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LIMIT THEOREMS
FOR NULL RECURRENT MARKOV PROCESSES

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Contents

Preface

The aim of this note is to give a self-contained treatment of weak convergence of martingales and integrable additive functionals in general Harris recurrent Markov processes in continuous time.

If a Harris process $X = (X_t)_{t \geq 0}$ has a recurrent atom, then necessary and sufficient conditions for weak convergence of martingales associated to X have two components: first, either ergodicity of X or – in case of null recurrence – regular variation at infinity of tails of lifecycle length distributions (life cycles are excursions of the process between suitably defined successive visits to the atom); second, an integrability condition (with respect to invariant mesure) on the predictable quadratic variation. The norming functions are expressed in terms of tails of the lifecycle length distribution; they vary regularly at infinity with some index $0 < \alpha \leq 1$.
Limit processes are either Brownian motion (case $\alpha = 1$), or Brownian motion subject to independent time change by a Mittag-Leffler process (the process inverse to a stable increasing process) of index $0 < \alpha < 1$. No other weak limits under linear scaling of time and suitable norming can occur. Brownian motion in the limit does not characterize ergodicity of the process X, but arises also in a null recurrent case on the frontier to ergodicity.

For general Harris processes, recurrent atoms and thus i.i.d life cycles for the process X do not exist. So we consider instead of X a family of Harris processes $(\widetilde{X}^m)_m$ where $\widetilde{X}^m$ for large m is very close to X, and where trajectories of $\widetilde{X}^m$ have from time to time flats of independent exponential length over which Nummelin splitting can be applied. In this way we get for every one of the processes $\widetilde{X}^m$ a recurrent atom, i.i.d life cycles and thus limit theorems as above, for martingales and integrable additive functionals of $\widetilde{X}^m$. These limit theorems depend on m only through a set of constants which converge to a limit as m tends to infinity. In this way, we can deduce the desired limit theorem for martingales and integrable additive functionals of X from the family of limit theorems for $(\widetilde{X}^m)_m$. Of course, since life cycles for $\widetilde{X}^m$ have been introduced artificially and are different at each stage m, we need an intrinsic representation of the norming function for X-martingales: this intrinsic norming function is given in terms of regular variation at 0 of resolvants of X.

This is a new look on a partially very old topic. A first famous paper on convergence of integrable additive functionals is by Darling and Kac in 1957 ([D-K 57], re-exposed in the book by Bingham, Goldie and Teugels [B-G-T 87]): they prove that under a 'Darling-Kac condition', norming functions for additive functionals of X are necessarily regularly varying, and limit laws (for one-dimensional marginals) are necessarily Mittag-Leffler laws. Weak convergence of martingales under a Lindeberg condition implies weak convergence of quadratic variation processes (for weak convergence of stochastic processes, we rely on the book by Jacod and Shiryaev [J-Sh 87]). So the Darling-Kac result remains a main argument in the 'necessary part' of the result on weak convergence of martingales (we note here that the case of slow variation of tails of life cycle length distributions, present in the Darling-Kac theorem, does not correspond to *weak* convergence, but only to convergence of finite-dimensional distributions: this explains the absence of the case $\alpha = 0$ in our treatment). In a spirit similiar to [D-K 57], additive functionals of null recurrent birth and death processes or branching processes with immigration were treated by Karlin and McGregor [K-MG 61], Zubkov [Zu 72] and Pakes [Pa 75]. For one-dimensional diffusions, Khasminskii ([Has 80], see also [Kh 00], [Kh 01]) took a completely different route – based on differential equation techniques – to limit theorems for integrable additive functionals.

The 'sufficient part' of the result of weak convergence of martingales is an assertion 'regular variation of tails of life cycle length distributions implies convergence of martingales to Brownian motion time-changed by an independent Mittag-Leffler process'. The key step for this appears in a paper by Greenwood and Resnick [Gr-R 78]: they consider joint convergence of bidimensional random walks where the first marginal is attracted to Brownian motion, the second to a stable process, and proved – with strong reference to P. Lévy – that necessarily Brownian motion and stable process involved in such limits are independent. In the sequel, similiar ideas reappear in Kasahara [Ka 84] and other papers.

The next important progress was the paper by Touati [Tou 88] considering completely general Harris processes. Touati argued that using Nummelin splitting along a sequence of independent exponential times, life cycles may always be introduced artificially, and he gave a very good argument allowing to avoid 'Darling-Kac conditions' – which for general processes are highly cumbersome and rather impossible to verify – by use of 'special functions'. The corresponding parts of our treatment below are entirely based on this idea. However, we do not follow Touati in his argument on artificial introduction of life cycles (in continuous-time setting) which seems to us problematic. Instead of this, we propose another approach via an 'accompanying family' $(\widetilde{X}^m)_m$ for X such that at every stage m, Nummelin-like splitting of $\widetilde{X}^m$ is possible. Touati was the first to enounce a result on weak convergence of martingales and integrable additive functionals of a general Harris process in complete generality and under minimal hypotheses; unfortunately, a final publication of his paper never took place, and some points in his preprint version (e.g. treatment of case $\alpha = 1$ where errors occur) have to be corrected.

We now state the general result under minimal hypotheses in a *preliminary* way; see section 3 (theorems 3.15 and 3.16 there) for the complete set of assumptions and the definitive formulation.

Theorem (preliminary version): Consider a strong Markov process $X = (X_t)_{t\geq 0}$, defined on $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$, with Polish state space $(E, \mathcal{E})$, and with càdlàg paths. Assume that X is Harris recurrent with invariant measure μ.

a) For $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞, the following i) and ii) are equivalent:

i) for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, one has regular variation at 0 of resolvants of the process X

$$\left(R_{1/t}g\right)(x) = E_x\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right) \quad \sim \quad t^\alpha \frac{1}{l(t)}\, \mu(g), \quad t \to \infty$$

for μ-almost all $x \in E$ (the exceptional set depending on g);

ii) for every integrable additive functional A of X, $0 < E_\mu(A_1) < \infty$, one has weak convergence

$$\frac{(A_{tn})_{t\geq 0}}{n^\alpha / \, l(n)} \quad \to \quad E_\mu(A_1)\, W^\alpha$$

in $D(I\!R_+, I\!R)$ as $n \to \infty$, under P_x, for all $x \in E$.

For $0 < \alpha < 1$, the process W^α occurring in ii) is the Mittag-Leffler process of index α, i.e. the process inverse of the stable increasing process S^α; for $\alpha = 1$, W^1 is the deterministic process $id := (t)_{t\geq 0}$.

b) The cases in a) are the only ones where weak convergence of $\dfrac{(A_{tn})_{t\geq 0}}{v(n)}$ to a continuous nondecreasing limit process W (with $W_0 = 0$ and $\mathcal{L}(W_1)$ not degenerate at 0) is available for some norming function v.

c) Consider a locally square integrable local martingale M on $(\Omega, \mathcal{A}, I\!F, P_x)$, càdlàg and with $M_0 = 0$. Assume that its predictable quadratic variation $\langle M \rangle$ is a locally bounded process which is an *integrable* additive functional of X.

If a)i) holds for some $0 < \alpha \leq 1$ and some $l(\cdot)$ varying slowly at ∞, we have

$$\frac{1}{\sqrt{n^\alpha / l(n)}} \, (M_{tn})_{t\geq 0} \quad \to \quad \left(E_\mu \left(\langle M \rangle_1\right)\right)^{1/2} B(W^\alpha)$$

(weak convergence in $D(I\!R_+, I\!R)$ as $n \to \infty$, under P_x).

If in addition the sequence $\frac{1}{\sqrt{n^\alpha / l(n)}} \, (M_{tn})_{t\geq 0}$, $n \geq 1$, satisfies a Lindeberg condition, we have weak convergence of pairs

$$\left(\frac{1}{\sqrt{n^\alpha / l(n)}} \, M_{tn} \, , \, \frac{1}{n^\alpha / l(n)} \, \langle M \rangle_{tn} \right)_{t\geq 0} \quad \to \quad \left(\left(E_\mu \left(\langle M \rangle_1\right)\right)^{1/2} B(W^\alpha) \, , \, \left(E_\mu \left(\langle M \rangle_1\right)\right) W^\alpha \right)$$

in $D(I\!R_+, I\!R \times I\!R)$ as $n \to \infty$, under P_x.

Here our notations are as usual in semimartingale theory, see e.g. the book Jacod and Shiryaev [J-Sh 87]; in particular, the predictable quadratic variation $\langle M \rangle$ of a locally square integrable local martingale M is the unique predictable increasing process such that $M^2 - \langle M \rangle$ is a local martingale. An extension of c) to multidimensional martingales M is straightforward: replace B in c) by a multidimensional standard Brownian motion, and the covariance by the matrix $\left(E_\mu \left(\langle M^i, M^j \rangle_1\right)\right)_{i,j}$, where M^i, M^j are the components of M. Also, by the ratio limit theorem, the second assertion of c) yields convergence of martingales together with arbitrary integrable additive functionals of X.

However, there is an essential difficulty related to this general formulation. Usually in applications, one specifies a Markov process by its infinitesimal generator, and – except some rare examples – there is no possibility to put hands – in a sense of explicit representations – on the semigroup itself. As a consequence, explicit calculation of resolvents from the semigroup seems possible only in very few cases, so condition a)i) is of rather limited practical interest. This is why the study of processes with life cycles presents an interest in itself: various tools to calculate explicit norming functions from tails of suitable life cycle distributions do exist. Some care is needed in order to define properly these life cycles in continuous time. We state a *preliminary rough version* of the result 'with life cycles', see section 3 for the definitive formulation with all details, in particular for our assumptions concerning life cycles (theorem 3.1 together with corollaries 3.2, 3.3, and proposition 3.4).

Theorem (preliminary version): Assume that the Harris process X has a recurrent atom. For suitably defined life cycles of X – with life cycle length distribution F – and appropriate norming of the invariant measure, condition a)i) on resolvents of X in the preceding theorem (with $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞) is equivalent to

$$r(t) := \int_0^t (1 - F(x))\, dx \ \sim \ \frac{1}{\Gamma(2-\alpha)}\, t^{1-\alpha}\, l(t)\,, \quad t \uparrow \infty\,,$$

with same α and $l(\cdot)$, and in case $\alpha < 1$ also equivalent to

$$1 - F(x) \ \sim \ \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha}\, l(x)\,, \quad x \uparrow \infty\,.$$

There are several points which are *not treated* in this text. First, we do not consider the case of slowly varying norming functions; this arises e.g. in connection with two-dimensional Brownian motion, see Kasahara and Kotani [K-K 79] or Hu and Yor [H-Y 98]. Here interesting time transformations are non-linear, and only finite-dimensional convergence can be obtained: our text is based on weak convergence techniques, functional in time, for semimartingales. Thus for the case $\alpha = 0$, we refer the reader to the work of Kasahara ([Ka 82], [Ka 86], [Ka 85]), and – relying on Kasahara here – Touati [Tou 88]. Next, we do not consider discrete time processes: for discrete time, there are recent results of Chen ([Che 99], [Che 00]) who uses Nummelin splitting and 'special functions', but is interested in convergence of one-dimensional marginals only. Touati [Tou 88] treated the continuous time case parallel to discrete time: he has the discrete-time versions of all above results. Third, there is work on strong approximation of additive functionals: see the papers by Csáki, Csörgö, Földes, Révész [C-C-F-R 92], and [C-C 95], [C-S 96].

The present text is organized as follows. First, there are two introductory sections: section 1 deals with Harris recurrence, and section 2 with stable processes and classical convergence to stable laws. All our main results are formulated in section 3. Here subsection 3.1 is devoted to processes X which admit a recurrent atom and thus i.i.d life cycles. Subsection 3.2 gives a family of examples which apply the result 'with life cycles' to classical one-dimensional diffusions, with strong reference to Khasminskii (his explicit representation of tails of life-cycle length distributions in null recurrent one-dimensional diffusions is a key tool here). Finally subsection 3.3 states the *general result* (without assuming existence of life cycles for X) under *minimal hypotheses*.

All proofs together then form the rest of our text: section 4 proves the 'sufficient part' in case of life cycles, section 5 the corresponding 'necessary part', section 6 recalls classical Nummelin splitting as introduced by [Num 78], and section 7 – devoted to general processes without life cycles – constructs the family $(\widetilde{X}^m)_m$ of processes 'accompanying' X such that Nummelin-like splitting can introduce atoms and life cycles artificially into $\widetilde{X}^m$, at every stage m, and then deduces the convergence theorem for X from the family of convergence theorems for $(\widetilde{X}^m)_m$.

We hope that the present text may contribute to make existing theorems on weak convergence of martingales and integrable additive functionals in null recurrent Markov processes better known in the probabilistic and statistical community, and may be useful as a self-contained reference in applications such as statistical inference for stochastic processes.

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1 Harris recurrence

This introductory section states some main facts about Harris recurrence. These facts will be used througout this text. An essential reference is Azéma-Duflo-Revuz [A-D-R 69].

Throughout this note, we consider a stochastic basis $(\Omega, \mathcal{A}, I\!F)$, $I\!F$ right-continuous, and on $(\Omega, \mathcal{A}, I\!F, (P_x)_x)$ a process $X = (X_t)_{t\geq 0}$ which is strongly Markov, taking values in a Polish space $(E, \mathcal{E})$, with càdlàg paths, and with $X_0 = x$ P_x-a.s., $x \in E$. We have shift operators $(\vartheta_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, I\!F)$, and write $(P_t)_{t\geq 0}$ for the semigroup of X.

1.1 Definition: ([A-D-R 69]) X is called *Harris recurrent* if there exists some σ-finite measure m on $(E, \mathcal{E})$ such that

$$(*) \qquad m(A) > 0 \quad \Longrightarrow \quad \forall\, x \in E: \quad P_x\left(\int_0^\infty 1_A(X_s)ds = \infty\right) = 1.$$

Sometimes also the terminology *m-irreducible* is used for $(*)$.

1.2 Theorem: ([A-D-R 69]) If X is Harris recurrent, then there is a unique (up to constant multiples) invariant measure μ for X (i.e. a σ-finite measure such that $\mu P_t = \mu$ for all $t \geq 0$), and property $(*)$ in 1.1 holds with μ in place of m.

Definition: A Harris recurrent process X with invariant measure μ is called *positive recurrent* (or *ergodic*) if $\mu(E) < \infty$, *null recurrent* if $\mu(E) = \infty$.

We give the major ideas of the proof of 1.2; the notions developped here will reappear as main tools in section 7.

Sketch of the proof of 1.2: For $\alpha > 0$, the α-potential kernel is

$$U^\alpha(x, A) = E_x(\int_0^\infty e^{-\alpha t} 1_A(X_t)dt) = \int_0^\infty e^{-\alpha t} P_t(x, A)dt$$

i) A first step is to prove that $\mu P_t = \mu$ for all t if and only if $\mu U^1 = \mu$. The nontrivial direction is $\Longleftarrow$. The proof starts from the resolvent equation (see [Chu 82, p.83])

$$U^\alpha = U^\beta + (\beta - \alpha) U^\alpha U^\beta = U^\beta + (\beta - \alpha) U^\beta U^\alpha$$

for all $\alpha > 0$, $\beta > 0$. In particular for $\alpha \geq 1$

$$U^1 = U^\alpha + U^1 I_{\alpha - 1} U^\alpha$$

where $I_{\alpha-1}$ is a multiplication kernel. Assume $\mu U^1 = \mu$, and $\alpha \geq 1$; consider sets $\Gamma \in \mathcal{E}$ with $\mu(\Gamma) < \infty$. Then $U^\alpha(x, \Gamma) \leq U^1(x, \Gamma)$ and thus $\mu U^\alpha(\Gamma) < \infty$; this gives

$$\mu(\Gamma) = \mu U^\alpha(\Gamma) + (\alpha - 1)\mu U^\alpha(\Gamma) = \alpha \mu U^\alpha(\Gamma) \quad \forall\, \alpha \geq 1.$$

Integrating the l.h.s with respect to the probability law $\alpha e^{-\alpha t} dt$ gives

$$\mu U^\alpha(\Gamma) = \int_0^\infty e^{-\alpha t} \mu(\Gamma) dt.$$

Since $\mu U^\alpha(\Gamma) = \int_0^\infty e^{-\alpha t} \mu P_t(\Gamma) dt$ by definition, we get

$$\forall\, \alpha \geq 1: \quad \int_0^\infty e^{-\alpha t} \mu P_t(\Gamma) dt = \int_0^\infty e^{-\alpha t} \mu(\Gamma) dt.$$

On open subsets of $(0, \infty)$, Laplace transforms characterize the underlying measures on $(0, \infty)$ uniquely, thus $\mu P_t(\Gamma) = \mu(\Gamma)$ for λ-almost all $t \geq 0$ (with notation λ for Lebesgue measure). Since X is strongly Markov, $t \to P_t(\Gamma)$ is continuous, and so we have $\mu P_t(\Gamma) = \mu(\Gamma)$ for all $t \geq 0$.

ii) Let $\overline{X}$ denote a discrete-time Markov chain with one-step transition kernel U^1: this means that the continuous-time process X is observed at the jump times of an independent Poisson process with rate 1 (a typical reasoning in order to transfer results available in discrete time to the continuous-time setting). By i), a σ-finite measure μ is thus invariant for $X = (X_t)_{t \geq 0}$ if and only if it is invariant for $\overline{X} = (\overline{X}_n)_{n \geq 0}$.

iii) Following Harris (see [Har 56]), the discrete time chain $\overline{X} = (\overline{X}_n)_{n \geq 0}$ is called Harris recurrent if there is some σ-finite measure m on $(E, \mathcal{E})$ such that

$$(+) \qquad m(A) > 0 \quad \Longrightarrow \quad \forall\, x \in E: \quad P_x\left(\sum_{n=1}^\infty 1_A(\overline{X}_n) = \infty \right) = 1.$$

Replacing the random variable in (+) by its expectation, the following *weaker* property

$$(++) \qquad \left[\exists\, x: \quad \sum_{n=1}^\infty (U^1)^n(x, A) = 0 \right] \quad \Longrightarrow \quad m(A) = 0$$

was used by Foguel; he showed ([Fog 66, thm. 4]) that (++) implies the existence of a σ-finite subinvariant measure μ for $(\overline{X}_n)_n$ which is dominating m:

$$\mu U^1 \leq \mu \ , \quad m \ll \mu \ .$$

iv) Note that we have

$$\sum_{n=1}^{\infty} (U^1)^n(x,A) = \int_0^\infty P_t(x,A)dt$$

since

$$(U^1)^n(x,A) = \int_0^\infty e^{-t} \frac{t^{n-1}}{(n-1)!} P_t(x,A)dt \ .$$

The continuous-time process $X = (X_t)_{t\geq 0}$ is by assumption Harris recurrent, so this equality shows that the discrete-time chain $(\overline{X}_n)_n$ has the property (++). By [Fog 66], there is a σ-finite subinvariant measure μ for U^1 which is dominating m. Again by Harris recurrence of $(X_t)_{t\geq 0}$, for sets $A \in \mathcal{E}$ meeting $m(A) > 0$ and (w.l.o.g.) $\mu(A) < \infty$, we have

$$\forall x \ : \quad \sum_{n=1}^{\infty} (U^1)^n(x,A) = \infty$$

together with finite bounds

$$\int (\mu - \mu U^1)(dx) \sum_{n=1}^{N} (U^1)^n(x,A) \quad = \quad \mu U^1(A) - \mu(U^1)^{N+1}(A) \quad \leq \quad \mu(A) < \infty$$

not depending on N: for μ subinvariant, this implies

$$\mu = \mu U^1 \ .$$

Hence μ is invariant for $(\overline{X}_n)_n$ and by ii) also invariant for $X = (X_t)_{t\geq 0}$.

v) For the remaining parts of the proof we refer to [A-D-R 69]: μ of iv) is the only invariant measure for X, μ is equivalent to mU^1, and the property $(*)$ in 1.1 holds with invariant measure μ in place of m. □

The above argument following [A-D-R 69] did not prove the discrete time chain $\overline{X} = (\overline{X}_n)_{n\geq 0}$ to be Harris if $X = (X_t)_{t\geq 0}$ is Harris, establishing only the weaker property (++) . The equivalence of both properties is proved in the next theorem; here we will make use of the somewhat simpler criterion $(\circ)$ below to check Harris recurrence of a continuous time process.

