\,\widetilde{K}(\vartheta,f)^{-1/2}\,\frac{B(W_1^{\alpha(\vartheta)})}{W_1^{\alpha(\vartheta)}}$$

(weak convergence in $I\!R$, as $n \to \infty$), with $\widetilde{K}(\vartheta,f)$ given by (6).

b) the sequence of statistical models

$$\left(C\,,\mathcal{G}_n\,,\{Q^\vartheta|\mathcal{G}_n : \vartheta \in \Theta\}\right)\,, \quad n \geq 1$$

is locally asymptotically mixed normal (LAMN) at $\vartheta \in \Theta$, with local scale $n^{-\alpha(\vartheta)/2}$, and $(\widehat{\vartheta}_n)_n$ is asymptotically efficient at ϑ in the sense of the convolution theorem (i.e. in the class of all asymptotically at ϑ regular estimator sequences).

c) for arbitrary sequences of $\mathcal{G}_n$-measurable estimates $\widetilde{\vartheta}_n$, for subconvex and bounded loss functions l, one has the local asymptotic minimax bound

$$\begin{aligned}\sup_{d<\infty}\,\liminf_{n\to\infty}\,\sup_{|h|<d}\, & E_{\vartheta+n^{-\alpha(\vartheta)/2}h}\left(l\left(n^{\alpha(\vartheta)/2}\left(\widetilde{\vartheta}_n - (\vartheta + n^{-\alpha(\vartheta)/2}h)\right)\right)\right) \\ & \geq\; E\left(l\left(\sigma\,\widetilde{K}(\vartheta,f)^{-1/2}\,\frac{B(W_1^{\alpha(\vartheta)})}{W_1^{\alpha(\vartheta)}}\right)\right)\end{aligned}$$

which is attained by the MLE sequence.

Proof : Part a) follows from proposition 1 and representation of MLE errors. We prove b). From the representation of the likelihood ratio in our model, we have local representations in small neighbourhoods of points $\vartheta \in \Theta$ with radius $n^{-\alpha(\vartheta)/2}$:

(8) $$\log\, L_{tn}^{\left(\vartheta+n^{-\alpha(\vartheta)/2}h\right)/\vartheta} = h\,\frac{1}{n^{\alpha(\vartheta)/2}}\,\Delta_{tn}(\vartheta) - \frac{h^2}{2}\,\frac{1}{n^{\alpha(\vartheta)}}\,I_{tn}\,, \quad t \geq 0\,.$$

Under Q^ϑ, the processes in (8) converge as $n \to \infty$ weakly in $D(I\!R^+, I\!R)$ to

$$h \, \frac{\widetilde{K}(\vartheta,f)^{1/2}}{\sigma} \, B(W^{\alpha(\vartheta)}) - \frac{h^2}{2} \, \frac{\widetilde{K}(\vartheta,f)}{\sigma^2} \, W^{\alpha(\vartheta)} \tag{9}$$

for arbitrary fixed $h \in I\!R$, again by proposition 1. Taking $t = 1$ fixed, this is local asymptotic mixed normality (LAMN) of $\left(C\,, \mathcal{G}_n\,, \{Q^{\vartheta'} | \mathcal{G}_n : \vartheta' \in \Theta\}\right)$ at ϑ as $n \to \infty$, with local scale $n^{-\alpha(\vartheta)/2}$, see Davies (1985), Jeganathan (1988), LeCam and Yang (1990, Sect. 5.6).

As a consequence of (8) and (9) with $t = 1$, the LAMN version of Hájek's convolution theorem given in Davies (1985, Cor. 7.2) (note that all assumptions A0 - A7 in Davies (1985) are met in our case, and see Hájek (1970) for the original LAN version) gives an asymptotic efficiency bound for regular estimator sequences. In $\{Q^{\vartheta'} | \mathcal{G}_n : \vartheta' \in \Theta\}$, a sequence $(\widetilde{\vartheta}_n)_n$ of estimates is termed regular at ϑ if we have joint convergence

$$\mathcal{L}\left(n^{\alpha(\vartheta)/2} \left(\widetilde{\vartheta}_n - (\vartheta + n^{-\alpha(\vartheta)/2} h) \right) , \frac{1}{n^{\alpha(\vartheta)}} \, I_n \;\mid\; Q^{\vartheta + n^{-\alpha(\vartheta)/2} h} \right)$$

as $n \to \infty$ to a bivariate limit law $\overline{P}$ which does not depend on the value $h \in I\!R$ of the local parameter. Write $(\widehat{h}, J)$ for a random variable having law $\overline{P}$. The convolution theorem states that for subconvex loss functions l, one has necessarily

$$\int l(\widehat{h}) \, d\overline{P} \;\geq\; \int \int l(z) \, \mathcal{N}(0, j^{-1})(dz) \, \overline{P}^J(dj) \, .$$

