Point process models and local asymptotics in statistics

I – Likelihood ratio processes for multivariate point processes

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The aim of this talk is to explain statistical models for multivariate point process models, based on

- Jacod, J.: Multivariate point processes: predictable projection,Radon-Nikodym derivatives, representation of martingales. Zeitschrift f. Wahrscheinlichkeitstheorie u. verw. Geb. **31**, 235–253 (1975)
- Kabanov, Y., Liptser, R., Shiryaev, A.: Criteria of absolute continuity of measures corresponding to multivariate point processes. In: Proceedings of the 3rd Japan-USSR symposium on probability theory, Lecture Notes in Math., Vol. 550, 232–252. Springer 1976.
- Bremaud, P.: Point processes and queues. Springer 1981.
- Jacod, J., Shiryaev, A.: Limit theorems for stochastic processes. Springer 1987.

and in the following order:

- Exponentials of pure jump semimartingales
- Detailed outline of a simple case (univariate point processes, constant intensities)
- Multivariate point processes
- One example: observing jumps ≥ 1 in a stable increasing process under unknown parameters
- Outlook towards Levy processes

1 Exponentials of pure jump semimartingales

Bremaud (1981) Appendix A4: For a deterministic function $t \to a(t)$ which is cadlag and of locally finite variation (BV) – thus we assume in particular that jumps of $a(\cdot)$ are absolutely summable – there is a unique solution $\ell(\cdot)$ to

$$\ell(t) \;=\; 1 \;+\; \int_0^t \ell(s-)\, da(s) \quad , \quad t \geq 0$$

given by

$$\ell(t) \;=\; \left[\prod_{0<s\leq t} (1+\Delta a(s)) \right] e^{\breve{a}(t)} \;=\; e^{a(t)-a(0)} \left[\prod_{0<s\leq t} (1+\Delta a(s))\, e^{-\Delta a(s)} \right]$$

with notation

$$\Delta a(s) \;:=\; a(s) - a(s-) \quad , \quad \breve{a}(t) \;:=\; a(t) - a(0) - \sum_{0<s\leq t} \Delta a(s)\ .$$

Jacod-Shiryaev (1987) I.4.61 – I.4.63: For a pure-jump semimartingale $X = (X_t)_{t\geq 0}$ (thus we assume: no continuous martingale part X^c, square-summability of small jumps over finite time intervals, finite number of big jumps over finite time intervals) there is a unique cadlag adapted solution $L = (L_t)_{t\geq 0}$ to

$$L_t \;=\; 1 \;+\; \int_0^t L_{s-}\, dX_s \quad , \quad t \geq 0$$

given by

$$L_t \;=\; e^{X_t - X_0} \left[\prod_{0<s\leq t} (1+\Delta X_s)\, e^{-\Delta X_s} \right] .$$

The process L is called the stochastic exponential of X, notation $\mathcal{E}(X) = L$. The product is well defined: by square-summability of small jumps, $(1+z)e^{-z} \sim (1+z)(1-z) = 1 - z^2$ as $z \to 0$.

2 Univariate case, constant intensities

1) We introduce a canonical probability space for Poisson processes. Write $\mathbb{M}$ for the set of all functions $f : [0,\infty) \to \mathbb{N}_0$, piecewise constant, cadlag, $f(0) = 0$, with jump heigth $\Delta f(s) = f(s) - f(s-)$ equal to 1 at the jump times. The canonical process is the process of coordinate projections:

$$N = (N_t)_{t\ge 0} \quad , \quad N_t(f) \;=\; f(t)\;,\; f \in \mathbb{M}\;,\; t \ge 0\;.$$

Equip $\mathbb{M}$ with the σ-field $\mathcal{M} := \sigma(N_t : t \ge 0)$ and the filtration

$$\mathbb{G} = (\mathcal{G}_t)_{t\ge 0} \quad , \quad \mathcal{G}_t \;:=\; \sigma(N_s : 0 \le s \le t)$$

which is right-continuous (Bremaud 1981, Appendix A2, T26). Write $(\tau_n)_{n\ge 1}$ for the sequence of jump times; we have $\tau_n \uparrow \infty$ by definition of $\mathbb{M}$.