1.3 Proposition: (cf. Revuz-Yor [R-Y 91, pp. 395-396]) If $X = (X_t)_{t\geq 0}$ is strongly Markov with invariant measure m and if

$$(\circ) \qquad m(A) > 0 \quad \Longrightarrow \quad \forall\, x \in E: \quad P_x\left(\limsup_{t\to\infty} 1_A(X_t) = 1\right) = 1\,,$$

then $(*)$ of 1.1. holds, and X is Harris.

Proof : Consider $A \in \mathcal{E}$ with $m(A) > 0$; put $B_t = \int_0^t 1_A(X_s)ds$, $t \geq 0$, and $\tau_\varepsilon = \inf\{t : B_t > \varepsilon\}$. m being invariant for X, we have $E_m(B_1) = m(A)$, so there is some $\varepsilon > 0$ with $P_m(B_1 > \varepsilon)$ strictly positive, hence

$$m\{x : P_x(\tau_\varepsilon < \infty) > a\} \;>\; 0$$

for some $a > 0$. Property $(\circ)$ then implies

$$\limsup_{t\to\infty} P_{X_t}(\tau_\varepsilon < \infty) \quad \geq a > 0 \quad P_x\text{-a.s. for all } x \in E\,.$$

Write $Y_t := 1_{\{t+\tau_\varepsilon\circ\vartheta_t<\infty\}}$. For all $x \in E$, $P_{X_t}(\tau_\varepsilon < \infty)$ is a version of $E_x(Y_t|\mathcal{F}_t)$. As $t \to \infty$, Y_t converges to $Y := 1_{\bigcap_t\{t+\tau_\varepsilon\circ\vartheta_t<\infty\}}$ which is $\mathcal{F}_\infty$-measurable. A corollary to classical martingale theorems ([R-Y 91, cor. II.2.4]) then shows

$$\lim_{t\to\infty} P_{X_t}(\tau_\varepsilon < \infty) \quad = \quad Y \geq a > 0 \quad P_x\text{-a.s. for all } x \in E\,.$$

But Y is the indicator of a set, thus $\bigcap_t\{t + \tau_\varepsilon \circ \vartheta_t < \infty\} = \Omega$ P_x-a.s. for all $x \in E$. This implies $\{B_\infty = \infty\} = \{\int_0^\infty 1_A(X_s)ds = \infty\} = \Omega$ P_x-a.s. for all $x \in E$, which is $(*)$ of 1.1. □

We resume the discussion of Harris properties.

1.4 Theorem: The assumption

(H1): $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ

is equivalent to any of the following properties (H2) or (H2$^\alpha$), $0 < \alpha < \infty$:

(H2): $\overline{X} = (X_{\sigma_n})_{n\geq 0}$ is Harris, with $\sigma_n - \sigma_{n-1}$ i.i.d $exp(1)$-waiting times independent of X

(H2$^\alpha$): $\overline{X}^\alpha = (X_{\rho_n})_{n\geq 0}$ is Harris, with $\rho_n - \rho_{n-1}$ i.i.d $exp(\alpha)$-waiting times independent of X

where we put $\sigma_0 = \rho_0 = 0$, and where the invariant measure for $\overline{X}$ or $\overline{X}^\alpha$ is μ.

Proof: We fix $0 < \alpha < \infty$. By proposition 1.3, (H2$^\alpha$) implies (H1); we prove the converse. Lift X to a standard extension $(\Omega', \mathcal{A}', \mathbb{F}' = (\mathcal{F}'_t)_{t\geq 0}, (P'_x)_{x\in E})$ of $(\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, (P_x)_{x\in E})$, with shifts again denoted by $(\vartheta_t)_{t\geq 0}$, on which X is strongly Markov and where $\rho_n - \rho_{n-1}$, $n \geq 1$, are i.i.d $exp(\alpha)$-waiting times independent of X.

(This is done as follows: let Ω'' denote the space of all functions $f : \mathbb{R}_+ \to \mathbb{N}_0$ which are càdlàg, piecewise constant, with jumps only of height $+1$ and $f(0) = 0$, equipped with σ-field and filtration generated by the coordinate projections $\pi_t(\omega'') = \omega''(t)$: $\mathcal{A}'' = \sigma(\pi_t : t \geq 0)$, $\mathbb{F}'' = (\mathcal{F}''_t)_{t\geq 0}$, $\mathcal{F}''_t = \sigma(\pi_s : 0 \leq s \leq t)$. Then $\mathbb{F}''$ is right-continuous. We take P'' the unique law on $(\Omega'', \mathcal{A}'')$ under which the canonical process $(\pi_t)_t$ is a Poisson process with parameter α. Then $\Omega' := \Omega \times \Omega''$, $\mathcal{A}' := \mathcal{A} \otimes \mathcal{A}''$, $\mathcal{F}'_t = \mathcal{F}_t \otimes \mathcal{F}''_t$, $P'_x := P_x \otimes P''$ is the desired extension, we take $X(\omega') := X(\omega)$, $\pi(\omega') := \pi(\omega'')$ if $\omega' = (\omega'', \omega)$, and $(\rho_n)_n$ the sequence of jump times of π.)

On $(\Omega', \mathcal{A}', \mathbb{F}', (P'_x)_{x\in E})$, we define processes

$$N = \left(\sum_{n\geq 1} 1_{\{X_{\rho_n} \in A\}} 1_{[[\rho_n, \infty[[}(t) \right)_{t\geq 0} \quad , \quad \hat{N} = \left(\int_0^t \alpha 1_A(X_s)\, ds \right)_{t\geq 0}$$

where $A \in \mathcal{E}$ is fixed. Then $N - \hat{N}$ is a $(\mathbb{F}', P'_x)$-martingale for every $x \in E$. Using Lepingle ([Le 78]), we know that N_t increases to ∞ P'_x-a.s. on the event $\{\lim_{t\to\infty} \hat{N}_t = \infty\}$. But this event equals $\{\int_0^\infty 1_A(X_s)\, ds = \infty\}$. If $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ, then $\mu(A) > 0$ implies $P'_x\left(\int_0^\infty 1_A(X_s)\, ds = \infty\right) = 1$ for all $x \in E$: so P'_x-a.s. for all $x \in E$, $(X_{\rho_n})_n$ visits the set A infinitely often. $\square$

Convention: From now on we *assume throughout this note* that $X = (X_t)_{t\geq 0}$ is Harris recurrent with invariant measure μ.

1.5 Definition: An *additive functional* of X is a process $A = (A_t)_{t\geq 0}$ with the properties

(i) A is $\mathbb{F}$-adapted, $A_0 \equiv 0$;

(ii) all paths of A are nondecreasing and right-continuous;

(iii) for every $x \in E$ and for all $s, t \geq 0$, we have $A_{t+s} = A_t + A_s \circ \vartheta_t \quad P_x -$ a.s..

See Revuz-Yor ([R-Y 91, p.371, p.78]). Examples of additive functionals of X are

$$A_t = \int_0^t g(X_s) ds$$

for $g \geq 0$ bounded measurable, or counting processes based on the point process of jumps of X

$$\mu^X = \sum_{s>0:|\Delta X|_s>0} \epsilon_{(s,X_{s-},X_s)}$$

where ϵ_a is Dirac measure sitting in a, or (suitable versions of) local time in case where X is a one-dimensional diffusion. For every additive functional A of X, $f(t) := E_\mu(A_t)$ is linear in t, and

$$\nu_A(B) := E_\mu(\int_0^1 1_B(X_s)dA_s) = \frac{1}{t}E_\mu(\int_0^t 1_B(X_s)dA_s)\ ,\ B \in \mathcal{E}$$

defines a measure ν_A on $(E, \mathcal{E})$. The additive functional A is termed *integrable* if

$$\|\nu_A\| := \nu_A(E) = E_\mu(A_1)$$

is finite. As an immediate consequence of this definition, we note

1.6 Remark: a) For $A = id$ (i.e. $A_t = t$, $t \geq 0$), the measure $\nu_{id}(B) = \mu(B)\,,\ B \in \mathcal{E}$, is the invariant measure μ for X.
b) For $A_t = \int_0^t 1_{A'}(X_s)ds$, $A' \in \mathcal{E}$, we have $\nu_A = \mu(\cdot \cap A')$ and thus $\|\nu_A\| = \mu(A')$.

We quote the ratio limit theorem (RLT) for additive functionals of X.

1.7 Ratio Limit Theorem: ([A-D-R 69]) For additive functionals A, B of X, $0 < \|\nu_B\| < \infty$,

(i) $\lim\limits_{t\to\infty} \dfrac{E_x(A_t)}{E_x(B_t)} = \dfrac{\|\nu_A\|}{\|\nu_B\|}$ $\mu-$a.s. (with exceptional set depending on A, B),

(ii) $\lim\limits_{t\to\infty} \dfrac{A_t}{B_t} = \dfrac{\|\nu_A\|}{\|\nu_B\|}$ P_x-a.s. $\forall\, x$.

Write $(R_\lambda)_{\lambda>0}$ for the resolvent of X:

$$(R_\lambda f)\,(x) = E_x\left(\int_0^\infty e^{-\lambda t} f(X_s)\,ds\right)\,, \quad \lambda > 0\,.$$

Then the RLT for additive functionals of X implies a RLT for resolvants of X as $\lambda \to 0$.

1.8 Corollary: For f, g nonnegative, $\mathcal{E}$-measurable, $0 < \mu(f) < \infty$,

$$\lim_{\lambda\to 0} \frac{(R_\lambda g)\,(x)}{(R_\lambda f)\,(x)} \;=\; \frac{\mu(g)}{\mu(f)} \quad \mu\text{-a.s.} \quad \text{(with exceptional set depending on } f \text{ and } g).$$

Proof: It is sufficient to consider f, g with $0 < \mu(f), \mu(g) < \infty$. By partial integration, write

$$(R_\lambda g)(x) = \int_0^\infty \lambda e^{-\lambda t} E_x(A_t^g)\, ds$$

with $A_t^g = \int_0^t g(X_s) ds$. By Harris recurrence, $E_x(A_t^g)$ increases to ∞ as $t \to \infty$, for all $x \in E$, thus $(R_\lambda g)(x)$ increases to ∞ as $\lambda \downarrow 0$. For fixed t_0 arbitrarily large, we have

$$(R_\lambda g)(x) = o(1) + \int_{t_0}^\infty \lambda e^{-\lambda t} E_x(A_t^g)\, ds\,, \quad \lambda \downarrow 0$$

whereas by the RLT for additive functionals, there is some μ-null set $N_{f,g}$ such that $\frac{E_x(A_t^g)}{E_x(A_t^f)}$ converges to $\frac{\mu(g)}{\mu(f)}$ as $t \to \infty$ for all $x \notin N_{f,g}$. Both arguments combined show 1.8. $\square$

In some cases, we have naturally a decomposition of the trajectory of X into i.i.d. excursions away from some recurrent atom - two examples are given below, two others at the end of this section. In some cases, Nummelin's splitting technique ([Num 78], see section 6) allows to introduce recurrent atoms artificially.

1.9.A Definition: We call *atom* for X a set $A \in \mathcal{E}$ such that
i) $\sigma_A := \inf\{t > 0 : X_t \in A\}$ and $\tau_A := \inf\{t > 0 : X_t \notin A\}$ are $I\!F$-stopping times;
ii) for $x \in A$, $\mathcal{L}(X_{\tau_A} | X_0 = x) =: \rho_A$ does not depend on $x \in A$.
An atom A is called *recurrent* if P_x-a.s. for all $x \in A$: $\forall N\ \exists t > N$ with $X_t \in A$.

Examples: a) Consider the one-dimensional Ornstein-Uhlenbeck diffusion $dX_t = -aX_t dt + dW_t$ with $a \geq 0$. The process is Harris (take m the Lebesgue measure in 1.1), $A = \{0\}$ is a recurrent atom, with ρ_A the Dirac measure at 0.
b) Fix some measurable function λ on $I\!R$ taking values in some interval $[a, b]$, $0 < a < b < \infty$, define a transition probability $\pi(\cdot, \cdot)$ on $(I\!R, \mathcal{B}(I\!R))$ by

$$\pi(x, \cdot) := \mathcal{N}(x - x_0, 1) \text{ if } x > x_0\,, \quad \pi(x, \cdot) := \mathcal{N}(0, 1) \text{ if } x \leq x_0$$

with $x_0 \geq 0$. Consider the Markov step process $X = (X_t)_{t \geq 0}$ having exponential holding times with parameter $\lambda(x)$ in states $x \in I\!R$, and successor states for x selected according to $\pi(x, \cdot)$. This process is Harris (sets of positive Lebesgue measure will be visited infinitely often in the sense of 1.1) and admits $A = (-\infty, x_0]$ as a recurrent atom, with ρ_A given by $\mathcal{N}(0, 1)$ conditioned on A^c, cf. 1.9.A ii).

To a recurrent atom, we can associate a sequence of $I\!F$-stopping times $(R_n)_n$ which decompose - by the strong Markov property - the path of X into i.i.d excursions $[\![R_i, R_{i+1}[\![$, $i = 1, 2, ...$, plus an initial segment $[\![0, R_1[\![$.

1.9.B Definition: A *life cycle decomposition* of X associated to a recurrent atom A is a sequence $(R_n)_n$ of $I\!F$-stopping times increasing to ∞ ($R_0 \equiv 0$) such that P_x-a.s. for every $x \in E$:
i) $\forall\, n \geq 1$: $R_n < \infty$ and $R_n = R_{n-1} + R_1 \circ \vartheta_{R_{n-1}}$;
ii) $\forall\, n \geq 1$: $(X_{R_n+t})_{t\geq 0}$ is independent of $\mathcal{F}_{R_n^-}$ with $\mathcal{L}(X_{R_n}) = \rho_A$
where ρ_A is given in 1.9.A (thus a.s. all R_n, $n \geq 1$, are times where the process *leaves* A).

Examples: a) In the Ornstein-Uhlenbeck example a) above, one may take

$$R_1 := \inf\{t > S_0 : X_t = 0\}$$

where S_0 is an independent exponential time. One may take as well

$$R_1 := \inf\{t > S_0 : X_t = 0\} \quad \text{with} \quad S_0 := \inf\{t > 0 : |X_t| \geq 1\}$$

or more generally $S_0 := \inf\{t > 0 : X_t \in B\}$ where $B \in \mathcal{B}(I\!R)$ has positive Lebesgue measure and does not intersect some ε-neighbourhood of 0, $\varepsilon > 0$. There are i.g. many ways to define stopping rules R_1 meeting 1.9.B.
b) Any atom with $\mu(A) > 0$ is a recurrent atom. In this case one may take

$$R_1 := \inf\{t > S_0 : X_t \notin A\}$$

where S_0 is an exponential waiting time spent in the atom A. In particular, this applies to the Markov step process example b) above.

1.10 Proposition: If X has a recurrent atom A, then for every life cycle decomposition $(R_n)_n$ associated to A:
a) the invariant measure μ (unique up to constant multiples) is given by

$$\mu(A') = cst\ E(\int_{R_1}^{R_2} 1_{A'}(X_s)ds)\ ,\ A' \in \mathcal{E}$$

b) X is positive recurrent if and only if $E(R_2 - R_1) < \infty$.

Proof: Consider additive functionals A, B, with $0 < \|\nu_B\| < \infty$. Then by SLLN

$$\lim_{t\to\infty} \frac{A_t}{B_t} = \lim_{n\to\infty} \frac{\frac{A_{R_n}}{n}}{\frac{B_{R_n}}{n}} = \frac{E(A_{R_2} - A_{R_1})}{E(B_{R_2} - B_{R_1})} \quad P_x\text{-a.s.} \quad \forall\, x.$$

The RLT yields

$$\lim_{t\to\infty} \frac{A_t}{B_t} = \frac{\|\nu_A\|}{\|\nu_B\|} \quad P_x\text{-a.s.} \quad \forall\, x$$

which together give

$$\|\nu_A\| = \text{cst } E(A_{R_2} - A_{R_1})$$

up to some constant which does not depend on A. Considering in particular $A_t = \int_0^t 1_{A'}(X_s)ds$, $A' \in \mathcal{E}$, assertion a) follows from 1.6.b); then μ has finite total mass iff $E(R_2 - R_1) < \infty$. □

We end this section with two more examples illustrating definition 1.9.A.

1.11 Example: The process $X = (X_t)_t$ under consideration is of the following type: piecewise on suitable random intervals, the first component X^1 is a Brownian motion; the second component X^2 attributes 'colours' 0 or 1 to the trajectory of X^1; this colour is initially 0, later changes to 1, finally a jump occurs in the first component; this jump time is a renewal time for the process, thus '$I\!R$ coloured 1' will be an atom for X.

a) Prepare on some $(\Omega, \mathcal{F}, I\!F, P)$ a real valued $I\!F$-Brownian motion and a $I\!F$-standard Poisson process N, independent and both starting from 0. Define a transition probability $K(\cdot,\cdot)$ on $E = I\!R \times \{0,1\}$ as follows:

$$K((x,0),\cdot) := \epsilon_x \otimes \left(\frac{1}{2}\epsilon_0 + \frac{1}{2}\epsilon_1\right) \;, \quad K((x,1),\cdot) := \nu \otimes \epsilon_0 \;, \quad x \in I\!R$$

for some fixed probability law ν on $(I\!R, \mathcal{B}(I\!R))$, and ϵ_a the Dirac measure at a. Let $(T_j)_j$ denote the sequence of jump times of N. The process X is constructed as follows: first, put

$$X_s = (B_s, 0)\,, \quad 0 \le s < T_1\,,$$

then successively for $j \ge 1$, select X_{T_j} according to $K(X_{T_j^-},\cdot)$, and put

$$X_s = \left(X^1_{T_j} + (B_s - B_{T_j}), X^2_{T_j}\right) \,, \quad T_j \le s < T_{j+1}\,.$$

The resulting process X is defined i.g. on an extension of the original $(\Omega, \mathcal{F}, I\!F, P)$. On this extension, let $I\!F^X$ denote the filtration generated by X: $\mathcal{F}_t^X = \bigcap_{r>t} \sigma(X_s : 0 \le s \le r)$, $t \ge 0$. Then X is strongly Markov w.r.t. $I\!F^X$, with Polish state space, and is Harris (take m in 1.1 such that its restriction to $I\!R \times \{0\}$ and $I\!R \times \{1\}$ is Lebesgue measure on $I\!R$).