By (9), the variance mixture of normals appearing on the right hand side is the limit law for rescaled MLE errors appearing in assertion a); since the property

$$n^{\alpha(\vartheta)/2} \left(\widetilde{\vartheta}_n - \vartheta \right) = \left(\frac{1}{n^{\alpha(\vartheta)/2}} \, \Delta_n(\vartheta) \right) / \left(\frac{1}{n^{\alpha(\vartheta)}} \, I_n \right) + o_{Q^\vartheta}(1) \, , \quad n \to \infty$$

characterizes estimator sequences $(\widetilde{\vartheta}_n)_n$ which are regular at ϑ and efficient in the sense of the convolution theorem, the proof of assertion b) is complete. Assertion c) is an application of LeCam and Yang (1990, Thm. 1 in Sect. 5.6). □

Remark 2 : A 'practically' useful version of the MLE convergence result is obtained with random norming: we get convergence in law

$$\sqrt{I_n} \left(\widehat{\vartheta}_n - \vartheta \right) \quad \Longrightarrow \quad \mathcal{N}(0,1)$$

under Q^ϑ as $n \to \infty$ at every point $\vartheta \in \Theta$. This unified result using random norming – due to the particular feature of our model that the information process I does not depend on ϑ – remains true in the larger 'recurrent' statistical model $\{Q^\vartheta : \vartheta < \frac{\sigma^2}{2}\}$: we do have a convergence theorem at the null recurrent point $\vartheta = \frac{-\sigma^2}{2}$ on the frontier to ergodicity (see Höpfner and Löcherbach, 2000, section 3.1) even if we are unable to specify the rate function appearing there explicitely (this function is regularly varying at infinity with index 1), and clearly we have in the ergodic points $\vartheta < -\frac{\sigma^2}{2}$ the usual martingale convergence theorem (Jacod and Shiryaev, 1987) and strong law of large numbers at our disposition. We are not able to treat the frontier case from null recurrence to transience $\vartheta = +\frac{\sigma^2}{2}$ where non-integrable additive functionals occur, and we do not know what happens in cases $\vartheta > +\frac{\sigma^2}{2}$ where the observed process is transient.

3 Extension: nuisance functions in the drift

Now we consider statistical models of type

$$(10) \qquad dX_t \;=\; \left(\vartheta \, \frac{X_t}{1+X_t^2} + g(X_t) \right) dt + \sigma \, dW_t \,, \quad t \geq 0 \,, \quad X_0 = 0$$

where ϑ is an unknown one-dimensional parameter of interest; $\sigma > 0$ is known and constant, and g is some function $\mathbb{R} \to \mathbb{R}$ with the property

$$\int_{-\infty}^{+\infty} |g|(x) dx < \infty \,.$$

We write $Q^{\vartheta,g}$ for the law on $(C \,, \mathcal{C} \,, \mathbb{G})$ of the process in (10), and η for the canonical process. The nuisance function g in (10) will appear in limit theorems for martingales and integrable additive functionals via a weight parameter for right and left tails

$$(11) \qquad \zeta_+ := \exp\left(\frac{2}{\sigma^2} \int_0^\infty g(x) dx \right) \,, \quad \zeta_- := \exp\left(-\frac{2}{\sigma^2} \int_{-\infty}^0 g(x) dx \right)$$

and via densities $\rho_\pm(y)$ w.r.to the measure μ^ϑ defined in (5) meeting $\rho_\pm(y) \to 1$ as $y \to \pm\infty$:

$$(12) \, \rho_+(y) := 1_{(0,\infty)}(y) \exp\left(-\frac{2}{\sigma^2} \int_y^\infty g(x) dx \right) \,, \quad \rho_-(y) := 1_{(-\infty,0)}(y) \exp\left(\frac{2}{\sigma^2} \int_{-\infty}^y g(x) dx \right) \,.$$