For all $\lambda > 0$, $(\mathbb{M}, \mathcal{M})$ carries a unique probability measure $Q^{(\lambda)}$ such that N under $Q^{(\lambda)}$ is Poisson with constant intensity λ. Equivalent characterizations of $Q^{(\lambda)}$ are:

$$M^{(\lambda)} \;:=\; (\,N_t - \lambda\, t\,)_{t\ge 0} \qquad \text{is a } (Q^{(\lambda)}, \mathbb{G})\text{-martingale} \;\;,$$

or: whenever $C = (C_s)_{s\ge 0}$ is nonnegative, $\mathbb{G}$-predictable and bounded,

$$(\diamond) \qquad E_{Q^{(\lambda)}}\left(\int_0^t C_s\, dN_s \right) \;=\; E_{Q^{(\lambda)}}\left(\int_0^t C_s\, \lambda\, ds \right) \quad , \quad t \ge 0\;.$$

This is our filtered statistical model

$$\left(\mathbb{M}\,,\, \mathcal{M}\,,\, \mathbb{G}\,,\, \left\{ Q^{(\lambda)} : \lambda > 0 \right\} \right)\;.$$

2) Fix $\chi > 0$ constant. Then the unique solution to

$$dL_t \;=\; L_{t-}\,(\chi-1)\,dM_t^{(\lambda)} \quad , \quad L_0 \equiv 1$$

is the exponential

$$L \;=\; \mathcal{E}_{Q^{(\lambda)}}\left(\int(\chi-1)dM^{(\lambda)}\right)$$

given in section 1. Since $\int(\chi-1)dM_s^{(\lambda)}$ has finitely many jumps over finite time intervals, the exponential takes the form

$$(\times) \qquad L_t \;=\; \left[\prod_{0<s\leq t}(1+(\chi-1)\Delta M_s^{(\lambda)})\right] e^{-\int_0^t(\chi-1)\,\lambda\,ds} \;=\; \chi^{N_t}\,e^{-\int_0^t(\chi-1)\,\lambda\,ds} \quad , \quad t \geq 0\,.$$

L is strictly positive by $\chi > 0$. The SDE shows that L is a local martingale. Laplace transforms for Poisson random variables

$$E_{Q^{(\lambda)}}\left(e^{\alpha N_t}\right) \;=\; \exp\{\lambda t(e^{\alpha}-1)\} \quad \forall \alpha \in \mathbb{R}$$

establish $E_{Q^{(\lambda)}}(L_t) = 1$ for all $0 \leq t < \infty$. We thus arrive at:

$$L = (L_t)_{0\leq t<\infty} \text{ is a strictly positive } (Q^{(\lambda)}, \mathbb{G})\text{-martingale}\,.$$

3) For the filtered statistical model $\left(\mathbb{M}, \mathcal{M}, \mathbb{G}, \{Q^{(\lambda)} : \lambda > 0\}\right)$ we prove the following theorem.

Theorem : i) For $\widetilde{\lambda} \neq \lambda$ in $(0,\infty)$, probability measures $Q^{(\widetilde{\lambda})}$ and $Q^{(\lambda)}$ are <u>locally equivalent relative to $\mathbb{G}$</u>, i.e.:

$$Q^{(\widetilde{\lambda})}|\mathcal{G}_t \;\sim\; Q^{(\lambda)}|\mathcal{G}_t \quad , \quad 0 \leq t < \infty\,.$$

ii) There is a unique $(Q^{(\lambda)}, \mathbb{G})$-martingale $L^{\widetilde{\lambda}/\lambda} = (L_t^{\widetilde{\lambda}/\lambda})_{0\leq t<\infty}$ such that

$$(*) \qquad 0 \leq t < \infty\,,\; A \in \mathcal{G}_t \;:\; Q^{(\widetilde{\lambda})}(A) \;=\; E_{Q^{(\lambda)}}\left(L_t^{\widetilde{\lambda}/\lambda}\,1_A\right) \;:$$

$L^{\widetilde{\lambda}/\lambda}$ is called the <u>likelihood ratio process of $Q^{(\widetilde{\lambda})}$ with respect to $Q^{(\lambda)}$ relative to $\mathbb{G}$</u>.