X admits $A = I\!R \times \{1\}$ as recurrent atom with $\rho_A = \nu \otimes \epsilon_0$, and the rule

$$R_1 := \inf\{T_{j+1} : j \ge 1, X_{T_j} \in A\}$$

generates a life cycle decomposition $(R_n)_{n \ge 1}$ for X according to 1.9.A+B.

Note that $(X_{R_n+t})_{t \ge 0}$ is independent of $\mathcal{F}^X_{R_n^-}$, but not of $\mathcal{F}^X_{R_n}$.

b) A more general variant of the example in a) could be formulated using suitable position dependent killing of X at rate $\kappa(\cdot)$ — if X is in position $(x, 0)$ at time t, colour will switch to 1 in a small time interval $(t, t+h]$ with probability $\kappa(x)\, h + o(h)$ — instead of killing at constant rate 1 as above.

1.12 Example: Consider a Markov step process $(X_t)_{t \ge 0}$ with Polish state space $(E, \mathcal{E})$, Harris recurrent, with $exp(\lambda(x))$-distributed holding times in states $x \in E$ (λ is measurable and takes values in $[a, b]$, $0 < a < b < \infty$), and with successor states for x selected according to a transition probability $\pi(\cdot, \cdot)$ on $(E, \mathcal{E})$. In general, X will not have a recurrent atom. Let $(T_j)_j$ denote the sequence of jump times of X. Since λ is bounded and bounded away from 0, also $(X_{T_j})_j$ is Harris with one-step transition probability π (compare with the completely different situation in Proof of 1.2, steps ii) and iii)). Let us assume that $\pi(\cdot, \cdot)$ satisfies Nummelin's minorization condition (M) with $k = 1$, see [Num 78]. Then Nummelin's splitting technique applied to $(X_{T_j})_j$ yields a representation of $(X_t)_{t \ge 0}$ as first component of a 'split' process $(X_t^*)_{t \ge 0}$ with state space $E^* = E \times \{0, 1\}$ such that X^* is again Harris and admits a recurrent atom $A^* \subset E^*$. See section 6.

2 Stable increasing processes and Mittag-Leffler processes

In this section, we collect some main facts about one-sided stable laws, their domains of attraction, stable increasing processes and their process inverse called Mittag-Leffler processes. The main references are Feller ([Fe 71]) and Bingham-Goldie-Teugels ([B-G-T 78]). For regularly varying functions and their properties, we refer always to [B-G-T 78].

2.1 Definition: A mesurable function $\ell : (0,\infty) \to (0,\infty)$ is *slowly varying at* ∞ if

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \ \ \forall \lambda > 0.$$

The class of slowly varying functions is denoted by RV_0. A mesurable function $r : (0,\infty) \to (0,\infty)$ is *regularly varying at* ∞ if it is of form

$$r(x) = \ell(x) \cdot x^{\varrho}, \quad x > 0, \quad \ell \in RV_0, \quad \varrho \in \mathbb{R},$$

where ϱ is termed index of regular variation. The class of functions varying regularly at ∞ with index ϱ is denoted by RV_ϱ, $\varrho \in \mathbb{R}$, and RV is the class of regularly varying functions with arbitrary index.

These notions go back to Karamata, about 1930. Examples of slowly varying functions are $\ell(x) = \log(x)$ and its iterates $\log_m(x)$; with $\ell(\cdot)$ also $\frac{1}{\ell(\cdot)}$ is slowly varying. We mention that for $r \in RV_\varrho$, the convergence $\frac{r(\lambda x)}{r(x)} \to \lambda^\varrho$ as $x \to \infty$ is (at least) uniform in $\lambda \in K$ for arbitray compacts K contained in $(0,\infty)$, see [B-G-T 78, thm. 1.5.2].

2.2 Definition: A probability law F on $(\mathbb{R}, B(\mathbb{R}))$ is called (strictly) *stable* if

$$\mathcal{L}(X_1 + \ldots + X_k) = \mathcal{L}(a_k X) \ , \ k \in \mathbb{N}$$

for $(X_n)_{n\geq 1}$ i.i.d. with $\mathcal{L}(X_1) = F$ and for suitable choice of a norming sequence $(a_k)_k$.

The word 'strictly' will be omitted in the sequel.

2.3 Theorem: ([Fe 71, XIII.6]) For $0 < \alpha < 1$ the function $\varphi_\alpha(\lambda) = e^{-\lambda^\alpha}$ is the Laplace transform of a probability law G_α with the properties

i) G_α is concentrated on $(0,\infty)$;

ii) G_α is stable and $a_n = n^{1/\alpha}$;
iii) $1 - G_\alpha(x) \sim \dfrac{x^{-\alpha}}{\Gamma(1-\alpha)} \quad (x \to \infty)$.
G_α is called the one sided stable law of index α, $0 < \alpha < 1$.

Domains of attraction of G_α, $0 < \alpha < 1$, are characterized as follows.

2.4.A Theorem: ([Fe 71, XIII.6]) Consider a probability law F concentrated on $\mathbb{R}_+$, a norming sequence $(a_n)_n$, and a probability law G on $[0, \infty)$ which is not a Dirac measure.

a) Assume weak convergence of rescaled convolutions:

$$(*) \qquad F^{*n}(a_n x) \to G(x) \quad (n \to \infty)$$

at all continuity points of G. Then there is some $0 < \alpha < 1$ and some $\ell \in RV_0$ such that

$$(**) \qquad 1 - F(x) \sim \frac{x^{-\alpha}\ell(x)}{\Gamma(1-\alpha)} \quad (x \to \infty).$$

b) If the tails of F satisfy $(**)$ for some $0 < \alpha < 1$ and some $\ell \in RV_0$, then $(*)$ holds with $G = G_\alpha$ and $a_n = a(n)$ where $a(\cdot)$ is an asymptotic inverse to

$$t \mapsto \frac{1}{\Gamma(1-\alpha)(1-F(t))} \quad \in RV_\alpha,$$

i.e. $n \cdot \ell(a_n) \sim a_n^\alpha$ as $n \to \infty$.

There is a case $\alpha = 1$, covering the SLLN and more generally 'relative stability' in the terminology of Bingham, Goldie and Teugels [B-G-T 87]. In 2.4.A, limit distributions G concentrated at one point in $(0, \infty)$ were excluded. Write G_1 for the Dirac measure sitting at 1; obviously G_1 meets 2.2 with $a_n = n$. Domains of attraction of G_1 are as follows.

2.4.B Theorem: ([B-G-T 87, 8.8]) Consider a probability law F concentrated on $\mathbb{R}_+$, with

$$r(t) := \int_0^t (1 - F(u))du \uparrow \int_0^\infty xF(dx) \leq \infty,$$

and a norming sequence $(a_n)_n$. Then weak convergence

$$(*) \qquad F^{*n}(a_n \cdot) \to G_1(\cdot) \quad (n \to \infty)$$

is equivalent to

$$(**) \qquad\qquad r \in RV_0 .$$

Under $(**)$, $(*)$ holds with $a_n = a(n)$ where $a(\cdot)$ is an asymptotic inverse to

$$t \mapsto \frac{t}{r(t)} \;\; \in RV_1 ,$$

i.e. $n \cdot r(a_n) \sim a_n$ as $n \to \infty$.

As a consequence of 2.4.A and 2.4.B, there are no other stable laws concentrated on $(0, \infty)$ except G_α, $0 < \alpha \leq 1$ (up to scaling by a constant).

2.5 Definition: A *stable increasing process of index* α, $0 < \alpha < 1$, is a process X with the following properties:
i) all paths of X are càdlàg and nondecreasing, and $X_0 \equiv 0$;
ii) X is a PIIS (independent and stationary increments) with $E(e^{-\lambda X_t}) = e^{-t\lambda^\alpha}$, $\lambda \geq 0$, $t \geq 0$.

We write S^α for the stable increasing process of index α, $0 < \alpha < 1$. Note that i)+ii) of 2.5 define a unique probability law on the Skorohod space $D(\mathbb{R}_+, \mathbb{R})$ with Borel-σ-field $\mathcal{D}$ and canonical filtration $\mathbb{G}$. Defined on a suitable stochastic basis (e.g. on $(D, \mathcal{D}, \mathbb{G})$), S^α is necessarily a Feller process and thus strongly Markov. In 2.7 below we will give a construction of S^α. Note that by definition, almost all paths $t \to S_t^\alpha$ increase to ∞ as $t \to \infty$ and do not have flats.

2.6 Definition: For $0 < \alpha < 1$, the process inverse of S^α

$$W_t^\alpha := \inf\{s > 0 : \; S_s^\alpha > t\}, \quad t \geq 0$$

is a process W^α with $W^\alpha \equiv 0$, nondecreasing, having almost all paths continuous and increasing to ∞ as $t \to \infty$. W^α is called *Mittag-Leffler process of index* α.

In the sequel, we shall always use versions of S^α where all paths $t \to S_t^\alpha$ increase to ∞ as $t \to \infty$ and do not have flats, and versions of W^α having all paths continuous and increasing to ∞ as $t \to \infty$. We shall also need a definition of S^α and W^α for $\alpha = 1$: we take $S^1 = W^1 = id$, the deterministic process.

In order to prepare for the essential point in the limit theorems of section 3 (the independence of Brownian motion and stable process involved in the limit, see 4.12 and 4.21 below), we discuss in detail the structure of the stable increasing process S^α.

2.7 Remark: (see Ito-McKean [I-MK 65, p.32]) Let $\mu(dt, dx)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}_+$ denote *Poisson random measure* (PRM) on some $(\Omega, \mathcal{A}, P)$ with intensity $\nu(dt, dx) = dt\, m(dx)$,

$$m(dx) := \frac{c\alpha 1_{(0,\infty)}(x)}{\Gamma(1-\alpha)x^{\alpha+1}} dx.$$

By definition (e.g. [I-W 89, I.8]), Poisson random measure $\mu(dt, dx)$ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}_+$, characterized by the properties:

i) for $F \in B(\mathbb{R}_+)\otimes B(\mathbb{R}_+)$: the r.v. $\mu(F)$ has Poisson law with parameter $\nu(F)$;

ii) for pairwise disjoint sets $F_1 \ldots, F_n \in B(\mathbb{R}_+)\otimes B(\mathbb{R}_+)$, $\mu(F_1), \ldots, \mu(F_n)$ are independent.

Define

$$S_t := \int_0^t \int_{(0,\infty)} x\mu(ds, dx)\ ,\ t \geq 0.$$

Up to the scaling factor c, this gives a version of the stable increasing process S^α with

$$E(e^{-\lambda S_t}) = e^{-ct\lambda^\alpha}\ ,\ \lambda \geq 0\ ,\ t \geq 0.$$

This is seen as follows: Approximate the process $X_t := e^{-sS_t}$ by

$$X_t^n := e^{-s \sum_{k=1}^{n\cdot 2^n} \int_0^t \int_{(\frac{k-1}{2^n}, \frac{k}{2^n}]} \frac{k-1}{2^n} \mu(ds,dx)}.$$

Then $X_t^n \downarrow X_t,\ n \to \infty$. By independence of $\mu\left((0,t] \times (\frac{k-1}{2^n}, \frac{k}{2^n}]\right)$ for $k = 1, ..., n2^n$, we see

$$\begin{aligned}
E(e^{-sS_t}) &= \lim_n E(X_t^n) \\
&= \lim_n \prod_{k=1}^{n\cdot 2^n} E\left[e^{-s\frac{k-1}{2^n}\mu\left((0,t]\times(\frac{k-1}{2^n}, \frac{k}{2^n}]\right)}\right] \\
&= \lim_n e^{t\sum_{k=1}^{n\cdot 2^n} m\left((\frac{k-1}{2^n}, \frac{k}{2^n}]\right)\cdot\left(e^{-s\frac{k-1}{2^n}}-1\right)} \\
&\overset{n\to\infty}{\longrightarrow} e^{-t\int_{(0,\infty)}(1-e^{-sx})m(dx)} = e^{-cts^\alpha}
\end{aligned}$$

where by partial integration

$$\int_{(0,\infty)} (1-e^{-sx})m(dx) = \int (1-e^{-sx})\alpha x^{-\alpha-1} dx \cdot \frac{c}{\Gamma(1-\alpha)} = c \cdot s^\alpha\,.$$

Independence and stationarity of the increments of S follow directly from the corresponding properties of PRM. We prove that $S_t < \infty$ for all $t \geq 0$, a.s.. Contribution of small jumps $\int_0^t \int_{(0,1]} x\mu(dt, dx)$ is summable a.s. since $0 < \alpha < 1$ implies

$$\int_0^t \int_{(0,1]} x\nu(dt, dx) = t \cdot \int_0^1 \alpha x^{-\alpha} dx \cdot \frac{c}{\Gamma(1-\alpha)} < \infty \, .$$

There are only finitely many big jumps over finite time intervals: $\mu\left((0,t] \times [1,\infty)\right) < \infty$ a.s. since

$$\nu\left((0,t] \times [1,\infty)\right) = t \cdot \int_1^\infty \alpha x^{-\alpha-1} dx \cdot \frac{c}{\Gamma(1-\alpha)} < \infty.$$

Thus $N^c := \{\omega \in \Omega : S_n(\omega) < \infty \; \forall \, n\}$ is a set of full measure, and paths of S are right-continuous and nondecreasing on N^c. Moreover $P(S_{t+h} = S_t) = P(\mu((t, t+h] \times (0,\infty)) = 0) = 0$, so paths of S a.s. do not have flats. S being a PIIS, the paths of S a.s. increase to ∞. Modifying the paths of S on a set of measure 0, we get all path properties required in 2.5. □

2.8 Remark: ([Fe 71, p.453]) For $0 < \alpha < 1$, W_t^α has Laplace transform

$$\psi_t(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(1+n\alpha)} \, t^{n\alpha}$$

and thus admits finite moments of arbitrary order $n \geq 1$

$$m_n(t) = \frac{n!}{\Gamma(1+n\alpha)} \, t^{n\alpha}.$$

Note also that $P(W_t^\alpha \leq x) = P(S_x^\alpha > t) = 1 - P(S_x^\alpha \leq t) = 1 - F\left(\frac{t}{x^{1/\alpha}}\right)$ where F is the distribution function of of S_1^α. Using the last expression one has

$$\mathcal{L}\left(W_1^\alpha\right) = \mathcal{L}\left((S_1^\alpha)^{-\alpha}\right);$$

this representation of the Mittag-Leffler law appears e.g. in Khasminskii [Has 80, Ch. IV.11].

2.9 Remark: For $\alpha = \frac{1}{2}$, stable increasing process $S^{1/2}$ and Mittag-Leffler process $W^{1/2}$ occur in well known connection with one-dimensional Brownian motion. First, by [R-Y 91, p. 76, p.102], the process of level crossing times of Brownian motion is equal in law to $2S^{1/2}$. The process inverse to $2S^{1/2}$ is $\frac{1}{\sqrt{2}}W^{1/2}$. Thus $\frac{1}{\sqrt{2}}W^{1/2}$ is equal in law to the maximum process of Brownian motion, or to local time of Brownian motion in 0 with choice of norming constant such that local time is an occupation time density: see [R-Y 91, p. 223, p. 207-209].

3 The main theorem

In this section, we state the main theorem on weak convergence of *integrable* additive functionals and local martigales whose predictable quadratic variation is an *integrable* additive functional of the Harris process X (the integrability assumption is crucial and in null recurrent cases indeed a restrictive condition).

The theorem has a long history. A key argument for one direction of the proof is the classical Darling-Kac theorem ([D-K 57]) on necessary conditions for convergence in law of (one-dimensional marginals of) additive functionals of X. In the other direction, the proof relies on a paper by Greenwood and Resnick ([R-Gr 79]) who study weak convergence of bivariate random walks where one component is attracted to a Gaussian and the other to a stable limit process, with strong reference to P. Lévy. In a highly interesting but unfortunately never published paper, Touati ([Tou 88]) gave the theorem in very general form (general state space, Nummelin splitting applied to continuous time, and avoiding restrictive Darling-Kac conditions; a gap left was the case of relative stability which was ignored there, and some lines of argument – namely for Nummelin splitting in continuous time – which seem problematic). Touati's arguments relied heavily on semimartingale theory and weak convergence of processes in the sense of the book Jacod and Shiryaev ([J-Sh 87]). For related work, see [Bin 71], [B-G-T 87, ch. 8.11]; see Khasminskii [Has 80, ch. IV.10-11] for one-dimensional diffusions; a note on Markov step processes with countable state space (where things are much simpler) was [Hö 88].

One application of this theorem is in a context of local asymptotic statistics where convergence of the score function martingale is essential for convergence of statistical experiments (weak convergence of filtered statistical experiments to Gaussian or Mixed Gaussian limit models), or simply when convergence of e.g. maximum likelihood estimators is considered: see [Lu 92], [Lu 94], [Lu 95] for general semimartingale models, see [H-J-L 90], [Hö 90 a, b], [Hö 93 a, b] for Markov step processes, [Lö 97], [Lö 99 a-c] for systems of diffusing particles with branching and killing; for ergodic diffusions, see the forthcoming book of Kutoyants [Ku 01]; there seem to be relatively few cases of models for null recurrent diffusions where the above integrability condition indeed holds, see [H-K 01] for an example. It is interesting to note that 'martingale convergence theorems' typically can not deal with general nullrecurrent cases (the reason is that martingale convergence theorems need convergence in probability of angle bracketts; in most null recurrent cases there is only convergence in law).

The section is organized as follows. In subsection 3.1, we state the theorems in case where the Harris process X has life cycles. Subsection 3.2 is devoted to examples. Subsection 3.3 states the theorems for general Harris processes where no life cycles exist. All proofs will be postponed to sections 4, 5, 7.

3.1 Processes with life cycles

If X has life cycles, the main result is theorem 3.1 together with its corollaries 3.2 and 3.3. The 'sufficient part' of the assertion (regular variation of tails of life cycle length distribution implies weak convergence of normed and linearly time-scaled martingales or additive functionals to suitable continuous limit processes) will be proved in section 4 below (see 4.12 and 4.22). The 'necessary part' (there are no other possibilities for weak convergence to continuous limit processes, under linear time-scaling and suitable norming) will be proved in section 5 (see 5.27).

In this subsection, we assume the following for the process X:
(H1): $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ;
(H3): X has a recurrent atom $A \in \mathcal{E}$ and a life cycle decomposition $(R_n)_{n\geq 1}$, see 1.9.A + 1.9.B;
(H4): There is some function f, bounded, nonnegative, $\mathcal{E}$-measurable, $0 < \mu(f) < \infty$, such that

$$y \quad \to \quad E_y\left(\int_0^{R_1} f(X_s)\,ds\right) \quad \text{is bounded on } E \ ;$$

below, functions f with this property will be called *weakly special for* X *and* R_1.