Presence of g has no qualitative effect on recurrence properties of the canonical process: as before, η under $Q^{\vartheta,g}$ is recurrent if and only if $\frac{2\vartheta}{\sigma^2} \leq 1$, the invariant measure now being

$$(13) \qquad \mu^{\vartheta,g}(dx) = (\zeta_- \rho_-(x) + \zeta_+ \rho_+(x)) \, \mu^\vartheta(dx) \,, \quad \frac{2\vartheta}{\sigma^2} \leq 1 \,;$$

$\Theta = \left(-\frac{\sigma^2}{2}, +\frac{\sigma^2}{2} \right)$ is the maximal open null recurrent submodel; spaces $L^2\left(\mu^{\vartheta,g}\right)$, $L^2\left(\mu^\vartheta\right)$ coincide.

Proposition 2 : For every $\vartheta \in \Theta$ and g with $\int |g|(x)dx < \infty$, for $F \in L^2\left(\mu^\vartheta\right)$, we have weak convergence of

$$\left(\frac{1}{n^{\alpha(\vartheta)/2}} \int_0^{tn} F(\eta_s) dM_s^{\vartheta,g} \,,\; \frac{1}{n^{\alpha(\vartheta)}} \int_0^{tn} F^2(\eta_s)ds \right)_{t\geq 0}$$

under $Q^{\vartheta,g}$ as $n\to\infty$ in $D(\mathbb{R}^+,\mathbb{R}^2)$ to

$$\left(\sigma\, \widehat{K}(\vartheta,g,F)^{1/2}\, B(W^{\alpha(\vartheta)}) \,,\; \widehat{K}(\vartheta,g,F)\, W^{\alpha(\vartheta)} \right)$$

where the constants $\widehat{K}(\vartheta,g,F)$ are given by

$$\widehat{K}(\vartheta,g,F) = \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)\,\alpha^{2\alpha}\,A^\alpha}\; \frac{1}{\zeta_- + \zeta_+}\; \mu^{\vartheta,g}(F^2)\,, \quad \alpha = \alpha(\vartheta)\,, \quad A = A(\vartheta) \tag{14}$$

with $A(\vartheta)$ defined in (7).

Proof : The proof modifies the proof of proposition 1. We start with the space transformation of section 1: with $f(x) = \frac{x}{1+x^2}$ we have

$$S^{\vartheta,g}(x) = \int_0^x s^{\vartheta,g}(y)dy\,, \quad s^{\vartheta,g}(y) = \exp\left(-\frac{2\vartheta}{\sigma^2}\int_0^y f(v)dv - \frac{2}{\sigma^2}\int_0^y g(v)dv \right) = \frac{1}{\zeta_\pm \rho_\pm(y)}\; s^\vartheta(y)$$

where s^ϑ is as in the proof of proposition 1

$$s^\vartheta(y) = \sqrt{1+y^2}^{\;-\frac{2\vartheta}{\sigma^2}}\,.$$

For $\vartheta\in\Theta$, writing $\alpha(\vartheta)$, $\beta(\vartheta)$, $\gamma(\vartheta)$ as in the proof of proposition 1, the diffusion coefficient $\widetilde{\sigma}^\vartheta$ of $\widetilde{\eta} = S^{\vartheta,g}(\eta)$ under $\vartheta\in\Theta$ is now

$$\widetilde{\sigma}^{\vartheta,g}(x) \;\sim\; \sigma\,\frac{1}{\zeta_\pm}\,|\left(S^{\vartheta,g}\right)^{-1}(x)|^{-\gamma(\vartheta)} \;\sim\; (\zeta_\pm)^{-(1-\gamma(\vartheta))^{-1}}\,\widetilde{\sigma}^\vartheta(x)$$

as $x\to\pm\infty$, where $\widetilde{\sigma}^\vartheta$ is as before

$$\widetilde{\sigma}^\vartheta(x) \;\sim\; \sigma\;\left((1-\gamma(\vartheta))\,|x|\right)^{\frac{-\gamma(\vartheta)}{1-\gamma(\vartheta)}}\,, \quad x\to\pm\infty\,.$$