iii) The likelihood ratio process of $Q^{(\widetilde{\lambda})}$ with respect to $Q^{(\lambda)}$ relative to $\mathbb{G}$ is the exponential

$$\mathcal{E}_{Q^{(\lambda)}}\left(\int(\chi-1)dM^{(\lambda)}\right) \quad \text{where} \quad \chi \;:=\; \frac{\widetilde{\lambda}}{\lambda}$$

whence by $(\times)$

$$L_t^{\widetilde{\lambda}/\lambda} \;=\; \chi^{N_t}\,e^{-\int_0^t(\chi-1)\,\lambda\,ds} \quad , \quad 0 \leq t < \infty\,.$$

Proof : Put $\chi = \frac{\widetilde{\lambda}}{\lambda}$ and use the strictly positive $(Q^{(\lambda)}, \mathbb{G})$-martingale L of 2) above

$$L_t \;=\; \mathcal{E}_{Q^{(\lambda)}}\left(\int (\chi - 1) dM^{(\lambda)} \right)_t \;=\; \chi^{N_t}\, e^{-\int_0^t (\chi-1)\,\lambda\, ds} \quad , \quad 0 \le t < \infty$$

to define a probability measure $\widetilde{Q}$ on $\left(\mathbb{M}\,,\, \mathcal{M} = \bigvee_{0 \le t < \infty} \mathcal{G}_t \right)$:

$$\widetilde{Q}(A) \;:=\; E_{Q^{(\lambda)}}\,(\, L_t\, 1_A\,) \quad \text{if} \quad 0 \le t < \infty \;\text{ and }\; A \in \mathcal{G}_t\;.$$

Then it remains to prove the following:

$\widetilde{Q}$ coincides on $(\mathbb{M}, \mathcal{M})$ with the probability measure $Q^{(\widetilde{\lambda})}$ defined in 1) above ;

recall from 1) that N admits constant intensity $\widetilde{\lambda}$ for one and only one probability measure on $(\mathbb{M}, \mathcal{M}, \mathbb{G})$. We proceed on the lines of Bremaud 1981, Ch. VI.2, using the criterion $(\diamond)$ from 1).

For $C = (C_s)_{s \ge 0}$ nonnegative, $\mathbb{G}$-predictable and bounded, we have to check that expectations

$$E_{\widetilde{Q}}\left(\int_0^t C_s\, dN_s \right) \;=\; E_{Q^{(\lambda)}}\left(L_t\, \int_0^t C_s\, dN_s \right)$$

coincide with

$$E_{\widetilde{Q}}\left(\int_0^t C_s\, \widetilde{\lambda}\, ds \right) \;=\; E_{Q^{(\lambda)}}\left(L_t\, \int_0^t C_s\, \widetilde{\lambda}\, ds \right)\;.$$

We use the integration by parts formula for cadlag BV functions f and g (Bremaud 1981, App. A4)

$$f(t)g(t) \;=\; f(0)g(0) + \int_0^t f(s-)\, dg(s) + \int_0^t g(s)\, df(s)\;.$$

Rewriting thus the second integrand in the first line of expectations, we have

$$\begin{aligned} L_t \int_0^t C_s\, dN_s \;&=\; 0 \;+\; \int_0^t (\int_0^{v-} C_s\, dN_s)\, dL_v \;+\; \int_0^t L_v\, C_v\, dN_v \\ &=\; 0 \;+\; \{\text{a local } (Q^{(\lambda)}, \mathbb{G})\text{-martingale}\} \;+\; \int_0^t \chi L_{v-}\, C_v\, dN_v \;: \end{aligned}$$

this holds since the process $(\int_0^{v-} C_s\, dN_s)_{s \ge 0}$ is predictable and locally bounded, since L is a $(Q^{(\lambda)}, \mathbb{G})$-martingale, and since L_{τ_n} equals $\chi L_{\tau_n -}$ at the jump times τ_n of N. Integration by parts in the second line of expectations gives

$$\begin{aligned} L_t \int_0^t C_s\, \widetilde{\lambda}\, dv \;&=\; 0 \;+\; \int_0^t L_{v-} C_v\, \widetilde{\lambda}\, dv \;+\; \int_0^t (\int_0^v C_s\, \widetilde{\lambda} ds)\, dL_v \\ &=\; 0 \;+\; \int_0^t \chi\, L_{v-} C_v\, \lambda\, dv \;+\; \{\text{a local } (Q^{(\lambda)}, \mathbb{G})\text{-martingale}\} \end{aligned}$$

again with $\chi = \frac{\widetilde{\lambda}}{\lambda}$. Replacing now C by $C 1_{[[0,\tau_n]]}$, the local $Q^{(\lambda)}$-martingales are martingales; taking $Q^{(\lambda)}$-expections and letting n tend to ∞, we obtain $(\diamond)$. $\square$

3 Multivariate point processes

Let Ω denote some probability space with right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, define $\mathcal{A} = \bigvee_{0\leq t<\infty} \mathcal{F}_t$.