We will describe at the end of this subsection (see proposition 3.4 below) a large class of life cycle decompositions $(R_n)_n$ which satisfy (H4).

Let $\mathcal{M}^{2,\mathrm{loc}}(P_x, I\!\!F)$ denote the class of locally square integrable local $(P_x, I\!\!F)$-martingales, càdlàg and with $M_0 = 0$. Here P_x is some probability measure on $(\Omega, \mathcal{A}, I\!\!F)$ as in the beginning of section 1. For $M \in \mathcal{M}^{2,\mathrm{loc}}(P_x, I\!\!F)$, the process $\langle M \rangle$ is (a version of) the predictable quadratic variation of M (or angle brackett) relative to P_x and $I\!F$, and $[M]$ is the quadratic variation (or square brackett) of M. We assume that M meets the following assumptions:

(H5^A): M has the property

$$\forall\, y\,, \forall\, s, t: \quad M_{t+s} - M_t = M_s \circ \vartheta_t \quad P_y\text{-a.s.}\,;$$

the processes $\langle M \rangle$ and $[M]$ are additive functionals of X, and $E_\mu\left(\langle M \rangle_1\right) < \infty$.

(H5^B): For the life cycle decomposition $(R_n)_n$ of (H3), M satisfies either $(*)$:

$(*)$ $\qquad M_{R_n}$ is measurable with respect to $\mathcal{F}_{R_n^-}$, for all $n \geq 1$

or the following $(**)$:

$(**)$ $\qquad R_{n+1} - R_n$ and $M - M^{R_n}$ are independent of $\mathcal{F}_{R_n}$, for all $n \geq 1$.

Assumption (H5^B) guarantees that the martingales under consideration accumulate independent and square integrable increments over life cycles of X. This is not obvious: for the Harris process of example 1.11 a), examples of martingales meeting or violating $(*)$ or $(**)$ of (H5^B) will appear in 4.27 below.

3.1 Theorem: For suitable choice of a norming function $v(\cdot) \uparrow \infty$, consider a rescaled sequence

$$M^n := \left(\frac{1}{\sqrt{v(n)}} M_{tn} \right)_{t \geq 0} ,$$

satisfying the Lindeberg condition

$$\frac{1}{v(n)} \int_0^{tn} \int |x|^2 \, 1_{\{|x| > \varepsilon \sqrt{v(n)}\}} \nu(ds, dx) \;\longrightarrow\; 0 \quad \text{in } P_x\text{-probability, for all } t, \text{ all } \varepsilon > 0$$

where $\nu(ds, dx)$ is the compensator of the point process of jumps of M under P_x.
a) If there is some limit process $W = (W_t)_{t \geq 0}$, with $W_0 \equiv 0$ and $\mathcal{L}(W_1)$ not concentrated at 0 such that

$$M^n \xrightarrow{\mathcal{L}} W$$

(weak convergence in $D(I\!R_+, I\!R)$, under P_x, as $n \to \infty$), then only the following cases can arise: either $W = J^{1/2}\, B$ with standard Brownian motion B, and with $J \in (0, \infty)$ a constant, or $W = J^{1/2}\, B \circ W^\alpha$ for some $0 < \alpha < 1$, where W^α is a Mittag Leffler process independent of B, acting as time change for the Brownian motion: $B \circ W^\alpha = (B(W_t^\alpha))_{t \geq 0}$.

b) One has

$$M^n \xrightarrow{\mathcal{L}} J^{1/2}\, B \quad \Longleftrightarrow \quad r \in RV_0$$

where r is the function

$$r(t) = \int_0^t P(R_2 - R_1 > x) dx.$$

In this case, norming function v – up to asymptotic equivalence – and limiting constant J are given by

$$v(t) \;\sim\; t/r(t)\,, \quad t \to \infty\,, \quad J = E(< M >_{R_2} - < M >_{R_1})\,.$$

c) For $0 < \alpha < 1$, one has

$$M^n \xrightarrow{\mathcal{L}} J^{1/2}\, B \circ W^\alpha \quad \Longleftrightarrow \quad t \to P(R_2 - R_1 > t) \;\in RV_{-\alpha}\,;$$

in this case, norming function and limiting constant are

$$v(t) \;\sim\; (\Gamma(1-\alpha) P(R_2 - R_1 > t))^{-1}\,, \quad t \to \infty\,, \quad J = E(< M >_{R_2} - < M >_{R_1})\,.$$

Remark : a) If X is ergodic, we have $r(\infty) = E(R_2 - R_1) < \infty$ and thus $\sqrt{n}$-norming for martingales $M \in \mathcal{M}^2_{\text{loc}}$:

$$v(n) \;\sim\; \frac{n}{E(R_2 - R_1)}\,, \quad n \to \infty\,.$$

b) In the ergodic case, the martingale limit theorem (see [J-Sh 87, VIII.3.22]) applies and the assertion of theorem 3.1 could be derived from it. The same is true in the limiting case of 'relative stability' (null recurrence with index $\alpha = 1$). In null recurrent cases with index $0 < \alpha < 1$ however, we do not have convergence in probability of angle bracketts of martingales, but only convergence in law: so a basic assumption needed in martingale convergence theorems fails.

3.2 Corollary: In parts b) and c) of theorem 3.1 we also have the stronger assertion (recall the convention $W^1 = id$)

$$(M^n\,, < M^n >) \xrightarrow{\mathcal{L}} \left(J^{1/2}\, B \circ W^\alpha\,,\, J\, W^\alpha\right)$$

(weak convergence in $D(I\!R_+, I\!R \times I\!R)$, under P_x, as $n \to \infty$).

We mention that the proof of the 'sufficient part' in the above assertions (regular variation at ∞ of tails of life-cycle length distributions implies weak convergence of rescaled martingales to Brownian motion or to Brownian motion time-changed by an independent Mittag-Leffler process) does not need all assumptions made above. The Lindeberg condition comes in to prove that *arbitrary* weak limits of sequences of rescaled martingales are again martingales, and that convergence of martingales implies weak convergence of their brackett processes. Condition (H4), introduced by Touati [Tou 88], is needed to prove that regular variation of tails of life-cycle length distributions is necessary for weak convergence: it replaces the original Darling-Kac condition which is rather intractable (except in simple cases such as countable state space). The following corollary 3.3 reduces to merely notational changes in the proofs leading to 3.1 and 3.2.

3.3 Corollary: 3.1 and 3.2 remain true for d-dimensional $M = (M^i)_{1\leq i\leq d} \in \mathcal{M}^2_{\mathrm{loc}}(P_x, I\!\!F)$ provided

$$E(< M^j >_{R_2} - < M^j >_{R_1}) < \infty \ , \ 1 \leq j \leq d :$$

it is sufficient to replace B by a d-dimensional standard Brownian motion and to take

$$J = \left(J^{(i,j)}\right)_{i,j=1,\ldots,d} = \left(E(< M^i, M^j >_{R_2} - < M^i, M^j >_{R_1})\right)_{i,j=1,\ldots,d} .$$

At the end of this subsection, we discuss a large class of life cycle decompositions which satisfies assumption (H4). For $\kappa(\cdot)$ $\mathcal{E}$-measurable, $[0,1]$-valued, $\mu(\kappa) > 0$, write $\widehat{T}_\kappa$ for the stopping time corresponding to position-dependent killing of X at rate κ: this means that conditionally on the event that $\widehat{T}_\kappa$ has not occurred up to time t, it will occur in a following small time interval $(t, t{+}h]$ will probability $\kappa(X_t)\, h + o(h)$, $h \downarrow 0$. If κ is of form 1_B, $B \in \mathcal{E}$, $\mu(B) > 0$, we write for short $\widehat{T}_B$ and speak of killing of X in B at rate 1; if $B = E$, $\widehat{T}_E$ is simply an exponential waiting time. In general, killing times are stopping times on an extension of the original $(\Omega, \mathcal{A}, I\!\!F)$, but this will not appear in our notations. For sets $B \in \mathcal{E}$, a first entry time to B is denoted by T_B; obviously one has $T_B \leq \widehat{T}_B$. The following proposition will be proved in section 5 (see 5.28).

3.4 Proposition: A sufficient condition for (H4) is as follows: the life cycle decomposition in

(H3) is defined from a stopping time R_1 of form

$$(+) \qquad R_1 \;\leq\; S_0 + \max_{1 \leq i \leq l} T_{B_i} \circ \vartheta_{S_0} \,, \quad S_0 \;\leq\; \max_{1 \leq j \leq m} \widehat{T}_{\kappa_j}$$

where B_i are sets in $\mathcal{E}$ with $\mu(B_i) > 0$, and $\kappa_j(\cdot)$ are $\mathcal{E}$-measurable, $[0,1]$-valued, with $\mu(\kappa_j) > 0$.

Examples: We continue the examples discussed in section 1 after the definitions 1.9.A + B.

a) For the one-dimensional Ornstein-Uhlenbeck diffusion $dX_t = -aX_t dt + dW_t$ with $a \geq 0$, $A = \{0\}$ is a recurrent atom; then (H4) holds for the three choices of life cycle decompositions specified there. We show this in case

$$R_1 = \inf\{t > S_0 : X_t = 0\} \,, \quad S_0 = \inf\{t > 0 : X_t \in B\}$$

where $B \in \mathcal{B}(I\!R)$ has positive Lebesgue measure (thus $\mu(B) > 0$, the invariant measure being $\mu = \mathcal{N}\left(0, \frac{1}{2a}\right)$) and does not intersect some ε-neighbourhood of 0. R_1 has form (+) in 3.4 since

$$S_0 \leq \max\{\widehat{T}_{B^+}, \widehat{T}_{B^-}\} \,, \quad R_1 \leq S_0 + \max\{T_{B^+}, T_{B^-}\} \circ \vartheta_{S_0}$$

in case where both sets $B^+ := B \cap (0,\infty)$, $B^- := B \cap (-\infty, 0)$ have positive μ-measure; if B coincides with B^+, this simplifies to $S_0 \leq \widehat{T}_{B^+}$ and $R_1 \leq S_0 + T_{B^-} \circ \vartheta_{S_0}$.

b) If the Harris process X meeting (H3) has an atom A of positive mass $\mu(A) > 0$, life cycles

$$R_1 = \inf\{t > S_0 : X_t \in A^c\} \,, \quad S_0 = \widehat{T}_A \,.$$

(first entry times to A^c after an independent exponential time spent in A) satisfy (H4).

3.2 Examples

For Harris processes with recurrent atom and life cycle decomposition meeting (H4), the theorems in subsection 3.1 require quite complete knowledge on regular variation of tails of life cycle length distributions in the null recurrent case, and on integrability with respect to invariant measure. In this subsection, we illustrate the results of subsection 3.1 by some examples. For one-dimensional diffusions, the necessary results on regular variation of tails of tails of life cycle length distributions have been proved by Khasminskii ([Has 80], [Kh 00], [Kh 01]). We give the details in examples 3.5 and 3.10 below; example 3.9 considers the 'classical' special case of one-dimensional Brownian motion. For birth and death processes (see Karlin and McGregor [K-MG 61]) and for branching processes with immigration (see Zubkov [Zu 72] and Pakes [Pa 75]), regular variation of tails of life cycle length distributions is available under conditions on the birth-, death-, or branching rates in large populations, in which case also asymptotic behaviour of invariant measure is known. More sophisticated examples can be treated on this background, e.g. for finite systems of diffusing particles with branching and immigration where the void configuration is an atom for the process; under suitable conditions, the particle process is Harris and has the void configuration as recurrent atom of positive mass under the invariant measure; see [Hö-Lö 99 a,b], [Lö 99 a,b,c] and the references quoted there.

3.5 Example: We consider one-dimensional diffusions.

a) A one-dimensional diffusion $dX_t = \sigma(X_t)dB_t$ with σ continuous and strictly positive (thus nonexploding in finite time, see [K-S 91, p. 332]) is Harris recurrent with invariant measure $\frac{2}{\sigma^2(x)}\,dx$ (write X as time-changed Brownian motion and use [Le 78]). Assuming in addition that σ is locally Lipschitz and satisfies a global linear growth condition, Khasminskii [Has 80, sections IV.10-11] gives a sufficient condition for regular variation of tails of life cycle length distributions with index $-\alpha,\ 0<\alpha<1$. He uses life cycle decompositions $(R_n)_n$ defined by

$$(3.5') \qquad R_n = \inf\{t > S_n : X_t = 0\}, \quad S_n = \inf\{t > R_{n-1} : X_t = 1\}, \quad n \geq 1, \quad R_0 = 0$$

(which satify (H4), see 3.4 above) and calculates ([Has 80, lemma 10.5])

$$(3.5'') \qquad E\left(\int_{R_1}^{R_2} f(X_s)ds\right) = \mu(f), \quad \mu(dx) = \frac{2}{\sigma^2(x)}dx\,,$$

f nonnegative, measurable, in $L^1(\mu)$. Khasminskii's condition is

$$(3.6) \qquad \frac{2}{\sigma^2(x)} \sim A^+x^\beta,\ x\to+\infty, \quad \frac{2}{\sigma^2(x)} \sim A^-|x|^\beta,\ x\to-\infty$$

with $\beta := -2+\frac{1}{\alpha} > -1$ and nonnegative constants A^+, A^- meeting $A^+ + A^- > 0$ (here $A^\pm = 0$ is written for $\frac{2}{\sigma^2(x)} = o(|x|^\beta)$ as $x\to\pm\infty$); he shows that (3.6) implies

$$(3.6') \qquad P(R_2-R_1>t) \quad \sim \quad \frac{\alpha^{2\alpha}\left((A^+)^\alpha + (A^-)^\alpha\right)}{\Gamma(1+\alpha)}\, t^{-\alpha}, \quad t\to\infty\,.$$

This is proved in [Has 80, theorem 11.2, corollary, remark 3, theorem 11.3], or in [Kh 00, theorem 2.2] with a different proof, see also [Kh 01, theorem 1.1]. So the result on convergence in law of integrable additive functionals of X

$$P(R_2 - R_1 > t) \sim c\,t^{-\alpha},\ t \to \infty \quad \Longrightarrow \quad \frac{1}{(\Gamma(1-\alpha)P(R_2-R_1>t))^{-1}} \int_0^t f(X_s)ds \to \mu(f) W_1^\alpha$$

([Has 80, theorem 11.1], [Kh 01, theorem 1.1]) is – via RLT – a special case of theorem 3.1 and corollary 3.2 above.

b) A one-dimensional diffusion $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ is Harris recurrent with invariant measure equivalent to Lebesgue measure if the function S

$$S(x) := \int_0^x s(y)dy\ ,\ s(y) := \exp\left(- \int_0^y \frac{2b}{\sigma^2}(v)dv \right) \tag{3.7}$$

is a space transformation on $I\!R$, i.e.

$$\lim_{x \to -\infty} S(x) = -\infty\ ,\ S(0) = 0\ ,\ \lim_{x \to +\infty} S(x) = +\infty$$

(see [Has 80, example 2 in section III.8]). In this case, the process $\widetilde{X} := (S(X_t))_{t \geq 0}$ is a diffusion without drift, with same passage times to 0 as X, and with diffusion coefficient

$$\widetilde{\sigma} = (s \cdot \sigma) \circ S^{-1} \tag{3.8}$$

where S^{-1} is the function inverse of S on $I\!R$; the invariant measure of X is given by

$$\mu(dx) = \frac{2}{\sigma^2(x)} \exp\left(\int_0^x \frac{2b}{\sigma^2}(v)dv \right) dx, \quad x \in I\!R. \tag{3.8'}$$

□

3.9 Example : We consider 'classical' results in case of one-dimensional Brownian motion.
a) In the special case $\sigma \equiv 1$ of 3.6.a), X is Brownian motion; for the life cycles as there one has

$$E(e^{-\lambda(R_2-R_1)}) = e^{-2\cdot\sqrt{2\lambda}}, \quad \lambda \geq 0$$

([R-Y 91, p. 67]) and thus $\mathcal{L}(R_2 - R_1) = \mathcal{L}(8\,S_1^{1/2})$. As a consequence (cf. 2.3),

$$P(R_2 - R_1 > t) \ \sim\ P(S_1^{1/2} > \frac{t}{8}) \ \sim\ 2\sqrt{\frac{2}{\pi}}\, t^{-1/2}\,, \quad t \to \infty\,.$$

By theorem 3.1 and corollary 3.2, for $f \in L^1(\mu)$, we have weak convergence in $D(I\!R_+, I\!R)$ as $n \to \infty$

$$\frac{1}{\sqrt{n}} \int_0^{(\cdot n)} f(X_s)ds \quad \to \quad \int_{I\!R} f(x)dx\, \frac{1}{\sqrt{2}} W^{1/2}\,, \quad t \to \infty.$$

Note that $\frac{1}{\sqrt{2}} W^{1/2}$ is the process inverse to $2\,S^{1/2}$. Since $2\,S^{1/2}$ is equal in law to the process of level crossing times of Brownian motion B ([R-Y 91, p. 67, p. 102]), $\frac{1}{\sqrt{2}} W^{1/2}$ is equal in law to the maximum process $B^* := (\max_{0\le s\le t} B_s)_{t\ge 0}$. For local time of Brownian motion defined as occupation time density ([R-Y 91, p. 207-209]), the process $\frac{1}{\sqrt{2}} W^{1/2}$ is thus equal in law to local time at 0 of Brownian motion ([R-Y 91, p. 223]). In this form, weak convergence of additive functionals of Brownian motion has been proved by Papanicolaou, Strook and Varadhan ([P-S-V 77]), reported by Hu and Yor in their survey [H-Y 98, theorem A.1].

b) [P-S-V 77] also prove that for f in $L^1(\mu)$ having compact support and $\mu(f) = 0$

$$\frac{1}{n^{1/4}} \int_0^{(\cdot n)} f(X_s) ds \quad \to \quad C^{1/2}\, B \circ (\frac{1}{\sqrt{2}} W^{1/2})$$

(weak convergence in $D(I\!R_+, I\!R)$, as $n \to \infty$) with

$$C := 4 \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{x} f(y) dy \right)^2 dx\,.$$

Applying the Ito formula to the semimartingale $2F(X)$, $F(x) := \int_{-\infty}^x dy \int_{-\infty}^y dz f(z)$ being bounded on $I\!R$, this result is again contained in theorem 3.1 above.

3.10 Example : We consider a typical family of null recurrent diffusions with drift.

a) With notations of 3.5, consider a process X solution of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with b, σ continuous and σ strictly positive. Assume that for a family of parameters ρ, γ ranging over the domain $\rho < 1$, $-1 + 2\rho < \gamma < 1$, drift and diffusion coefficient have representations

$$(3.11) \qquad \sigma(x) \sim cst_{\pm}\, |x|^{\rho}, \quad x \to \pm\infty; \quad b(x) = \frac{\sigma^2(x)}{2} \left(\gamma \frac{1}{x} + \delta(x) \right), \quad |x| > 1$$

where $\delta(\cdot)$ is some function with $\int_{|x|>1} |\delta(x)| dx < \infty$ (which may also depend on ρ and γ). In (3.11) and below, all occuring constants 'cst' – varying from line to line – can be calculated for given b and σ using the methods of example 3.5 a)+b); see [H-K 01] for an application.