Using $\frac{2}{1-\gamma(\vartheta)} = \frac{1}{\alpha(\vartheta)}$ and the definition of $\beta(\vartheta)$, we arrive at

$$\frac{2}{\left(\widetilde{\sigma}^{\vartheta,g}\right)^2(x)} \;\sim\; = \; A_\pm(\vartheta,g)|x|^{\beta(\vartheta)} \tag{15}$$

with

$$A_+(\vartheta,g) = \zeta_+^{\frac{1}{\alpha(\vartheta)}}\,A(\vartheta)\,, \quad A_-(\vartheta,g) = \zeta_-^{\frac{1}{\alpha(\vartheta)}}\,A(\vartheta)$$

where $A(\vartheta)$ is given in (7). As in the proof of proposition 1, we use (15) and theorem A to conclude the proof of proposition 2. □

Observations of type (10) induce semiparametric models $\left(C\,, \mathcal{C}\,, \mathbb{G}\,, \{\, Q^{\vartheta,g} : \vartheta \in \Theta\,, g \in \mathcal{H}\,\}\right)$ where $\mathcal{H}$ is a suitable class of bounded measurable functions (for existence of strong solutions to (10) in this case see Krylov and Zvonkin (1981, p.42)). The estimate $(\widehat{\vartheta}_t)_{t\geq 0}$ of section 1 for $\vartheta \in \Theta$ turns out to be inconsistent in such larger models: combining the explicit representation of $\widehat{\vartheta}_t$ with proposition 2 and the ratio limit theorem, we have convergence in $Q^{\vartheta,g}$-probability

$$\widehat{\vartheta}_t \quad \to \quad \vartheta + \frac{\int f g \, d\mu^{\vartheta,g}}{\int f^2 \, d\mu^{\vartheta,g}} \qquad \text{as } t \to \infty$$

with $f(x) = \frac{x}{1+x^2}$. In the following, we consider $\mathcal{H} := \mathcal{H}_c$, the class of bounded measurable functions with support contained in $[-c, c]$, and write

$$\mathcal{E}^c := \left(C\,, \mathcal{C}\,, \mathbb{G}\,, \{\, Q^{\vartheta,g} : \vartheta \in \Theta\,, g \in \mathcal{H}_c\,\}\right) . \tag{16}$$

Theorem 2 : a) In the model $\mathcal{E}^c$, the sequence

$$\widehat{\vartheta}_n^c \;:=\; \int_0^n f^c(\eta_s)\, d\eta_s \;/\; \int_0^n (f^c)^2(\eta_s)\, ds\,, \quad f^c(x) := \frac{x}{1+x^2}\, 1_{\{|x|>c\}}$$

is consistent for $\vartheta \in \Theta$, and converges under $Q^{\vartheta,g}$ in law to a mixed normal limit

$$n^{\alpha(\vartheta)/2}\left(\widehat{\vartheta}_n^c - \vartheta\right) \quad \to \quad \sigma\, \widetilde{K}(\vartheta, f^c)^{-1/2}\, \frac{B(W_1^{\alpha(\vartheta)})}{W_1^{\alpha(\vartheta)}}\,, \quad n \to \infty$$

where the constant

$$\widetilde{K}(\vartheta, f^c) := \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)\,\alpha^{2\alpha}\,2A^{\alpha}}\, \mu^{\vartheta}((f^c)^2)\,, \quad \alpha = \alpha(\vartheta)\,,\, A = A(\vartheta) \tag{17}$$

(cf. (6)+(7)) does not depend on the nuisance function $g \in \mathcal{H}_c$.

b) At every point $(\vartheta_0, g_0) \in \Theta \times \mathcal{H}_c$ with the property $\mathrm{supp}(g_0) = [-c, +c]$, the model $\mathcal{E}^c$ contains a least favorable one-dimensional submodel – parametrized by $\vartheta \in \Theta$, and passing through (ϑ_0, g_0) – such that *in the submodel*: i) $(\widehat{\vartheta}_n^c)_n$ is the MLE sequence for $\vartheta \in \Theta$; ii) LAMN holds at ϑ_0 with local scale $n^{-\alpha(\vartheta_0)/2}$; iii) $(\widehat{\vartheta}_n^c)_n$ is asymptotically efficient at ϑ_0 in the sense of the convolution theorem or the local asymptotic minimax bound.

c) Write $E_{(\vartheta_0,g_0),\,n,\,h,\,\widetilde{g}}$ for expectation under $Q^{\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h\,,\, g_0 + n^{-\alpha(\vartheta_0)/2} h\, \widetilde{g}}$, $\widetilde{g} \in \mathcal{H}_c$.