Let $(E, \mathcal{E})$ denote a Polish space. We assume that $(\Omega, \mathcal{A})$ carries

- a process $\lambda = (\lambda_s)_{s\geq 0}$ nonnegative, $\mathbb{F}$-predictable, bounded
- a strictly increasing sequence $(\tau_n)_{n\geq 1}$ of $I\!\!F$-stopping times
- a set of $\mathcal{F}_{\tau_n}$-measurable random variables Z_n, $n \geq 1$, taking values in $(E, \mathcal{E})$.

Consider the random measure

$$\mu(ds, dz) \;=\; \sum_{n\geq 1} \epsilon_{(\tau_n, Z_n)}(ds, dz)$$

and associate counting processes

$$N(t, F) \;:=\; \mu((0,t] \times F) \quad , \quad F \in \mathcal{E} \;,\; 0 \leq t < \infty \;.$$

Finally, let $K(s, dz) = k(s,z)ds$ denote a transition probability from $[0, \infty)$ to E.

Assuming that the filtration $I\!\!F$ is 'small enough', e.g.

$$\mathcal{F}_t \;=\; \sigma\,(\, N(v,F) \,:\, 0 \leq v \leq t \,,\, F \in \mathcal{E} \,) \quad , \quad 0 \leq t < \infty \;,$$

there is one and only one probability measure $Q^{(\lambda,k)}$ on $(\Omega, \mathcal{A})$ such that all

$$M^{(\lambda,k,F)} \;:=\; \left(N(t,F) - \int_0^t \lambda_s \, K(s,F)\, ds \right)_{t\geq 0} \quad , \quad F \in \mathcal{E}$$

are $(Q^{(\lambda,k)}, I\!\!F)$-martingales. This is Jacod (1975), uniqueness theorem (3.4).

Consider $\widetilde{\lambda} = (\widetilde{\lambda}_s)_{s \geq 0}$ and $\widetilde{k}(.,.)$ with the same properties as $\lambda = (\lambda_s)_{s \geq 0}$ and $k(.,.)$ above, related by

$$\widetilde{\lambda}_s \, \widetilde{k}(s,z) \;=\; \chi(s,z)\, \lambda_s \, k(s,z)$$

where $\chi : [0,\infty) \times E \to [0,\infty)$ is some function. Then the following holds.

Theorem : (Kabanov-Liptser-Shiryaev 1976, theorem 1; Jacod 1975, section 5)

i) $Q^{(\widetilde{\lambda},\widetilde{k})}$ is locally absolutely continuous with respect to $Q^{(\lambda,k)}$ relative to $I\!\!F$, i.e.:

$$Q^{(\widetilde{\lambda},\widetilde{k})}|\mathcal{G}_t \;\ll\; Q^{(\lambda,k)}|\mathcal{G}_t \quad , \quad 0 \leq t < \infty \,.$$

ii) The likelihood ratio process $L^{(\widetilde{\lambda},\widetilde{k})/(\lambda,k)}$ of $Q^{(\widetilde{\lambda},\widetilde{k})}$ with respect to $Q^{(\lambda,k)}$ relative to $I\!\!F$ is given by the stochastic exponential

$$\mathcal{E}_{Q^{(\lambda,k)}}\left(\int_0^\bullet \int_E (\chi(s,z)-1)\, (\mu(ds,dz) - \lambda_s k(s,z) dz ds) \right) \quad \text{with} \quad \chi(s,z) \;=\; \frac{\widetilde{\lambda}_s \widetilde{k}(s,z)}{\lambda_s k(s,z)}$$

where the martingale in parentheses has paths of locally finite variation.