Since $\gamma < 1$, S of (3.7) is a space transformation, and X is Harris. The invariant measure μ of X normed as in (3.8') behaves as

$$(3.12) \qquad \mu(dx) \sim cst_{\pm}\, |x|^{\gamma - 2\rho}\, dx, \quad x \to \pm\infty.$$

Since $\gamma - 2\rho > -1$, it has infinite total mass on $I\!R$, so X is recurrent null. Calculating $\widetilde{\sigma}$ of (3.8)

$$\widetilde{\sigma}(x) \sim cst_{\pm}\, |x|^{\frac{\rho-\gamma}{1-\gamma}}, \quad x \to \pm\infty$$

the invariant measure $\widetilde{\mu}$ of $\widetilde{X} = S(X)$ has density

$$(3.13) \qquad \frac{2}{\widetilde{\sigma}^2(x)} \sim cst_{\pm}\, |x|^{-2+\frac{1}{\alpha}}, \quad x \to \pm\infty, \quad \alpha := \frac{1-\gamma}{2(1-\rho)}$$

where $\alpha = \alpha(\rho, \gamma)$ ranges over the full interval $(0,1)$ since $\rho < 1$, $-1 + 2\rho < \gamma < 1$.

b) Define life cycles for X by

$$R_n = \inf\{t > S_n : X_t = 0\}, \quad S_n = \inf\{t > R_{n-1} : X_t = S^{-1}(1)\}, \quad n \geq 1, \quad R_0 = 0$$

where S^{-1} is the function inverse of S. By (3.5") applied to $\widetilde{X}$ and by (3.7)-(3.8'), we see that

$$E\left(\int_{R_1}^{R_2} f(X_s)ds\right) = E\left(\int_{R_1}^{R_2} (f \circ S^{-1})(\widetilde{X}_s)ds\right) = \widetilde{\mu}(f \circ S^{-1}) = \mu(f)$$

for $f \in L^1(\mu)$; moreover, (3.13) is condition (3.6) relative to $\widetilde{X} = S(X)$, and so we can calculate as in (3.6') the factor $C(\alpha)$ such that

$$(3.14) \qquad P(R_2 - R_1 > t) \sim C(\alpha)\, t^{-\alpha}, \quad t \to \infty.$$

c) Thus for functions $h \in L^2(\mu)$, theorem 3.1 and corollary 3.2 yield weak convergence of

$$\left(\frac{1}{n^{\alpha/2}} \int_0^{tn} h(X_s)dB_s,\, \frac{1}{n^{\alpha}} \int_0^{tn} h^2(X_s)ds\right)_{t \geq 0}$$

(B can be recovered from the observed X) as $n \to \infty$ to

$$\left(K^{1/2}\, B(W^{\alpha}),\ K\, W^{\alpha}\right), \quad K = K(h, \alpha) = \frac{\mu(h^2)}{C(\alpha)\, \Gamma(1-\alpha)}$$

with α of (3.13) and $C(\alpha)$ of (3.14). Note that the condition $h \in L^2(\mu)$ – depending on ρ and γ via (3.12) – is a very strong condition if ρ and γ range over the domain $\rho < 1$, $-1 + 2\rho < \gamma < 1$: essentially, we are reduced to consider $h \in \mathcal{C}_{\mathcal{K}}$, $\mathcal{C}_{\mathcal{K}}$ the class of continuous functions with compact support.

d) In analogy to 3.9 b), we consider also the case of integrable additive functionals with $\mu(f) = 0$, $f \in \mathcal{C}_{\mathcal{K}}$. With s of (3.7), the function

$$F(x) = \int_{\infty}^{x} s(y)\, \mu(1_{(-\infty, y]} f)\, dy$$

is bounded on $\mathbb{R}$ and solves

$$\mathcal{A}F = f\,, \quad \mathcal{A}F := bF' + \frac{1}{2}\sigma^2 F''\,.$$

From Ito formula for $F(X)$ together with the result of c) applied to the martingale part of $F(X)$, we get weak convergence as $n \to \infty$

$$\left(\frac{1}{n^{\alpha/2}} \int_0^{tn} f(X_s) ds \right)_{t \geq 0} \quad \to \quad \widetilde{K}^{1/2} \, B(W^\alpha) \, , \qquad \widetilde{K} := \frac{\mu\left((F'\sigma)^2 \right)}{C(\alpha)\, \Gamma(1-\alpha)}$$

in case $\mu(f) = 0$. $\square$

3.3 General Harris processes

Many interesting Harris processes do not have recurrent atoms; thus life cycle decompositions as used in the preceding subsection are not available. However, it is possible to consider instead of X itself a family of new Harris processes which are arbitrarily close to the original one; in this family, life cycles can be introduced artificially via Nummelin's splitting technique. Using this idea, the above results carry over to general Harris processes where life cycles do not exist.

In this general setting, conditions on regular variation at 0 of resolvants of X replace the former conditions on regular variation at ∞ of tails of life cycle length distributions; note that we could have formulated theorems 3.1 - 3.3. already in this way. A slight disadvantage of resolvant conditions remains: unless using resolvants for very particular functions of X (the 'special functions' of 5.28 which are essentially nonconstructive), the required regular variation holds only μ-a.s. in x. In this subsection, we do not need more than the basic condition

(H1): $X = (X_t)_{t \geq 0}$ is Harris with invariant measure μ.

The proofs of the two theorems 3.15 and 3.16 stated in this subsection is the aim of sections 6 and 7, and is given in theorems 7.16 and 7.20 there. The results are in complete analogy to subsection 3.1 although we choose a different presentation.

3.15 Theorem : a) For $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞, the following i) and ii) are equivalent:

i) for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, one has regular variation at 0 of resolvants in X

$$\left(R_{1/t}g\right)(x) = E_x\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right) \quad \sim \quad t^\alpha \frac{1}{l(t)}\, \mu(g), \quad t \to \infty$$

for μ-almost all $x \in E$ (the exceptional set depending on g);

ii) for every additive functional A of X with $0 < E_\mu(A_1) < \infty$, one has weak convergence

$$\frac{(A_{tn})_{t\geq 0}}{n^\alpha/\, l(n)} \quad \to \quad E_\mu(A_1)\, W^\alpha$$

(in $D(I\!R_+, I\!R)$ as $n \to \infty$, under P_x for all $x \in E$) where W^α is the Mittag-Leffler process of index α.

b) The cases in a) are the only ones where weak convergence of $\dfrac{(A_{tn})_{t\geq 0}}{v(n)}$ to a continuous non-decreasing limit process W (with $W_0 = 0$ and $\mathcal{L}(W_1)$ not degenerate at 0) is available for some norming function v.

We turn to martingales $M \in \mathcal{M}^{2,\mathrm{loc}}(P_x, I\!F)$ with the property that $\langle M\rangle$ is a locally bounded process (this slight restriction coming in here was not needed in subsection 3.1). We require only

(H5^A): M has the property

$$\forall\, y\,, \forall\, s, t: \quad M_{t+s} - M_t = M_s \circ \vartheta_t \quad P_y\text{-a.s.}\,,$$

the processes $\langle M\rangle$ and $[M]$ are additive functionals of X, and $E_\mu\left(\langle M\rangle_1\right) < \infty$.

3.16 Theorem : Consider $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞. Assume that 3.15 a)i) holds for α and $l(\cdot)$. Then we have weak convergence

$$\frac{1}{\sqrt{n^\alpha/l(n)}}\, \left(M_{tn}\right)_{t\geq 0} \quad \to \quad \left(E_\mu\left(\langle M\rangle_1\right)\right)^{1/2} B(W^\alpha)$$

in $D(I\!R_+, I\!R)$ as $n \to \infty$, under P_x.

Under a Lindeberg condition on $\frac{1}{\sqrt{n^\alpha/l(n)}}\, \left(M_{tn}\right)_{t\geq 0}$, we can again deduce from the last assertion of theorem 3.16 weak convergence of pairs (martingale, angle brackett) as in corollary 3.2, and then conclude from 3.15 that no other weak limits (under linear time scaling, with continuous limit process as in 3.1, and for some sequence of norming constants) can arise. The extension to multidimensional martingales (as in corollary 3.3) is obvious.

It might look strange that the seemingly simpler case with life cycles required more assumptions than the general case. The reason is the following. In our proof, we switch from the process X of interest to a family of new Harris processes, arbitrarily close to X, where life cycles are introduced artificially: so we can use the degrees of freedom in this construction to make sure that all additional assumptions needed in subsection 3.1 are satisfied at these auxiliary stages, and things become surprisingly simple at the level of the final result.

4 Proofs for subsection 3.1 - sufficient condition

In this section, we prove the 'sufficient' part of theorem 3.1 and its corollaries given in subsection 3.1: for processes X with life cycles, regular variation of tails of life cycle length distributions implies convergence of rescaled martingales to either Brownian motion or Brownian motion time-changed by an independent Mittag-Leffler process. This result is formulated in theorem 4.12 and in proposition 4.22, see also the remarks 4.25 and 4.26.

We work in the setting of subsection 3.1, but under weaker conditions: we list the assumptions on X and on the martingales M to be considered in this subsection, and then retrace the arguments given by Greenwood and Resnick ([R-Gr 79]) on weak convergence of bidimensional random walks. From this, weak convergence of martingales follows by time change. An extension of this argument yields weak convergence of pairs (martingale, angle brackett). (H4) is never needed in the present section.

Let us recall – for later use in sections 4 and 5 – the arguments proving theorems 2.4.A and 2.4.B, see [Fe 71] or [B-G-T 87]. Classical facts about regular variation (like Tauberian theorems etc.) are quoted from the first chapter of [B-G-T 87].

4.1 Proof of 2.4.A and 2.4.B : The proof is in several steps. Notations F, G, G_α are as in 2.4.A and 2.4.B: F, G are probability laws on $[0,\infty)$, G is not a Dirac measure at 0. $\widehat{F}$, $\widehat{G}$ denotes the Laplace transform (LT) of F, G.

A) (cf. [B-G-T 87, Cor. 8.1.7]) For $0 \le \alpha \le 1$ and $\ell \in R_0$, the assertion

$$1 - \widehat{F}(s) \sim s^\alpha \ell(1/s) \quad s \downarrow 0 \tag{4.2}$$

is equivalent to

$$r(t) = \int_0^t (1 - F(x))\, dx \sim \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} \ell(t), \quad t \uparrow \infty; \tag{4.3}$$

in case $\alpha < 1$, the last assertion is again equivalent to

$$1 - F(x) \sim \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} \ell(x), \quad x \uparrow \infty. \tag{4.4}$$

This is seen as follows: the function r defines a measure on $I\!R^+$; partial integration gives

$$\int_0^\infty e^{-\lambda x}(1 - F(x))\, dx \quad = \quad \frac{1}{\lambda}(1 - \widehat{F}(\lambda))$$

$$= \int_0^\infty e^{-\lambda x} r(dx) \quad = \quad \widehat{r}(\lambda).$$

The Tauberian theorem ([B-G-T 87, p. 37]) shows that for $0 \leq \alpha \leq 1$

$$\begin{array}{lcll} \widehat{r}(s) & \sim & s^{\alpha-1}\ell(1/s) & s \downarrow 0 \\ \Longleftrightarrow & & & \\ r(t) & \sim & \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}\ell(t) & t \uparrow \infty; \end{array}$$

this shows (4.2) $\Longleftrightarrow$ (4.3); (4.3) $\Longleftrightarrow$ (4.4) for $\alpha < 1$ is the monotone density theorem ([B-G-T 87, p. 39]) und the Karamata theorem ([B-G-T 87, p. 26]).

B) We determine possible limit laws for $F^{*n}(a_n \cdot)$ for suitable norming sequences $a_n \uparrow \infty$. The convergence

$$F^{*n}(a_n x) \longrightarrow G(x) \quad \forall \quad x \text{ continuity point of } G$$

is equivalent to convergence of LT

$$-n \log \widehat{F}(\lambda/a_n) \quad \longrightarrow \quad -\log \widehat{G}(\lambda), \quad \forall \quad \lambda > 0$$

and thus to

$$n(1 - \widehat{F}(\lambda/a_n)) \quad \longrightarrow \quad -\log \widehat{G}(\lambda), \quad \forall \quad \lambda > 0. \tag{4.5}$$

Consider $U := 1 - \widehat{F}$ which is nondecreasing: then for $a_n \leq x \leq a_{n+1}$

$$\frac{U(\lambda/a_{n+1})}{U(1/a_n)} \leq \frac{U(\lambda/x)}{U(1/x)} \leq \frac{U(\lambda/a_n)}{U(1/a_{n+1})}$$

and (4.5) implies

$$\forall \quad \lambda > 0 : \quad \frac{U(\lambda/x)}{U(1/x)} \quad \longrightarrow \quad \frac{-\log \widehat{G}(\lambda)}{-\log \widehat{G}(1)}, \quad x \to \infty. \tag{4.5$'$}$$

This is regular variation of the function $U = 1 - \widehat{F}$ in 0, and at the same time determines (cf. [B-G-T 87, p.17]) the possible limits in (4.5'):

$$1 - \widehat{F} \quad RV_\varrho \quad \text{in } 0, \quad -\log \widehat{G}(\lambda) = c \cdot \lambda^\varrho, \quad \lambda > 0 \tag{4.6}$$

for some $\varrho \in I\!R$ and some constant $c > 0$. We have necessarily $\varrho \geq 0$ since $\widehat{G}$ is nonincreasing as LT of a probability law concentrated on $[0, \infty)$; necessarily $\varrho \leq 1$ since otherwise $\lambda \to e^{-\lambda^\varrho}$ would not be 'completely monotone' and thus not a LT of a measure on $[0, \infty)$ ([Fe 71, p. 439]); necessarily also $\varrho > 0$ since G by assumption is not the Dirac measure at 0. Finally, a constant

c in (4.6) can always be absorbed into the norming sequence $(a_n)_n$, so we put $c = 1$. The only remaining possibilities are

$$G = G_\alpha, \quad 0 < \alpha < 1, \quad \text{with } 1 - \widehat{F} \quad RV_\alpha \text{ in } 0, \tag{4.7}$$

$$G = G_1, \quad \text{with } 1 - \widehat{F} \quad RV_1 \text{ in } 0. \tag{4.8}$$

According to A) we have for $0 \le \alpha \le 1$

$$1 - \widehat{F} \quad RV_\alpha \quad \text{in} \quad 0 \quad \Longleftrightarrow \quad r \quad RV_{1-\alpha} \quad \text{in} \quad \infty.$$

C) Comparison of (4.5) and (4.5') shows: with $c = 1$ in (4.6), the norming sequence $(a_n)_n$ satisfies

$$n \sim \frac{1}{1 - \widehat{F}(1/a_n)}, \quad n \to \infty, \tag{4.9}$$

whence $a_n = a(n)$ where $a(\cdot)$ is an asymptotic inverse to

$$t \quad \longrightarrow \quad \frac{1}{1 - \widehat{F}(1/t)} \sim \frac{t}{\widehat{r}(1/t)} \sim \frac{t}{\Gamma(2-\alpha) r(t)} \tag{4.10}$$

where we have used A); for $0 < \alpha < 1$ the function in (4.10) is asymptotically equivalent to

$$t \quad \longrightarrow \quad \frac{1}{\Gamma(1-\alpha)(1 - F(t))}. \tag{4.11}$$

D) Steps B)+C) prove part a) of theorem 2.4.A, and the corresponding direction in 2.4.B. The converse is proved by using the arguments of B)+C) in reverse order. □

On this basis, we turn to the topic of subsection 3.1. The proof of the 'sufficient' part of 3.1 is contained in the following theorem 4.12. Our arguments follow the references Greenwood and Resnick [R-Gr 79], Touati [Tou 88], [Hö 88]. For background on semimartingales and weak convergence we refer to Jacod and Shiryaev [J-Sh 87], Ikeda and Watanabe [I-W 89], Billingsley [Bill 68].

For the rest of of this section, the following assumptions on the process X will be in force:
(H1): $X = (X_t)_{t \ge 0}$ is Harris with invariant measure μ;
(H3): X has a recurrent atom $A \in \mathcal{E}$ and a life cycle decomposition $(R_n)_{n \ge 1}$, see 1.9.A + 1.9.B.

We consider martingales $M \in \mathcal{M}^{2,\text{loc}}(P_x, I\!F)$ for some P_x on $(\Omega, \mathcal{A}, I\!F)$ as in section 1, meeting (H5^A): M has the property

$$\forall\, y\,, \forall\, s, t: \quad M_{t+s} - M_t = M_s \circ \vartheta_t \quad P_y\text{-a.s.}\,,$$

the processes $\langle M \rangle$ and $[M]$ are additive functionals of X, and $E_\mu\left(\langle M \rangle_1\right) < \infty$;

(H5^B): For the life cycle decomposition $(R_n)_n$ of (H3), M satisfies either $(*)$:

$$(*) \qquad M_{R_n} \text{ is measurable with respect to } \mathcal{F}_{R_n^-}\,, \text{ for all } n \geq 1\ ;$$

or the following $(**)$:

$$(**) \qquad R_{n+1} - R_n \text{ and } M - M^{R_n} \text{ are independent of } \mathcal{F}_{R_n}, \text{ for all } n \geq 1\ .$$

Only these assumptions will be needed in the remaining parts of this section.

With respect to the sequence $(R_n)_{n\geq 1}$ of (H3), we write $r(\cdot)$ for the function

$$r(t) := \int_0^t P(R_2 - R_1 > s)\, ds\,,$$

and we fix the norming constant for μ (cf. 1.10) by

$$(4.11') \qquad \mu(F) = E\left(\int_{R_1}^{R_2} 1_F(X_s)\, ds\right), \quad F \in \mathcal{E}.$$

We recall also the convention $W^1 = S^1 = id$.