For $(\vartheta_0, g_0) \in \Theta \times \mathcal{H}_c$ with $\mathrm{supp}(g_0) = [-c, +c]$, the local asymptotic minimax bound at ϑ_0 is

$$\begin{aligned}
&\sup_{\widetilde{g}\in\mathcal{H}_c} \left[\sup_{d<\infty} \lim_{n\to\infty} \inf_{\widetilde{\vartheta}_n} \sup_{|h|<d} E_{(\vartheta_0,g_0),\,n,\,h,\,\widetilde{g}} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right) \right] \\
&= \sup_{d<\infty} \lim_{n\to\infty} \inf_{\widetilde{\vartheta}_n} \sup_{|h|<d} E_{(\vartheta_0,g_0),\,n,\,h,\,-f 1_{[-c,+c]}} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right) \\
&\qquad = E \left(l \left(\sigma\, \widetilde{K}(\vartheta_0, f^c)^{-1/2}\, \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}} \right) \right)
\end{aligned}$$

(with inf over arbitrary $\mathcal{G}_n$-measurable estimates for ϑ). The bound is attained by $\widetilde{\vartheta}_n := \widehat{\vartheta}_n^c$ which is regular at ϑ_0 with respect to every fixed 'direction' $\widetilde{g} \in \mathcal{H}_c$.

Proof : 1) Since $f^c g = 0$ for all $g \in \mathcal{H}_c$, we have under $Q^{\vartheta, g}$ via (10)

$$\widehat{\vartheta}_t^c - \vartheta \;=\; \int_0^t f^c(\eta_s)\, dM_s^{\vartheta,g} \;/\; \int_0^t (f^c)^2(\eta_s)\, ds\,;$$

since μ^ϑ has a symmetric density, $\rho_- \equiv 1$ on $(-\infty, -c)$ and $\rho_+ \equiv 1$ on $(+c, +\infty)$, we have also

$$\mu^{\vartheta,g}\left((f^c)^2\right) \;=\; \int_{-\infty}^{+\infty} (\zeta_-\rho_-(x) + \zeta_+\rho_+(x))\,(f^c)^2(x)\,\mu^\vartheta(dx) \;=\; \frac{\zeta_- + \zeta_+}{2}\,\mu^\vartheta\left((f^c)^2\right)$$

for $g \in \mathcal{H}_c$. Hence assertion a) is a consequence of proposition 2 .

2) Fix some reference point $(\vartheta_0, g_0) \in \Theta \times \mathcal{H}_c$ such that the support of g_0 is the full interval $[-c, +c]$. For $g' \in \mathcal{H}_c$, $-f 1_{[-c,+c]} + g'$ is in $\mathcal{H}_c$, and arbitrary functions $\widetilde{g}$ in $\mathcal{H}_c$ can be written in this form: so the scope of possible directions away from (ϑ_0, g_0) in $\Theta \times \mathcal{H}_c$ is described by pairs

$$\left(1, -f 1_{[-c,+c]} + g'\right)\,, \quad g' \in \mathcal{H}_c\,.$$

Correspondingly, we consider one-dimensional submodels of $\Theta \times \mathcal{H}_c$ passing through (ϑ_0, g_0) which are parametrized by ϑ:

$$\text{(18)} \qquad \mathcal{S}^{g'}_{(\vartheta_0, g_0)} := \left(C\,, \mathcal{C}\,, I\!\!G\,, \{\, Q^{\vartheta,\, g_0 + (\vartheta - \vartheta_0)(-f 1_{[-c,+c]} + g')} : \vartheta \in \Theta \,\}\right)$$

where we call for short $g' \in \mathcal{H}_c$ the 'direction' of the submodel $\mathcal{S}^{g'}_{(\vartheta_0, g_0)}$. For fixed g', we localize the model around ϑ_0, introducing a local parameter h by $\vartheta = \vartheta_0 + n^{-\alpha(\vartheta_0)/2} h$, and write

$$\text{(19)} \quad Q_n^h := Q^{\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h\,,\, g_0 + (n^{-\alpha(\vartheta_0)/2} h)(-f 1_{[-c,+c]} + g')}\,, \qquad h \in I\!\!R \text{ s.t. } \vartheta_0 + n^{-\alpha(\vartheta_0)/2} h \in \Theta\,.$$