Thus by section 1

$$L_t^{(\widetilde{\lambda},\widetilde{k})/(\lambda,k)} \;=\; \left[\prod_{n:\tau_n \leq t} \chi(\tau_n, Z_n) \right] \exp\left\{ - \int_0^t \int_E (\chi(s,z)-1)\, \lambda_s k(s,z)\, dz ds \right\}$$

or, $k(s,z)dz$ and $\widetilde{k}(s,z)dz$ being transition probabilities,

$$L_t^{(\widetilde{\lambda},\widetilde{k})/(\lambda,k)} \;=\; \left[\prod_{n:\tau_n \leq t} \chi(\tau_n, Z_n) \right] \exp\left\{ - \int_0^t (\widetilde{\lambda}_s - \lambda_s)\, ds \right\} \,, \quad 0 \leq t < \infty \,.$$

4 Example

- Höpfner, R., Jacod, J.: Some remarks on joint estimation of index and scale parameter for stable processes. In: Asymptotic statistics, Proc. 5th Prague symp., 273–284. Physica-Verlag 1994.

Write S for a stable increasing process of some index $0 < \alpha < 1$ and some weight parameter $\xi > 0$, constructed from Poisson random measure on $(0,\infty)\times(0,\infty)$ with intensity $ds\, \xi\, \alpha z^{-\alpha-1}\, dz$.

Fixing some $\varepsilon > 0$, let X denote the sum of all jumps $\geq \varepsilon$ in the trajectory of S:

$$X_t \;=\; \sum_{\substack{0<s\leq t \\ \Delta S_s \geq \varepsilon}} \Delta S_s \quad , \quad t \geq 0$$

and consider the statistical model related to inference on α and ξ based on observation of the trajectory of X. This example will be continued (statistical properties) in the last part of the lectures.

View X as canonical process on the space of all piecewise constant nondecreasing cadlag functions with jumps $\geq \varepsilon$, starting in $X_0 = 0$, write $\mu(ds, dz)$ for the jump measure of X:

$$X_t \;=\; \int_0^t \int_{[\varepsilon,\infty)} z\, \mu(ds,dz) \quad , \quad t \geq 0\,.$$

We consider the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t<\infty}$ generated by X and take $\mathcal{A} := \bigvee_{0\leq t<\infty} \mathcal{F}_t$.

Then, for every $\alpha \in (0,1)$ and $\xi \in (0,\infty)$, there is a unique probability measure $Q^{(\alpha,\xi)}$ on $(\Omega, \mathcal{A})$ such that $\mu(ds, dz)$ is Poisson random measure with intensity

$$ds\, \xi\, \alpha z^{-\alpha-1} 1_{\{z\geq\varepsilon\}}\, dz \;=:\; \lambda\, ds\, k(z) dz \quad \text{with} \quad \lambda \;:=\; \xi\varepsilon^{-\alpha}$$

under $Q^{(\alpha,\xi)}$. Since λ is a constant and $k(\cdot)$ a probability density, we are in the setting of section 3.

The theorem in section 3 shows that all probability measures in

$$\mathcal{P} \;:=\; \left\{ Q^{(\alpha,\xi)} \;:\; \alpha \in (0,1)\,,\, \xi \in (0,\infty) \right\}$$

are locally equivalent relative to $\mathbb{F}$, with

$$L_t^{(\widetilde{\alpha},\widetilde{\xi})/(\alpha,\xi)} \;=\; \left[\prod_{0<s\leq t} \frac{\widetilde{\alpha}\widetilde{\xi}}{\alpha\xi}\,(\Delta X_s)^{\alpha-\widetilde{\alpha}} \right] \exp\left\{ -t(\,\widetilde{\xi}\varepsilon^{-\widetilde{\alpha}} - \xi\varepsilon^{-\alpha}\,) \right\} \;, \quad 0 \leq t < \infty$$

the likelihood ratio process of $Q^{(\widetilde{\alpha},\widetilde{\xi})}$ with respect to $Q^{(\alpha,\xi)}$ relative to $\mathbb{F}$.

Remark : This extends to $\mathbb{F}$-stopping times T which are $\mathcal{P}$-almost surely finite:

$$Q^{(\widetilde{\alpha},\widetilde{\xi})}(A) \;=\; E_{Q^{(\alpha,\xi)}}\left(L_T^{(\widetilde{\alpha},\widetilde{\xi})/(\alpha,\xi)}\, 1_A \right) \quad , \quad A \in \mathcal{F}_T\,.$$

Remark : Local equivalence of probability measures associated to different values of $(\widetilde{\alpha},\widetilde{\xi})$ and (α,ξ) breaks down once we are able to observe arbitrarily small jumps in the trajectory of the stable increasing process S.