4.12 Theorem: Assume regular variation $r(\cdot) \in RV_{1-\alpha}$ at ∞, for some $0 < \alpha \leq 1$:

$$\begin{cases} P(R_2 - R_1 > \cdot) & \in RV_{-\alpha} \quad \text{falls } 0 < \alpha < 1, \\ r(\cdot) & \in RV_0 \quad \text{falls } \alpha = 1. \end{cases}$$

Define

$$v(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)P(R_2-R_1>t)}, & 0 < \alpha < 1, \\ t/r(t), & \alpha = 1. \end{cases}$$

Then one has

$$M^n := \left(\frac{1}{\sqrt{v(n)}} M_{tn}\right)_{t\geq 0} \qquad \longrightarrow \qquad J^{1/2}\, B \circ W^\alpha.$$

(weak convergence in $D(I\!R^+, I\!R)$, under P_x, as $n \to \infty$), where Brownian motion B and Mittag Leffler process W^α are independent, and where

$$J := E\left(\, \langle M \rangle_{R_2} - \langle M \rangle_{R_1}\right)\ .$$

Proof: 0) First we mention that due to 1.9.A+B and (H5^A)+(H5^B),

$$(4.12') \qquad (M_{R_n} - M_{R_1}, R_n - R_1)_{n\geq 1}$$

is a random walk either w.r.t. $(\mathcal{F}_{R_n^-})_n$ or w.r.t. $(\mathcal{F}_{R_n})_n$ (see example 4.27 for illustration of some typical problems). (H5^A) and Markov property give

$$E_{P_x}\left(e^{-\lambda_1(R_{n+1}-R_n)-\lambda_2(M_{R_{n+1}}-M_{R_n})}|\mathcal{F}_{R_n}\right) = E_{X_{R_n}}\left(e^{-\lambda_1 R_1 - \lambda_2 M_{R_1}}\right), \quad \lambda_1, \lambda_2 \geq 0\,.$$

Conditioning w.r.t. $\mathcal{F}_{R_n^-}$ according to 1.9.A+B, we see

$$E_{P_x}\left(e^{-\lambda_1(R_{n+1}-R_n)-\lambda_2(M_{R_{n+1}}-M_{R_n})}|\mathcal{F}_{R_n^-}\right) = E_{\rho_A}\left(e^{-\lambda_1 R_1 - \lambda_2 M_{R_1}}\right) =: \psi_A(\lambda_1, \lambda_2)\,.$$

Thus we have always

$$\left(R_{n+1} - R_n, M_{R_{n+1}} - M_{R_n}\right) \quad \text{is independent of } \mathcal{F}_{R_n^-}\text{, for all } n \geq 1\ .$$

If $(*)$ of (H5^B) holds, the r.v. $(M_{R_j} - M_{R_1}, R_j - R_1)$ is $\mathcal{F}_{R_j^-}$-measurable (R_j as a stopping time is always $\mathcal{F}_{R_j^-}$-measurable), thus (4.12') is a random walk w.r.t. $(\mathcal{F}_{R_n^-})_n$ under P_x.
If $(**)$ of (H5^B) holds, then – much simpler – (4.12') is a random walk w.r.t. $(\mathcal{F}_{R_n})_n$.
This holds under every starting law for the process X.
1) We consider the bivariate random walk $\left(M_{R_j} - M_{R_1}, R_j - R_1\right)_{j\geq 1}$ under P_x.
Writing $S_j := M_{R_j}$, we rescale the components of this random walk separately

$$(4.13) \qquad Y^n := \left(\frac{1}{\sqrt{n}} S_{[\cdot n]}, \frac{1}{a(n)} R_{[\cdot n]}\right)$$

according to 2.4.A and 2.4.B where $a(\cdot)$ is an asymptotic inverse for $v(\cdot)$. For every $n \in I\!N$, the bivariate process Y^n is a PII, and has approximately as $n \to \infty$ stationary increments. Considering the components of Y^n separately, we have weak convergence under P_x as $n \to \infty$

$$\frac{1}{\sqrt{n}} S_{[\cdot n]} \quad \longrightarrow \quad J^{1/2}\, B$$

in $D(I\!R^+, I\!R)$ according to Donsker's theorem, and convergence of finite-dimensional distributions

$$\frac{1}{a(n)} R_{[\cdot n]} \quad \longrightarrow \quad S^\alpha$$

as $n \to \infty$ by independence of increments combinded with theorems 2.4.A and 2.4.B: so possible limits for the sequence $(Y_n)^n$ in (4.13) under P_x, in the sense of finite-dimensional distributions, are

$$(4.14) \qquad Y = (J^{1/2} \cdot B, S^\alpha)$$

for some bivariate process having marginals B and S^α.

2) The essential point is to prove that the components B, S^α of the limit process Y in (4.14) are necessarily independent. A short argument, see Greenwood and Resnick [R-Gr 79], is as follows. Being limit of a bivariate random walk, Y is necessarily a PIIS. The Lévy-Khintchine formula shows that Y can be represented as independent sum of a Gaussian and a non-Gaussian (sum of big jumps and compensated sum of small jumps) Levy process plus a deterministic linear term:

$$Y = C \cdot id + B^Y + K^Y.$$

Comparison with (4.14) gives

$$C = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B^Y = \begin{pmatrix} J^{1/2} \cdot B \\ 0 \end{pmatrix}, \quad K^Y = \begin{pmatrix} 0 \\ S^\alpha \end{pmatrix}.$$

Thus B, S^α in (4.14) are independent. (We will give a detailed and more general argument following Ikeda and Watanabe [I-W 89, p. 77-78] - a Poisson random measure and a Brownian motion defined with respect to the same filtration are necessarily independent - in 4.21 below.)

3) By step 2) we know that under P_x

$$Y^n = \left(\frac{1}{\sqrt{n}} S_{[\cdot n]}, \frac{1}{a(n)} R_{[\cdot n]} \right) \quad \xrightarrow{\text{f.d.}} \quad Y = (J^{1/2} \cdot B, S^\alpha)$$

where Brownian motion B and stable increasing process S^α are necessarily independent. Write N_t for the number of life cycles of X (including the initial segment) completed at time t:

$$N_t = \sup\{n \in I\!N_0 : R_n \le t\}.$$

Then we have

$$\widetilde{Y}^n := \left(\frac{1}{\sqrt{n}} S_{[\cdot n]}, \frac{1}{n} N_{\cdot a(n)} \right) \quad \xrightarrow{\text{f.d.}} \quad \widetilde{Y} := (J^{1/2} \cdot B, W^\alpha) \tag{4.15}$$

where W^α is the process inverse to S^α and thus B and W^α are independent: for $0 \le t_1 < t_2 < \ldots < t_\ell < \infty$, for $x_1, \ldots, x_n \in (0, \infty)$, as $n \to \infty$

$$\begin{aligned} \left\{ \frac{N_{t_i a(n)}}{n} < x_i, \ 1 \le i \le \ell \right\} &= \left\{ R_{[x_i n]} > t_i a(n), \ 1 \le i \le \ell \right\} \triangle N_n \\ &= \left\{ \frac{R_{[x_i n]}}{a(n)} > t_i, \ 1 \le i \le \ell \right\} \triangle N_n \end{aligned}$$

up to symmetric difference with small sets N_n meeting $P_x(N_n) \to 0$; since $n \sim v(a(n))$ with $a(\cdot) \in RV_{1/\alpha}$, we identify $\frac{1}{n} N_{\cdot a(n)}$ with a subsequence of $\frac{1}{v(n)} N_{\cdot n}$ as $n \to \infty$. We show that the sequence $(\widetilde{Y}^n)_n$ in (4.15) is tight in $D(I\!R^+, I\!R^2)$ under P_x: tightness of the first component in

$D(\mathbb{R}^+, \mathbb{R})$ is clear from step 1); for tightness of the second component in $D(\mathbb{R}^+, \mathbb{R})$, we use ([J-Sh 87, VI.3.37]): since the second components of $(\widetilde{Y}^n)_n$ are increasing processes and since W^α is continuous, finite-dimensional convergence (4.15) implies

$$\frac{1}{n} N_{\cdot a(n)} \quad \overset{n\to\infty}{\longrightarrow} \quad W^\alpha \quad \text{(weak convergence in } D(\mathbb{R}^+, \mathbb{R}) \text{ under } P_x).$$

Thus both components of $(\widetilde{Y}^n)_n$ form tight sequences in $D(\mathbb{R}^+, \mathbb{R})$, and so the bivariate sequence $(\widetilde{Y}^n)_n$ is tight in $D(\mathbb{R}^+, \mathbb{R}^2)$ ([J-Sh 87, VI.3.33]). By (4.15), there is a unique limit law for arbitrary subsequences of $(\widetilde{Y}^n)_n$, so we have

$$\text{(4.16)} \qquad \begin{cases} \widetilde{Y}_n = \left(\frac{1}{\sqrt{n}} S_{[\cdot n]}, \frac{1}{n} N_{\cdot a(n)} \right) \quad \longrightarrow \quad \widetilde{Y} = (J^{1/2} \cdot B, W^\alpha) \\ \text{weak convergence in } D(\mathbb{R}^+, \mathbb{R}^2) \text{ under } P_x \text{ as } n \to \infty. \end{cases}$$

By Billingsley [Bill 68, p.145], both components of the limit process in (4.16) being continuous, the second component in (4.16) may be used as a time transformation for the first: so we get

$$\text{(4.17)} \qquad \begin{cases} \frac{1}{\sqrt{n}} S_{[(\frac{1}{n} N_{\cdot a(n)})n]} \quad \longrightarrow \quad J^{1/2} \cdot B(W^\alpha) \\ \text{weak convergence in } D(\mathbb{R}^+, \mathbb{R}) \text{ under } P_x \end{cases}$$

and after replacing n by $v(n)$ - which amounts to an insertion of members into the sequence - one arrives at

$$\text{(4.18)} \qquad \begin{cases} \frac{1}{\sqrt{v(n)}} S_{N_{\cdot n}} \quad \longrightarrow \quad J^{1/2} \cdot B(W^\alpha) \\ \text{weakly in } D(\mathbb{R}^+, \mathbb{R}) \text{ under } P_x \text{ as } n \to \infty. \end{cases}$$

4) It remains to show that (4.18) implies

$$\text{(4.19)} \qquad \frac{1}{\sqrt{v(n)}} M_{\cdot n} \quad \longrightarrow \quad J^{1/2} \cdot B(W^\alpha) \quad \text{weakly in } D(\mathbb{R}^+, \mathbb{R}) \text{ under } P_x \text{ as } n \to \infty\,.$$

The proof of (4.19) is in three parts.

i) For every starting point x for the Harris process X, the measure

$$t \quad \to \quad E_x(N_t) \; = \; \sum_{l \geq 1} P_x(R_l \leq t)$$

is (with notations of 1.9.A+B) a convolution

$$P_x(R_1 \in \cdot) * \left(\sum_{m=0}^{\infty} P_{\rho_A}(R_m \in \cdot) \right) .$$

The asymptotic behaviour of its Laplace transform (cf. [B-G-T 87, p. 361)

$$\lambda \quad \to \quad E_x\left(e^{-\lambda R_1}\right) \frac{1}{1 - E_{\rho_A}\left(e^{-\lambda R_1}\right)} \,, \quad \lambda > 0$$

as $\lambda \downarrow 0$ does not depend on x. Combining (4.2)-(4.4) and Karamata's Tauberian theorem (B-G-T 87, p. 37), we see that

$$E_x(N_t) \quad \sim \quad \frac{1}{\Gamma(1+\alpha)}\, v(t) \qquad \text{as } \ t \to \infty$$

independently of the starting point x.

ii) We show that for every $t > 0$, $\varepsilon > 0$ fixed,

$$(+) \qquad P_x\left(\frac{1}{\sqrt{v(n)}} |S_{N_{tn}} - M_{tn}| > \varepsilon \right) \quad \to \quad 0$$

as $n \to \infty$. Since $\lim_{n\to\infty} P_x(R_1 > nt) = 0$, is is sufficient to consider

$$(++) \qquad P_x\left(\sup_{s\geq 0} \left(M_{s\wedge R_{N_{tn}+1}} - M_{s\wedge R_{N_{tn}}} \right)^2 > \varepsilon^2 v(n) \, , \, R_1 \leq nt \right) .$$

Note that the last renewal time $R_{N_{tn}}$ before tn is not a stopping time: an event $\{R_{N_{tn}} \leq c\}$ with $0 < c < tn$ does not belong to the σ-field generated by observation of X only up to time c. By 1.9.A+B and (H5), rewrite (++) as

$$\begin{aligned}
&\sum_{l=1}^{\infty} E_x \left(1_{\{R_l \leq nt\}} \, E_x \left(1_{\{R_{l+1}-R_l > nt - R_l\}} \, 1_{\{\sup_{s\geq 0} \left(M_{s\wedge R_{l+1}} - M_{s\wedge R_l}\right)^2 > \varepsilon^2 v(n)\}} \mid \mathcal{F}_{R_l^-} \right) \right) \\
= \quad & \int_0^{nt} d_u \left(E_x(N_u)\right) P_{\rho_A} \left(R_1 > nt - u \, , \, \sup_{s\leq R_1} (M_s)^2 > \varepsilon^2 v(n) \right) \\
= \quad & \int_0^{\infty} P_{\rho_A}(R_1 \in dr) \, f^{(n)}(r) \, \left(E_x(N_{nt}) - E_x(N_{(nt-r)\vee 0}) \right)
\end{aligned}$$

with notation

$$f^{(n)}(r) := P_{\rho_A} \left(\sup_{s\leq R_1} (M_s)^2 > \varepsilon^2 v(n) \mid R_1 = r \right) .$$

By assumption in (H5),

$$\begin{aligned}
J \quad &= \quad E\left(\langle M \rangle_{R_2} - \langle M \rangle_{R_1} \right) = E\left(\sup_{s\geq 0} \left(M_{s\wedge R_2} - M_{s\wedge R_1}\right)^2 \right) = E_{\rho_A}\left(\sup_{s\leq R_1} M_s^2 \right) \\
&= \quad \int_0^{\infty} P_{\rho_A}(R_1 \in dr) \, E_{\rho_A} \left(\sup_{s\leq R_1} M_s^2 \mid R_1 = r \right) \; < \; \infty \, ,
\end{aligned}$$

thus we have for $P_{\rho_A}^{R_1}$- a.a. $r > 0$ as $n \to \infty$

$$\varepsilon^2 \, v(n) \, f^{(n)}(r) \leq E_{\rho_A} \left((\sup_{s\leq R_1} M_s^2) \, 1_{\{\sup_{s\leq R_1} (M_s)^2 > \varepsilon^2 v(n)\}} \mid R_1 = r \right) \quad \to \quad 0$$

and from this by dominated convergence

$$\int_0^{\infty} P_{\rho_A}(R_1 \in dr) \, v(n) \, f^{(n)}(r) \quad \to \quad 0 \, .$$

By part i) above and regular variation of v, we thus have proved

$$\int_0^\infty P_{\rho_A}(R_1 \in dr)\, f^{(n)}(r)\, \left(E_x(N_{nt}) - E_x(N_{(nt-r)\vee 0})\right) \quad \to \quad 0$$

which via (++) establishes (+).

iii) Part ii) together with (4.18) implies

$$\frac{1}{\sqrt{v(n)}} M_{\cdot n} \quad \stackrel{\text{f.d.}}{\longrightarrow} \quad J^{1/2} B(W^\alpha).$$

It remains to prove tightness of this sequence in $D(I\!R^+, I\!R)$ under P_x in order to complete the proof of theorem 4.12. By [J-Sh 87, VI.4.13], it is enough to verify

$$\left(\left\langle \frac{1}{\sqrt{v(n)}} M_{\cdot n} \right\rangle \right)_n \qquad \text{is } C\text{-tight in } D(I\!R^+, I\!R);$$

we will prove weak convergence

$$\left\langle \frac{1}{\sqrt{v(n)}} M_{\cdot n} \right\rangle \longrightarrow J \cdot W^\alpha \quad \text{weakly in } D(I\!R^+, I\!R) \text{ as } n \to \infty. \tag{4.20}$$

Again by [J-Sh 87, VI.3.37], it is enough to show finite-dimensional convergence in (4.20) - the pre-limiting processes are increasing, and the limit process is continuous - and this is a consequence of (4.15) (with n replaced by $v(n)$)

$$\frac{1}{v(n)} N_{\cdot n} \quad \stackrel{\text{f.d.}}{\longrightarrow} \quad W^\alpha$$

and the ratio limit theorem

$$P - \text{a.s.} : \lim_{t \to \infty} \frac{< M >_t}{N_t} \; = \; E\left(< M >_{R_2} - < M >_{R_1}\right) \; = \; J.$$

So (4.20) is proved, and thus (4.19): this concludes the proof of 4.12. □

By 4.12, we have proved the 'sufficent' direction in 3.1. Before proceding to joint convergence of pairs (martingale, angle brackett) in proposition 4.22, we give an alternative argument for step 2) of the preceding proof.

4.21 Remark : We give an alternative argument replacing step 2) of the previous proof, based on [I-W 89, pp. 77-78]. Consider any possible càdlàg limit process Y for $(Y^n)_n$ of (4.13), in the sense of finite-dimensional distributions: Y is defined on some $(\Omega', \mathcal{A}', I\!F', P')$ where $I\!F'$ is the filtration generated by Y

$$I\!F' = (\mathcal{F}_t)_{t \geq 0}, \quad \mathcal{F}'_t = \bigcap_{T > t} \mathcal{F}^0_T, \quad \mathcal{F}^0_T := \sigma(Y_s : \ 0 \leq s \leq T);$$

by (4.14), its first marginal denoted by B is a Brownian motion (for simplicity, we put $J = 1$); its second marginal denoted by S^α is a stable increasing process with index α. Since Y^n has independent increments which asymptotically as $n \to \infty$ are stationary, the limit process Y is a PIIS w.r.to $I\!F^0$: for $0 \leq s < t < \infty$, the conditional law $P^{Y_t - Y_s | \mathcal{F}^0_s} = \mathcal{L}(Y_{t-s})$ is independent of $\mathcal{F}^0_s$. Since Y is right-continuous, Y is also a PIIS w.r.to the filtration $I\!F'$: for arbitrary Z nonnegative and $\mathcal{F}'_s$-measurable, $\mathcal{F}'_s = \bigcap_n \mathcal{F}^0_{s+1/n}$, for $h \in \mathcal{C}_b(I\!R^2)$, we have

$$\begin{aligned} E(Zh(Y_t - Y_s)) &= \lim_n \ E(Zh(Y_t - Y_{s+1/n})) \\ &= \lim_n \ E\left(ZE\left(h(Y_t - Y_{s+1/n}) \mid \mathcal{F}^0_{s+1/n}\right)\right) \\ &= EZ \ \lim_n \ E(h(Y_t - Y_{s+1/n})) \\ &= EZ \ \ E(h(Y_t - Y_s)) \end{aligned}$$

which proves that $P^{Y_t - Y_s | \mathcal{F}'_s} = \mathcal{L}(Y_{t-s})$ is independent of $\mathcal{F}'_s$. By this argument, the first component of Y is a $I\!F'$-Brownian motion:

$$P^{B_t - B_s | \mathcal{F}'_s} = \mathcal{N}(0, t-s)$$

and the point process μ of jumps of the second component of Y is a $I\!F'$-Poisson random measure:

$$\begin{cases} P^{(\mu(]s,t] \times U_i)_{1 \leq i \leq \ell}) | \mathcal{F}'_s} = \bigotimes_{i=1}^{\ell} \mathcal{P}((t-s)\Lambda_\alpha(U_i)) \\ \text{for disjoint sets } U_1, \ldots, U_\ell \text{ in } B(I\!R) \text{ having } \Lambda_\alpha(U_i) < \infty \end{cases}$$

where we write $\Lambda_\alpha(dx)$ for the measure $\alpha x^{-\alpha-1} dx$ on $(0, \infty)$, see remark 2.7. Following [I-W 89, pp. 77-78], we will show that Poisson random measure and Brownian motion defined on the same $(\Omega', \mathcal{A}', P')$ with respect to the same filtration $I\!F'$ are necessarily independent (which implies the desired independence of B and S^α). To this aim we consider transforms

$$I\!R \times (I\!R^+)^\ell \ni (\xi, \lambda_1, \ldots, \lambda_\ell) \to \varphi(\xi, \lambda_1, \ldots, \lambda_\ell) := E\left(e^{i\xi(B_t - B_s) - \sum_{i=1}^{\ell} \lambda_i \mu(]s,t] \times U_i)} \mid \mathcal{F}'_s\right)$$

($0 \leq s < t < \infty$, U_i disjoint sets having $\Lambda_\alpha(U_i) < \infty$, $\ell \in I\!N$); if we prove that

$$(*) \qquad \varphi(\xi, \lambda_1, \ldots, \lambda_\ell) = e^{(t-s)[-\frac{1}{2}\xi^2 + \sum_{i=1}^{\ell}(e^{-\lambda_i} - 1)\Lambda_\alpha(U_i)]},$$

then $B_t - B_s$, $\mu(]s,t] \times U_i)$, $1 \leq i \leq \ell$ will be independent and independent of $\mathcal{F}'_s$, and as a consequence, the σ-fields $\sigma(B_t : t \geq 0)$ and $\sigma(\mu(]s,t] \times U : 0 \leq s < t < \infty, U \in B(I\!R^d))$ generated by B and μ will be independent, which concludes the proof.