The log-likelihood ratio process of $Q_n^h = Q_n^h(\vartheta_0, g_0; g')$ to Q_n^0 relatively to $(\mathcal{G}_{tn})_{t \geq 0}$ is

$$\text{(20)} \qquad h\, \frac{1}{n^{\alpha(\vartheta_0)/2}} \int_0^{tn} \frac{(f^c + g')(\eta_s)}{\sigma^2} dM_s^{\vartheta_0, g_0} - \frac{1}{2}\, h^2\, \frac{1}{n^{\alpha(\vartheta_0)}} \int_0^{tn} \frac{(f^c + g')^2(\eta_s)}{\sigma^2} ds\,, \quad t \geq 0\,.$$

By proposition 2, we have LAMN at ϑ_0 with local scale $n^{-\alpha(\vartheta_0)/2}$ in the submodel $\mathcal{S}^{g'}_{(\vartheta_0, g_0)}$. As a consequence, the local asymptotic minimax bound at ϑ_0 in $\mathcal{S}^{g'}_{(\vartheta_0, g_0)}$ is

$$\sup_{d < \infty}\, \liminf_{n \to \infty}\, \inf_{\widetilde{\vartheta}_n}\, \sup_{|h| < d}\, E_{Q_n^h(\vartheta_0, g_0; g')}\left(l\left(n^{\alpha(\vartheta_0)/2}\left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h)\right)\right)\right)$$
$$\geq\; E\left(l\left(\sigma\, \widehat{K}\,(\vartheta_0, g_0, f^c + g')^{-1/2}\; \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}}\right)\right)$$

(the inf is over arbitrary $\mathcal{G}_n$-measurable estimates $\widetilde{\vartheta}_n$ for ϑ) with $Q_n^h = Q_n^h(\vartheta_0, g_0; g')$ of (19). We have also the LAMN version of the convolution theorem for regular estimates: the optimal limit distribution at ϑ_0 of rescaled estimation errors for regular estimates of ϑ in $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$ is the law

$$\mathcal{L}\left(\sigma\, \widehat{K}\,(\vartheta_0, g_0, f^c{+}g')^{-1/2}\ \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}} \right) \tag{21}$$

of the central variable in the limit of local experiments (19) as $n \to \infty$. Finally, sequences $\widetilde{\vartheta}_n$ of $\mathcal{G}_n$-measurable estimates for the parameter ϑ in the submodel $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$ are regular and efficient at ϑ_0, and achieve the local asymptotic minimax bound, if and only if the rescaled estimation error $n^{\alpha(\vartheta_0)/2}\left(\widetilde{\vartheta}_n - \vartheta_0\right)$ admits an expansion

$$\left(\frac{1}{n^{\alpha(\vartheta_0)}}\int_0^n (f^c+g')^2(\eta_s)ds\right)^{-1}\left(\frac{1}{n^{\alpha(\vartheta_0)/2}}\int_0^n (f^c+g')(\eta_s)dM_s^{\vartheta_0,g_0}\right) \ + \ o_{Q^{\vartheta_0,g_0}}(1) \tag{22}$$

as $n \to \infty$. By step 1) we do have the preliminary estimator sequence $(\widehat{\vartheta}^c_n)_n$ for ϑ which at ϑ_0 is tight at rate $n^{\alpha(\vartheta_0)/2}$. From this preliminary sequence we can construct a modified sequence $(\check{\vartheta}_n)_n$ which is efficient at ϑ_0 in the submodel $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$, i.e. whose rescaled estimation error $n^{\alpha(\vartheta_0)/2}\left(\check{\vartheta}_n - \vartheta_0\right)$ has the expansion (22). Hence the above bounds are attainable: we arrive at

$$\sup_{d<\infty}\lim_{n\to\infty}\inf_{\widetilde{\vartheta}_n}\sup_{|h|<d} E_{Q_n^h(\vartheta_0,g_0;g')}\left(l\left(n^{\alpha(\vartheta_0)/2}\left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2}h)\right)\right)\right)$$
$$= \ E\left(l\left(\sigma\,\widehat{K}\,(\vartheta_0, g_0, f^c{+}g')^{-1/2}\ \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}}\right)\right)$$

See LeCam and Yang (1990, section 5.6), or Davies (1985), or Jeganathan (1988).