5 Outlook towards Levy processes

We consider pure-jump Levy processes $X = (X_t)_{t\geq 0}$ in a semimartingale representation

$$X_t \;=\; \int_0^t \int_{\mathbb{R}\setminus\{0\}} h(z)\,(\mu(ds,dz) - ds\Lambda(dz)) \;+\; \int_0^t \int_{\mathbb{R}\setminus\{0\}} (z - h(z))\,\mu(ds,dz)$$

where $h(\cdot)$ is a truncation function (i.e.: $h \in \mathcal{C}_{\mathcal{K}}^\infty$ such that $h(z) = z$ on some ball $B_\varepsilon(0)$), and Λ a Levy measure, i.e. a σ-finite measure on $\mathbb{R}\setminus\{0\}$ with the property

$$\int_{\mathbb{R}\setminus\{0\}} (|z|^2 \wedge 1)\;\Lambda(dz) \;<\; \infty\,.$$

Hence $\mu(ds,dx)$ is the jump measure of X, small jumps of X are square integrable over finite time intervals, and there is a finite number of big jumps over finite time intervals.

X lives on some $(\Omega,\mathcal{A})$ and is adapted to some filtration $\mathbb{F}$. If we take $\mathbb{F}$ small enough (e.g., the right continuous filtration generated by the process X) and $\mathcal{A} = \bigvee_{0\leq t<\infty} \mathcal{F}_t$, there is one and only one probability measure $Q^{(\Lambda)}$ on $(\Omega,\mathcal{A})$ (Jacod-Shiryaev 1987, II.4.25) such that the random measure $\mu(ds,dz)$ is compensated by $ds\,\Lambda(dz)$.

Consider now another Levy measure $\widetilde{\Lambda}$ satisfying

$$\widetilde{\Lambda}(dz) \;=\; \chi(z)\Lambda(dz) \;, \quad \int_{\mathbb{R}\setminus\{0\}} |h(z)(\chi(z)-1)|\,\Lambda(dz) \;<\; \infty \;, \quad \int_{\mathbb{R}\setminus\{0\}} \left|\sqrt{\chi(z)}-1\right|^2 \Lambda(dz) \;<\; \infty \;.$$

This holds in particular if we define $\widetilde{\Lambda}(dz) := \chi(z)\Lambda(dz)$ from some function $\chi : \mathbb{R} \to [0,\infty)$ with the properties

$$\chi(0) := 1 \;,\; |\chi(z)-1| \leq L|z| \;\text{ on } B_\varepsilon(0) \;,\; \int_{\{|z|\geq\varepsilon\}} \chi(z)\,\Lambda(dz) < \infty \;\; .$$

Then the following holds:

i) Jacod-Shiryaev (1987), theorem IV.4.39 :
$Q^{(\widetilde{\Lambda})}$ is locally absolutely continuous with respect to $Q^{(\Lambda)}$ relative to $\mathbb{F}$.

ii) Jacod-Shiryaev (1987), lemma III.5.17, theorem III.5.19:
the likelihood ratio process $L^{\widetilde{\Lambda}/\Lambda}$ of $Q^{(\widetilde{\Lambda})}$ with respect to $Q^{(\Lambda)}$ relative to $\mathbb{F}$ is given by the stochastic exponential

$$\mathcal{E}_{Q^{(\Lambda)}}\,(\,U\,) \quad , \quad U \;:=\; \int_0^\bullet \int_{\mathbb{R}\setminus\{0\}} (\chi(z)-1)\;(\mu(ds,dz) - ds\Lambda(dz))\;.$$

iii) An explicit representation of the LRP is

$$L_t^{\widetilde{\Lambda}/\Lambda} \;=\; \exp\{\,U_t\,\} \left[\prod_{0<s\leq t} (1+\Delta U_s)\, e^{-\Delta U_s}\right] \quad , \quad 0 \leq t < \infty$$

cf. section 1, since small jumps of U are square summable over finite time intervals,