To prove $(*)$, we consider the bounded and complex-valued semimartingale $F(Z)$

$$\begin{aligned} F(Z_t) &:= e^{i\xi B_t - \sum_{i=1}^{\ell} \lambda_i \mu(]0,t] \times U_i)} \\ Z &= (B, \mu(]0,\cdot] \times U_1), \ldots \mu(]0,\cdot] \times U_\ell)) . \end{aligned}$$

Itô's formula (see e.g. [J-Sh 87, p. 57]) for $F(Z)$ gives

$$\begin{aligned} F(Z_t) - F(Z_s) &= i\xi \int_s^t F(Z_{u^-}) dB_u + (-\frac{1}{2}\xi^2) \int_s^t F(Z_{u^-}) du \\ &\quad + \int_s^t \underbrace{\int_{I\!R} \left[F(Z_{u^-}) \left(e^{-\sum_{i=1}^{\ell} \lambda_i 1_{U_i}(x)} - 1 \right) \right] \mu(ds,dx)}_{= \sum_{u \in (s,t]} (F(Z_u) - F(Z_{u^-}))} \\ &= (M_t - M_s) + (-\frac{1}{2}\xi^2) \int_s^t F(Z_u) du \\ &\quad + \int_s^t \int_{I\!R} F(Z_u) \left(e^{-\sum_{i=1}^{\ell} \lambda_i 1_{U_i}(x)} - 1 \right) du\, \Lambda_\alpha(dx) \end{aligned}$$

for some local martingale M such that $< M >_t$ is bounded; since the sets U_i are disjoint, the last term on the r.h.s equals

$$\int_s^t F(Z_u) \left[\sum_{i=1}^{\ell} (e^{-\lambda_i} - 1) \Lambda_\alpha(U_i) \right] du.$$

For Y nonnegative, bounded, $\mathcal{F}'_s$-measurable we consider $E[Y(F(Z_t) - F(Z_s))]$: with notation

$$\varphi_Y(t) := E\left(Y\ e^{i\xi(B_t - B_s) - \sum_{i=1}^{\ell} \lambda_i \mu(]s,t] \times U_i)} \right), \quad t \geq s$$

(thus $\varphi_Y(s) = E(Y)$) we get from $E(M_t - M_s | \mathcal{F}'_s) = 0$ after absorption of the factor $F(Z_s)$ in Y

$$\begin{aligned} E\left(Y \left[e^{i\xi(B_t - B_s) - \sum_{i=1}^{\ell} \lambda_i \mu(]s,t] \times U_i)} - 1 \right] \right) &= \varphi_Y(t) - \varphi_Y(s) \\ &= \int_s^t du\ \varphi_Y(u) \left[-\frac{1}{2}\xi^2 + \sum_{i=1}^{\ell} (e^{-\lambda_i} - 1) \Lambda_\alpha(U_i) \right] du. \end{aligned}$$

The solution of this differential equation is well known

$$\varphi_Y(t) = E(Y) \cdot e^{(t-s)\left[-\frac{1}{2}\xi^2 + \sum_{i=1}^{\ell} (e^{-\lambda_i} - 1) \Lambda_\alpha(U_i) \right]}, \quad t \geq s.$$

Taking in particular $Y = 1_A$, for arbitrary $A \in \mathcal{F}'_s$, we get

$$E\left(e^{i\xi(B_t - B_s) - \sum_{i=1}^{\ell} \lambda_i \mu(]s,t] \times U_i)} | \mathcal{F}'_s \right) = e^{(t-s)\left[-\frac{1}{2}\xi^2 + \sum_{i=1}^{\ell} (e^{-\lambda_i} - 1) \Lambda_\alpha(U_i) \right]}.$$

This is $(*)$, and concludes the proof. $\square$

The argument leading to theorem 4.12 can be strengthened to obtain weak convergence of pairs (martingale, angle brackett). Note that we do not require a Lindeberg condition here.

4.22 Proposition : Under all assumptions of theorem 4.12, we have

$$(M^n\,,\,< M^n >) \xrightarrow{\mathcal{L}} \left(J^{1/2}\,B\circ W^\alpha\,,\,J\,W^\alpha\right)$$

(weak convergence in $D(I\!R_+, I\!R \times I\!R)$, as $n \to \infty$, under P_x).

Proof : We replace the bivariate random walk in step 1) of the proof of 4.12 by a trivariate one

$$\left(M_{R_j} - M_{R_0},\; < M >_{R_j} - < M >_{R_0},\; R_j - R_0\right)_{j\in I\!N_0}$$

and consider

$$Y^n := \left(\frac{1}{\sqrt{n}}S_{[\cdot n]},\; \frac{1}{n}K_{[\cdot n]},\; \frac{1}{a(n)}R_{[\cdot n]}\right) \tag{4.23}$$

where $K_j :=< M >_{R_j}$. The second component converges weakly in $D(I\!R_+, I\!R)$ to the deterministic process $(J\cdot t)_{t\geq 0}$. Inverting the last component and using it as a time change for the pair of first components in analogy to (4.16)-(4.18), we get

$$\left\{\begin{array}{l} \left(\frac{1}{\sqrt{v(n)}}S_{N\cdot n}\,,\,\frac{1}{v(n)}K_{N\cdot n}\right) \quad \longrightarrow \quad \left(J^{1/2}\cdot B(W^\alpha), J\cdot W^\alpha\right) \\ \text{weakly in } D(I\!R^+, I\!R\times I\!R) \text{ as } n\to\infty. \end{array}\right. \tag{4.24}$$

Up to obvious changes, the remaining parts of the proof are on the lines of 4.12. $\square$

4.25 Remark : In the same way, we may consider d-dimensional local martingales $M \in \mathcal{M}^2_{\text{loc}}$ by including all components M^i of M and $< M^i, M^j >$ of $< M >$ into the random walk (4.23).

4.26 Remark : Via the RLT, see 1.7, the last result implies joint weak convergence of martingales with arbitrary integrable additive functionals.

We end this section with an example illustrating why we need assumption (H5^B) to make sure that increments of the martingale over life cycles of X form indeed a random walk.

4.27 Example: We continue example 1.11 a), will all notations as there.

a) For F open in $I\!R$ with $0 < \nu(F) < 1$, consider the $I\!F^X$-counting process

$$N_t = \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_{\{X_{T_n} \in F \times \{0\}\}}$$

and let M denote the compensated counting process

$$M_t = N_t - \int_0^t \left(\frac{1}{2} 1_{F \times \{0\}}(X_s) + \nu(F) 1_A(X_s) \right) ds$$

where $A = I\!R \times \{1\}$ is the atom of X. By definition of $(R_n)_n$ in 1.11 as passage times from A to A^c, X_{R_n} is always in $I\!R \times \{0\}$, is distributed according to $\nu \otimes \epsilon_0$, and we have

$$X_{R_n} \in F \times \{0\} \quad \text{if and only if} \quad M_{R_n+t} - M_{R_n} = -\frac{1}{2} t \quad \text{for } t \text{ sufficiently small}$$

$$X_{R_n} \in F^c \times \{0\} \quad \text{if and only if} \quad M_{R_n+t} - M_{R_n} = 0 \quad \text{for } t \text{ sufficiently small}$$

as well as

$$M_{R_n} = M_{R_n^-} + 1_{F \times \{0\}}(X_{R_n}) \, .$$

So both $(*)$ and $(**)$ of (H5^B) are violated, and it is clear that $M - M^{R_n}$ and M_{R_n} are dependent. So $\left(M_{R_j} - M_{R_1}, R_j - R_1 \right)_{j \geq 1}$ is not a random walk.

b) In example 1.11 a) we have constant intensity for 'change of colour' on $E \times \{0\}$ and $E \times \{1\}$, thus $R_{n+1} - R_n$ is independent of $\mathcal{F}_{R_n}$. Consider

$$N_t^1 = \sum_{n \geq 1} 1_{\{R_n \leq t\}} 1_{\{X_{R_n} \in F \times \{0\}\}} \, , \quad M_t^1 = N_t^1 - \int_0^t \nu(F) 1_A(X_s) ds \, .$$

M^1 is a martingale such that $M^1 - (M^1)^{R_n}$ is independent of $\mathcal{F}_{R_n}$, $n \geq 1$. So $(**)$ of (H5^B) holds for M^1.

c) Consider now

$$N_t^2 = \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_{\{X_{T_n} \in F \times \{1\}\}} \, , \quad M_t^2 = N_t^2 - \int_0^t \left(\frac{1}{2} 1_{F \times \{0\}}(X_s) \right) ds$$

Then $(M^2)_{R_n}$ is $\mathcal{F}_{R_n^-}$-measurable, so $(*)$ of (H5^B) holds for M^2. $\square$

5 Proofs for subsection 3.1 - necessary condition

In this section, we consider processes with life cycles and prove that the conditions on regular variation of tails of life-cycle length distributions in theorem 3.1 are necessary conditions for weak convergence of rescaled martingales under a Lindeberg condition - this condition implies that the limit process is a continuous local martingale, and that we have also weak convergence of (predictable) quadratic variations. To these one applies the classical Darling-Kac theorem ([D-K 57], see [B-G-T 87, ch. 8.11]) which states that norming functions are necessarily regular varying, and that limit laws for (one-dimensional marginals of) rescaled additive functionals of X are necessarily Mittag-Leffler laws. However, the Darling-Kac theorem needs a uniformity condition (see [B-G-T 87, p. 390]) which is rather restrictive except for simple situations such as Markov step processes with countable state space. Touati ([Tou 88]) proposed to avoid 'Darling-Kac conditions' by use of 'special functions'. We give the argument exactly in this way.

In a first part of this section, we shall use only assumptions on the process X:
(H1): $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ;
(H3): X has a recurrent atom $A \in \mathcal{E}$ and a life cycle decomposition $(R_n)_{n\geq 1}$, see 1.9.A + 1.9.B;
(H4): There is some function f, bounded, nonnegative, $\mathcal{E}$-measurable, $0 < \mu(f) < \infty$, such that

$$x \quad \to \quad E_x\left(\int_0^{R_1} f(X_s)\,ds\right) \quad \text{is bounded on } E$$

(a 'weakly special function for X and R_1').
With respect to $(R_n)_n$, we fix the norming constant for μ as in (4.11').

(H1), (H3) and (H4) allow to prove a variant of the classical Darling-Kac theorem without Darling-Kac conditions: weak convergence of (linearly time-scaled and suitably normed) additive functionals of X implies regular variation of tails of life-cycle length distributions (theorems 5.6.A and 5.6.B below). We will use in this section the following abuse of language: we write $E_A(\cdot) := E_{\rho_A}(\cdot)$ with ρ_A the law of X_{R_n} as in 1.9.A+B, and we term functions f meeting (H4) for short weakly special without explicit reference to X and R_1.

As in (4.2)-(4.4), write $\widehat{F}$ for the Laplace transform of the life cycle length distribution:

$$\widehat{F}(\lambda) = E(e^{-\lambda(R_2-R_1)}) = E_A(e^{-\lambda R_1})\ ,\ \lambda \in \mathbb{R}_+$$

and introduce a function $\mathbf{v} : I\!R_+ \to I\!R_+$

$$\mathbf{v}(t) = \left(1 - E_A\left(e^{-\frac{1}{t}R_1}\right)\right)^{-1} = \left(1 - \widehat{F}(\frac{1}{t})\right)^{-1} \tag{5.1}$$

which is nondecreasing, with $\mathbf{v}(0) = 1$ and $\mathbf{v}(t) \uparrow \infty$ as $t \to \infty$. This function $\mathbf{v}$ - which was implicit already in (4.2)-(4.4) - will play a key role in the sequel. We will work with resolvants and define for f nonnegative, bounded, measurable

$$R_\lambda f(x) = \int_0^\infty \lambda e^{-\lambda t} E_x\left(\int_0^t f(X_s)ds\right) dt = E_x\left(\int_0^\infty e^{-\lambda t} f(X_t)dt\right).$$

5.2 Lemma : ([Tou 88]) R_λ admits the decomposition

$$R_\lambda f(x) \;=\; R^1_\lambda f(x) + R^2_\lambda f(x)$$

where

$$R^1_\lambda f(x) \;=\; E_x\left(\int_0^{R_1} e^{-\lambda t} f(X_t)dt\right)$$

and

$$R^2_\lambda f(x) \;=\; \mathbf{v}(\frac{1}{\lambda}) E_x\left(e^{-\lambda R_1}\right) E_A\left(\int_0^{R_1} e^{-\lambda t} f(X_t)dt\right).$$

Proof: We start from

$$\begin{aligned} R_\lambda f(x) &= E_x\left(\int_0^\infty e^{-\lambda t} f(X_t)dt\right) \\ &= E_x\left(\int_0^{R_1} e^{-\lambda t} f(X_t)dt\right) + E_x\left(\sum_{n\ge1}\int_{R_n}^{R_{n+1}} e^{-\lambda t} f(X_t)dt\right). \end{aligned}$$

Here, the first term on the r.h.s is $R^1_\lambda f(x)$. For the second one, note that

$$\begin{aligned} E_x\left(\int_{R_n}^{R_{n+1}} e^{-\lambda t} f(X_t)dt\right) &= E_x\left(e^{-\lambda R_n}\left\{\int_0^{R_1} e^{-\lambda v} f(X_v)dv\right\}\circ\theta_{R_n}\right) \\ &= E_x\left(e^{-\lambda R_n}\right) E_A\left(\int_0^{R_1} e^{-\lambda v} f(X_v)dv\right) \end{aligned}$$

by the strong Markov property, where

$$E_x\left(e^{-\lambda R_n}\right) = E_x\left(e^{-\lambda R_1}\right) E_x\left(e^{-\lambda(R_n - R_1)}\right) = E_x\left(e^{-\lambda R_1}\right) E_A\left(e^{-\lambda R_1}\right)^{n-1}.$$

By definition of $\mathbf{v}$ in (5.1), the assertion follows. □

For weakly special functions, lemma 5.2 can be strengthened.

5.3 Lemma : ([Tou 88]) For f weakly special and $C := \sup_x E_x\left(\int_0^{R_1} f(X_s)ds\right) < \infty$, for μ normed according to (4.11'), one has for arbitrary $x \in E$

$$(i) \qquad \frac{R_\lambda f(x)}{\mathbf{v}(\frac{1}{\lambda})} \underset{\lambda \downarrow 0}{\longrightarrow} \mu(f)$$

$$(ii) \qquad \forall\, \lambda > 0: \quad \mathbf{v}(\frac{1}{\lambda})\left[\frac{R_\lambda f(x)}{\mathbf{v}(\frac{1}{\lambda})} - \mu(f)\right] \leq C \quad .$$

Proof: For $\lambda \downarrow 0$, this follows from inspection of the terms

$$\frac{R_\lambda f(x)}{\mathbf{v}(\frac{1}{\lambda})} = \frac{E_x \overbrace{\left(\int_0^{R_1} e^{-\lambda t} f(X_t)dt\right)}^{\leq C}}{\mathbf{v}(\frac{1}{\lambda})} + \underbrace{E_x\left(e^{-\lambda R_1}\right)}_{\leq 1 \quad \uparrow 1} \cdot \underbrace{E_A\left(\int_0^{R_1} e^{-\lambda t} f(X_t)dt\right)}_{\leq \mu(f), \quad \uparrow \mu(f)}$$

arising in the decomposition of lemma 5.2. □

Next, for f weakly special, we consider moments of arbitrary order for additive functionals $A_t = \int_0^t f(X_s)ds$. Define

$$M_n(t,x) := E_x\left[\left(\int_0^t f(X_s)ds\right)^n\right]$$

and

$$\widehat{M}_n(\lambda, x) = \int_0^\infty \lambda e^{-\lambda t} M_n(t,x)dt, \quad \lambda > 0.$$

Then the following modification of [D-K 57] or [B-G-T 87] – due to Touati – holds.

5.4 Lemma : ([Tou 88]) For f weakly special, one has for all $x \in E$:

$$(i) \qquad \frac{\widehat{M}_n(\lambda, x)}{n!\,(\mathbf{v}(\frac{1}{\lambda}))^n} \underset{\lambda \downarrow 0}{\longrightarrow} (\mu(f))^n,$$

$$(ii) \qquad \forall\, \lambda > 0: \quad \mathbf{v}(\frac{1}{\lambda})\left[\frac{\widehat{M}_n(\lambda, x)}{n!\,(\mathbf{v}(\frac{1}{\lambda}))^n} - \mu(f)^n\right] \leq C(C + 2\mu(f))^{n-1}$$

where C is the constant of lemma 5.3.