3) Now we compare the scaling factors in (21) for different directions $g' \in \mathcal{H}_c$. Since $f^c g' \equiv 0$ for all directions $g' \in \mathcal{H}_c$, we have

$$\mu^{\vartheta_0,g_0}\left((f^c+g')^2\right) \geq \mu^{\vartheta_0,g_0}\left((f^c)^2\right) = \frac{\zeta_- + \zeta_+}{2}\,\mu^{\vartheta_0}\left((f^c)^2\right) \tag{23}$$

(by (11), the weights $\zeta_- = \zeta_-(g_0)$, $\zeta_+ = \zeta_+(g_0)$ depend on g_0); via (14) and (17), (23) yields

$$\min_{g'\in\mathcal{H}_c} \widehat{K}\,(\vartheta_0, g_0, f^c{+}g') = \widehat{K}\,(\vartheta_0, g_0, f^c) \ = \ \widetilde{K}(\vartheta_0, f^c)$$

where $\widetilde{K}(\vartheta_0, f^c)$ does not depend on g_0.

Hence the submodel $\mathcal{S}^0_{(\vartheta_0,g_0)}$ with direction $g' \equiv 0$ is asymptotically least informative at ϑ_0 among all submodels $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$, g' in $\mathcal{H}_c$. Since all bounds in step 2) are attainable bounds, this gives

$$\sup_{g'\in\mathcal{H}_c}\left[\sup_{d<\infty}\lim_{n\to\infty}\inf_{\widetilde{\vartheta}_n}\sup_{|h|<d} E_{Q_n^h(\vartheta_0,g_0;g')}\left(l\left(n^{\alpha(\vartheta_0)/2}\left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2}h)\right)\right)\right)\right]$$

$$= \sup_{d<\infty} \lim_{n\to\infty} \inf_{\widetilde{\vartheta}_n} \sup_{|h|<d} E_{Q_n^h(\vartheta_0,g_0;0)} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widetilde{\vartheta}_n - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right)$$

$$= E \left(l \left(\sigma \, \widetilde{K}(\vartheta_0, f^c)^{-1/2} \, \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}} \right) \right) .$$

We have obtained an expression which does not depend on the nuisance component g_0 of the reference point (ϑ_0, g_0) fixed in the beginning of step 2) (except that g_0 has full support $[-c, +c]$).

4) Consider the estimator sequence $(\widehat{\vartheta}_n^c)_n$ for ϑ in $\mathcal{S}^0_{(\vartheta_0,g_0)}$. This is the maximum likelihood estimate for ϑ in $\mathcal{S}^0_{(\vartheta_0,g_0)}$ since $f^c g_0 \equiv 0$. From 1) above, rescaled estimation errors $n^{\alpha(\vartheta_0)/2} \left(\widehat{\vartheta}_n^c - \vartheta_0 \right)$ do have the representation (22) with $g' \equiv 0$. Hence by step 2), the sequence $(\widehat{\vartheta}_n^c)_n$ achieves the local asymptotic minimax bound at ϑ_0 in the least favorable submodel $\mathcal{S}^0_{(\vartheta_0,g_0)}$

$$\sup_{d<\infty} \lim_{n\to\infty} \sup_{|h|<d} E_{Q_n^h(\vartheta_0,g_0;0)} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widehat{\vartheta}_n^c - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right)$$

$$= E \left(l \left(\sigma \, \widetilde{K}(\vartheta_0, f^c)^{-1/2} \, \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}} \right) \right) .$$

and also the efficieny bound for regular estimates in $\mathcal{S}^0_{(\vartheta_0,g_0)}$ at ϑ_0. All assertions of parts b) and c) of theorem 2 are proved by steps 3) and 4), except that we have not yet studied the estimate $(\widehat{\vartheta}_n^c)_n$ in directions other than the least favorable $g' \equiv 0$.

5) Fix an arbitrary direction g' in $\mathcal{H}_c$ and consider $(\widehat{\vartheta}_n^c)_n$ as estimate for ϑ in $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$. We shall show that for convergent sequences $h_n \to h$, the limit distribution in

$$\mathcal{L} \left(n^{\alpha(\vartheta_0)/2} \left(\widehat{\vartheta}_n^c - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h_n) \right) \mid Q_n^{h_n}(\vartheta_0, g_0; g') \right) \quad \to: \quad F^h \tag{24}$$

does not depend on the value of the local parameter h. Then $(\widehat{\vartheta}_n^c)_n$ is regular for ϑ in $\mathcal{S}^{g'}_{(\vartheta_0,g_0)}$ at ϑ_0; as a consequence, we get

$$\lim_{n\to\infty} \sup_{|h|<d} E_{Q_n^h(\vartheta_0,g_0;g')} \left(l \left(n^{\alpha(\vartheta_0)/2} \left(\widehat{\vartheta}_n^c - (\vartheta_0 + n^{-\alpha(\vartheta_0)/2} h) \right) \right) \right)$$

$$= E \left(l \left(\sigma \, \widetilde{K}(\vartheta_0, f^c)^{-1/2} \, \frac{B(W_1^{\alpha(\vartheta_0)})}{W_1^{\alpha(\vartheta_0)}} \right) \right) .$$

for arbitrary g' in $\mathcal{H}_c$ and all $d < \infty$, and thus the last assertion of theorem 2 c).

Write Λ_n^h for the log-likelihood ratio process (20) of $Q_n^h = Q_n^h(\vartheta_0, g_0; g')$ to Q_n^0 relative to $(\mathcal{G}_{tn})_{t\geq 0}$, write Y_n for the martingale arising in the rescaled estimation error of $\widehat{\vartheta}_n^c$ under Q_n^0:

$$Y_n(t) = \frac{1}{n^{\alpha(\vartheta_0)/2}} \int_0^{tn} \frac{(f^c)(\eta_s)}{\sigma^2} dM_s^{\vartheta_0,g_0} , \quad t \geq 0 .$$

Since $f^c g' = 0$, the quadratic covariation $\langle \Lambda_n^h, Y_n \rangle$ of Λ_n^h and Y_n is

$$h \, \frac{1}{n^{\alpha(\vartheta_0)}} \int_0^{tn} \frac{(f^c)^2(\eta_s)}{\sigma^2} ds .$$

Using this together with proposition 2, we get for the pairs $\mathcal{L}\left(\Lambda_n^{h_n}(1), Y_n(1) \mid Q_n^0\right)$ as $n \to \infty$ $(h_n \to h)$ a limit of type

$$\int \mathcal{L}(W_1^{\alpha(\vartheta_0)})(dj)\, \mathcal{N}\left(\begin{pmatrix} -\frac{1}{2}\sigma_1^2 j \\ 0 \end{pmatrix} \begin{pmatrix} \sigma_1^2 j & \sigma_1\sigma_2 j\rho \\ \sigma_1\sigma_2 j\rho & \sigma_2^2 j \end{pmatrix}\right)$$

with suitable σ_1, σ_2, $-1 < \rho < 1$. The well known mean shift argument in LeCam's third lemma then shows that the limit of $\mathcal{L}\left(\Lambda_n^{h_n}(1), Y_n(1) \mid Q_n^{h_n}\right)$ as $n \to \infty$ is

$$\int \mathcal{L}(W_1^{\alpha(\vartheta_0)})(dj)\, \mathcal{N}\left(\begin{pmatrix} +\frac{1}{2}\sigma_1^2 j \\ \sigma_1\sigma_2 j\rho \end{pmatrix} \begin{pmatrix} \sigma_1^2 j & \sigma_1\sigma_2 j\rho \\ \sigma_1\sigma_2 j\rho & \sigma_2^2 j \end{pmatrix}\right) .$$

Convergence of $Y_n(1)$ under $Q_n^{h_n}$ gives (after division by $\frac{1}{n^{\alpha(\vartheta_0)}} \int_0^n \frac{(f^c)^2(\eta_s)}{\sigma^2} ds$ whose limit law under Q_n^h is independent of h) $F^h = F^0$ in (24). The proof is complete. $\square$

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