Proof : 1) We start from

$$E_x\left[\left(\int_0^t f(X_s)ds\right)^n\right] = \int_0^t \ldots \int_0^t E_x(f(X_{u_1})\ldots f(X_{u_n}))du_1\ldots du_n$$
$$= n!\int_0^t du_1 \int_{u_1}^t du_2 \ldots \int_{u_{n-1}}^t du_n E_x(f(X_{u_1})\ldots f(X_{u_n}))$$

which gives

$$\widehat{M}_n(\lambda,x) = \int_0^\infty \lambda e^{-\lambda t} M_n(t,x)dt$$
$$= n!\int_0^\infty du_1 \int_{u_1}^\infty du_2 \ldots \int_{u_{n-1}}^\infty du_n e^{-\lambda u_n} E_x(f(X_{u_1})\ldots f(X_{u_n})).$$

Conditioning on $X_{u_{n-1}}$ and using the strong Markov property we get

$$\widehat{M}_n(\lambda,x) = n!\int_0^\infty du_1 \ldots \int_{u_{n-2}}^\infty du_{n-1} e^{-\lambda u_{n-1}} E_x(f(X_{u_1})\ldots f(X_{u_{n-1}})R_\lambda f(X_{u_{n-1}}))$$

where we have used that

$$E_x\left(\int_{u_{n-1}}^\infty e^{-\lambda(u_n-u_{n-1})} f(X_{u_n})du_n \middle| X_{u_{n-1}}\right) = E_x\left(\int_0^\infty e^{-\lambda u} f(X_{u_{n-1}+u})du \middle| X_{u_{n-1}}\right)$$
$$= E_{X_{u_{n-1}}}\left(\int_0^\infty e^{-\lambda u} f(X_u)du\right)$$
$$= R_\lambda f(X_{u_{n-1}}).$$

Iterating this argument, we arrive at

$$\widehat{M}_n(\lambda,x) = n!\int R_\lambda(x,dx_1)f(x_1)\int R_\lambda(x_1,dx_2)\ldots f(x_{n-1})\int R_\lambda(x_{n-1},dx_n)f(x_n)$$

which in short notation

$$\psi_0(\lambda,x) := 0, \quad \psi_1(\lambda,x) := R_\lambda f(x), \ldots, \quad \psi_n(\lambda,x) := \int R_\lambda(x,dx_1)f(x_1)\psi_{n-1}(\lambda,x_1)$$

takes the form

$$\widehat{M}_n(\lambda,x) = n!\,\psi_n(\lambda,x).$$

2) In a next step we prove that there is some sequence of constants $(K_n)_n$ such that for all n the following (+) and (++) hold:

$$(+) \qquad \frac{\psi_n(\lambda,x)}{(\mathbf{v}(\frac{1}{\lambda}))^n} \leq K_n \quad \forall \lambda, x$$

$$(++) \qquad \frac{\psi_n(\lambda,x)}{(\mathbf{v}(\frac{1}{\lambda}))^n} \underset{\lambda\downarrow 0}{\longrightarrow} (\mu(f))^n \quad \forall x.$$

The proof is by induction on n. The case $n=1$ is lemma 5.3 together with $\mathbf{v}(\frac{1}{\lambda}) \geq 1$. For arbitrary $n>1$, we write

$$(5.5)\quad \frac{\psi_n(\lambda,x)}{(\mathbf{v}(\frac{1}{\lambda}))^n} - (\mu(f))^{n-1}\frac{\psi_1(\lambda,x)}{\mathbf{v}(\frac{1}{\lambda})} = \frac{1}{\mathbf{v}(\frac{1}{\lambda})}\int R_\lambda(x,dx_1)f(x_1)\left[\frac{\psi_{n-1}(\lambda,x_1)}{(\mathbf{v}(\frac{1}{\lambda}))^{n-1}} - (\mu(f))^{n-1}\right]$$

and decompose again

$$R_\lambda = R_\lambda^1 + R_\lambda^2$$

according to lemma 5.2. Assuming (+) and (++) for $n-1$, the expression $[\ldots]$ in square bracketts in (5.5) converges to 0 pointwise in x_1 as $\lambda \downarrow 0$, and is bounded by $K_{n-1} + (\mu(f))^{n-1}$. Thus $f \in \mathcal{L}^1_+(\mu)$ implies

$$x_1 \;\mapsto\; f(x_1)\,|[\ldots]| \in \mathcal{L}^1_+(\mu)$$

and the last function is weakly special since f is weakly special. Using this property, we have

$$\left|\frac{1}{\mathbf{v}(\frac{1}{\lambda})}\int R_\lambda^1(x,dx_1)f(x_1)[\ldots]\right| \leq \frac{K_{n-1}+\mu(f)^{n-1}}{\mathbf{v}(\frac{1}{\lambda})}\cdot C \to 0 \quad (\lambda\downarrow 0)$$

for all x, and dominated convergence and the definition of R_λ^2 give

$$\left|\frac{1}{\mathbf{v}(\frac{1}{\lambda})}\int R_\lambda^2(x,dx_1)f(x_1)[\ldots]\right| \leq \int \mu(dx_1)f(x_1)\cdot|[\ldots]| \to 0 \quad (\lambda\downarrow 0)$$

which implies (+) and (++) for n.

3) Assertion (i) in lemma 5.4 is proved by (++). From (5.5), we then prove by induction also assertion (ii) of 5.4, using exactly the same arguments as in step 2) above. □

For sake of completeness, we now include the proof of the Darling-Kac theorem, under assumptions (H1), (H3), and (H4), and thus in a version where the use of weakly special functions avoids Darling-Kac conditions. The principal assertion of theorems 5.6.A and 5.6.B is that if we have weak convergence of (one-dimensional marginals of) additive functionals of X, then norming functions are automatically regularly varying.

5.6.A Theorem : ([D-K 57], [B-G-T 87, ch. 8.11], [Tou 88]) Consider an additive functional $(A_t)_{t\geq 0}$ of X, μ-integrable and such that $E_\mu(A_1)>0$. If we have convergence in law under P_x

$$(*)\qquad \frac{A_t}{v(t)} \xrightarrow{w} Y$$

to some limit variable Y such that $\mathcal{L}(Y)$ is not a Dirac measure, for some norming function $v(\cdot)$ ($v: I\!R_+ \to I\!R_+$ nondecreasing, $v(t)\uparrow\infty$ as $t\to\infty$), then we have necessarily

(i) $v(t) \sim c \cdot \mathbf{v}(t)$ as $t \to \infty$ for some $c > 0$, where $\mathbf{v}$ is given by (5.1);

(ii) $\mathbf{v} \in RV_\alpha$ for some $0 \leq \alpha < 1$;

(iii) $\frac{1}{\mathbf{v}(t)} A_t \quad \xrightarrow{w} \quad E_\mu(A_1) \cdot W_1^\alpha \quad$ under P_x

where the norming constant for μ is as in (4.11'), and where $W_1^0 \sim \exp(1)$ is defined for $\alpha = 0$ in accordance with 2.8 (the special case of a Mittag-Leffler law with parameter 0).

5.6.B Theorem : Under all assumptions of 5.6.A except that the limit variable Y in $(*)$ is replaced by a constant $y \in (0, \infty)$, we have the following:

(i) $v(t) \sim c \cdot \mathbf{v}(t)$ as $t \to \infty$ for some $c > 0$, where $\mathbf{v}$ is given by (5.1);

(ii) $\mathbf{v} \in RV_1$;

(iii) $\frac{1}{\mathbf{v}(t)} A_t \quad \xrightarrow{w} \quad E_\mu(A_1) \quad$ under P_x

with norming constant of (4.11').

We add a remark before proving the theorems.

5.7 Remark : a) Remark 2.8 shows that the limiting case $\alpha = 0$ of a Mittag-Leffler process is a process with time-independent marginals on the strictly positive half axis: the process has the form $W^0 = \xi 1_{(0,\infty)}$ where ξ is exponentially distributed with parameter 1. This process is not continuous and thus will not arise in the setting of theorem 3.1 where - under the Lindeberg condition - limit processes will be continuous.

b) For $0 < \alpha \leq 1$, (4.2)-(4.4) guarantee that the norming function $\mathbf{v}$ of (5.1) in theorems 5.6.A and 5.6.B coincides (up to asymptotic equivalence) with the norming functions used in theorems 3.1 and 3.2 (or in 4.12).

c) By (i) in lemma 5.3, we can replace the norming function $\mathbf{v}$ of (5.1) by a resolvent of a special function - for an arbitrary starting point x - and thus give a version of 5.6.A and 5.6.B where a life cycle decomposition of the process X does not appear in the formulation of the theorem. This will be important for section 7.

5.8 Proof of 5.6.A and 5.6.B : By the ratio limit theorem, it is sufficient to prove the theorems for additive functionals of form

$$A_t = \int_0^t f(X_s)ds$$

where the function f is weakly special. W.l.o.g., we can take $\mu(f) = 1$. This will be assumed in the sequel. The proof, following [B-G-T 87, p. 392], is in several steps. During the first ones, we consider a limit variable Y in $(*)$ whose law (certainly concentrated on $[0, \infty)$) is not a Dirac measure at 0; this is the common assumption in 5.6.A and 5.6.B.

1) Lemma 5.4 gives

$$\int_0^\infty \lambda e^{-\lambda t} E_x \left(\frac{(A_t)^n}{(\mathbf{v}(\frac{1}{\lambda}))^n} \right) dt \longrightarrow n! \quad (\lambda \downarrow 0)$$

or after substituting $u = \lambda t$

$$\int_0^\infty e^{-u} E_x \left(\frac{(A_{u/\lambda})^n}{(\mathbf{v}(1/\lambda))^n} \right) du \longrightarrow n! \, .$$

Choose some r.v. T exponentially distributed with parameter 1, and independent of the process X. Then the last convergence is

$$\forall n: \quad E_x \left(\frac{(A_{T/\lambda})^n}{(\mathbf{v}(1/\lambda))^n} \right) \to n! \quad (\lambda \downarrow 0). \tag{5.9}$$

But $(n!)$ is the sequence of moments of the exponential law $exp(1)$ with parameter 1, which is uniquely determined by its moments: by the method of moments, we have weak convergence

$$\frac{A_{T/\lambda}}{\mathbf{v}(1/\lambda)} \longrightarrow \xi \quad (\lambda \downarrow 0)$$

where $\xi \sim exp(1)$, or

$$\int_0^\infty e^{-t} P_x \left(\frac{A_{t/\lambda}}{\mathbf{v}(1/\lambda)} \leq c \right) dt \longrightarrow 1 - e^{-c} \quad \forall c \geq 0. \tag{5.10}$$

2) We have assumed weak convergence $\frac{A_{t/\lambda}}{v(t/\lambda)} \xrightarrow{w} Y$ as $\lambda \downarrow 0$, for some r.v Y whose law is not a Dirac measure at 0. Consider functions $g^\lambda(t) := \frac{v(t/\lambda)}{\mathbf{v}(1/\lambda)}$ which are nondecreasing in t for fixed λ. Helly's selection procedure applied to families $\{g^{\lambda_n} : n \geq 0\}$ (distribution functions of σ-finite measures) allows to select for every sequence $(\lambda_n)_n$ with $\lambda_n \downarrow 0$ a subsequence $(\lambda_{n'})_{n'}$ and some non-decreasing function g taking values in $[0, \infty]$ such that at all continuity points t of g

$$\frac{v(t/\lambda_{n'})}{\mathbf{v}(1/\lambda_{n'})} \underset{(n')}{\longrightarrow} g(t). \tag{5.11}$$

We take g right-continuous. Let G denote the distribution function of Y. Along the sequence $(\lambda_{n'})_{n'}$, we write the l.h.s of (5.10) as

$$\int_0^\infty e^{-t} P_x\left(\frac{A_{t/\lambda_{n'}}}{v(t/\lambda_{n'})} \le c\frac{\mathbf{v}(1/\lambda_{n'})}{v(t/\lambda_{n'})}\right) dt. \tag{5.12}$$

At continuity points t of both G and g and along $(\lambda_{n'})_{n'}$, the probability in the integrand converges to $P(Y \le c/g(t)) = G(c/g(t))$ (with $c/0 = \infty$, $c/\infty = 0$); there are at most countably many such discontinuities. So (5.10) gives

$$\forall\, c \ge 0: \quad \int_0^\infty e^{-t} G(c/g(t)) dt = 1 - e^{-c}. \tag{5.13}$$

From (5.13) we see that g is $(0,\infty)$-valued: $g \equiv \infty$ on some half axis $[t_0, \infty)$ would imply that the integrand in (5.13) equals $G(0)e^{-t}$ on $[t_0,\infty)$ which is impossible: let $c \uparrow \infty$ in (5.13), and recall that $G(0) < 1$ by assumption. A similiar argument excludes cases where $g \equiv 0$ on some $[0, t_0)$. So g takes values in $(0,\infty)$, and (5.13) reads

$$P(Y\cdot g(T) \le c) = E(G(c/g(T))) = 1 - e^{-c}, \quad c \ge 0. \tag{5.14}$$

Here we have used that T and Y are independent since T was independent of the process X. (5.14) with $c = 0$ then shows that also Y is $(0,\infty)$-valued; thus we may take logarithms and have

$$\log Y + \log g(T) \stackrel{d}{=} \log T. \tag{5.15}$$

3) Consider characteristic functions φ_Y of $\log Y$, $\varphi_{g(T)}$ of $\log g(T)$, φ_T of $\log T$, then

$$\varphi_Y(u)\varphi_{g(T)}(u) = \varphi_T(u), \quad u \in I\!R. \tag{5.16}$$

None of these can take the value 0 since

$$\varphi_T(u) = \int x^{iu} e^{-x} dx = \Gamma(1 + iu) \neq 0, \quad u \in I\!R.$$

So $\varphi_{g(T)} = \varphi_T/\varphi_Y$ is uniquely determined from Y, so the distribution function of $g(T)$ and thus the (right-continuous) function g itself are uniquely determined from Y. In particular, g does not depend on choice of subsequences $(\lambda_{n'})_{n'}$ of sequences $(\lambda_n)_n$, so (5.11) is improved to

$$\frac{v(t/\lambda)}{\mathbf{v}(1/\lambda)} \underset{\lambda\downarrow 0}{\longrightarrow} g(t) \quad \text{for almost all } t \tag{5.17}$$

which gives

$$\frac{v(t/\lambda)}{v(1/\lambda)} \underset{\lambda\downarrow 0}{\longrightarrow} \frac{g(t)}{g(1)} \quad \text{for almost all } t\ . \tag{5.18}$$

But (5.17) and (5.18) imply

$$(5.19) \qquad v \in RV_\alpha, \quad g(t) = g(1)t^\alpha, \quad \mathbf{v} \in RV_\alpha, \quad v(t) \sim g(1)\mathbf{v}(t) \quad \text{as} \quad t \uparrow \infty$$

for some $\alpha \in I\!\!R$; necessarily $\alpha \geq 0$ since g is nondecreasing.

4) It remains to show that not all cases $\alpha \geq 0$ can occur in (5.19), and to identify the limit law $\mathcal{L}(Y)$; by (5.17) - (5.19) and in virtue of (5.14), we have specified the initial assumption on convergence in law to

$$(5.20) \qquad \frac{A_t}{\mathbf{v}(t)} \stackrel{w}{\to} g(1)Y \quad \text{as } t \uparrow \infty,$$

for some r.v Y concentrated on $(0, \infty)$.

i) If $\alpha = 0$, then the function g is constant by (5.19), so (5.14) shows that $g(1)Y$ has law $exp(1)$.

ii) Consider the case $0 < \alpha < 1$. Since $A_t = \int_0^t f(X_s)ds$ where f is special and $\mu(f) = 1$, we apply lemma 5.4 a) which gives the asymptotics as $\lambda \downarrow 0$ of the Laplace transform $\widehat{U}_n$ of the measure $U_n(ds) := M_n(s,x)ds$:

$$\widehat{U}_n(\lambda) \quad \stackrel{\lambda \downarrow 0}{\sim} \quad \frac{1}{\lambda} n! \left(\mathbf{v}(\frac{1}{\lambda}) \right)^n .$$

Since $\mathbf{v} \in RV_\alpha$, the Tauberian theorem ([B-G-T 87, p. 37]) gives

$$U_n([0,t]) \quad \stackrel{t \uparrow \infty}{\sim} \quad \frac{t\, n!\, (\mathbf{v}(t))^n}{\Gamma(2 + \alpha n)};$$

since $M_n(\cdot, x)$ is by definition monotone, the monotone density theorem ([B-G-T 87, p. 39]) shows

$$M_n(t,x) \quad \stackrel{t \uparrow \infty}{\sim} \quad \frac{n!\, (\mathbf{v}(t))^n}{\Gamma(1 + \alpha n)}$$

which we write in the form

$$(5.21) \qquad E_x \left(\left(\frac{A_t}{\mathbf{v}(t)} \right)^n \right) \longrightarrow \frac{n!}{\Gamma(1 + \alpha n)}, \quad t \to \infty$$

for arbitrary $n \in I\!\!N$. On the r.h.s of (5.21), we find the sequence of moments of the Mittag-Leffler variable W_1^α, cf. 2.8: so the method of moments gives convergence in law under P_x

$$(5.22) \qquad \frac{A_t}{\mathbf{v}(t)} \stackrel{w}{\to} W_1^\alpha, \quad t \to \infty$$

and thus specifies the limit law in (5.20).

iii) We show that under the assumptions of theorem 5.6.A, other cases $\alpha > 0$ except $0 < \alpha < 1$ are impossible. Indeed, the ratio limit theorem and (5.20) - where $\mu(f) = 1$ - imply convergence in law

$$(5.23) \qquad \frac{N_t}{\mathbf{v}(t)} \stackrel{w}{\to} g(1)Y, \quad t \to \infty,$$

with notations as in the proof of theorem 4.12. $\mathbf{v}$ being regularly varying by (5.19) with positive index, the arguments in step 3) of the proof of 4.12 show that we have weak convergence of $\frac{R_n}{a(n)}$ as $n \to \infty$ to some limit law which is concentrated on $(0, \infty)$ and which is not a Dirac measure; here $a(\cdot)$ is an asymptotic inverse of $\mathbf{v}$. Then theorem 2.4.A combined with (4.2)-(4.4) and (5.1) show that the index α of regular variation of $\mathbf{v}$ is necessarily in $(0, 1)$. So all assertions of theorem 5.6.A are proved.

iv) We show that under the assumptions of theorem 5.6.B, all cases $\alpha \neq 1$ are impossible. Steps i) and ii) above exclude $0 \leq \alpha < 1$. With the same arguments as in iii) except that $\frac{R_n}{a(n)}$ now converges in probability as $n \to \infty$ to some stricly positive constant, we apply theorem 2.4.B combined with (4.2)-(4.4) and (5.1) to show that the index α of regular variation of $\mathbf{v}$ necessarily equals 1. Then g in (5.19) is linear, so $g(1)Y = 1$ by (5.14), and all assertions of theorem 5.6.B are proved. $\square$

Now we turn to convergence of martingales $M \in \mathcal{M}^{2,\mathrm{loc}}(P_x, \mathbb{F})$, on a space $(\Omega, \mathcal{A}, \mathbb{F})$ as in section 1. Theorems 5.6.A and 5.6.B contain one essential argument for the proof of the 'necessary' part of theorem 3.1; the other is the following.

5.24 Theorem : Consider $M \in \mathcal{M}^2_{\mathrm{loc}}(P_x, \mathbb{F})$ whose angle and square brackett are μ-integrable additive functionals of X. For some norming function $v(\cdot)$, let

$$M^n = \left(\frac{1}{\sqrt{v(n)}} M_{tn} \right)_{t \geq 0}$$

converge (weakly in $D(\mathbb{R}_+, \mathbb{R})$, under P_x, as $n \to \infty$) to some limit process $W = (W_t)_{t \geq 0}$ such that $W_0 = 0$ and $\mathcal{L}(W_1)$ is not the Dirac measure at 0, and assume that the sequence $(M^n)_n$ satisfies the Lindeberg condition

$$(5.25) \quad \frac{1}{v(n)} \int_0^{tn} \int |x|^2 \, 1_{\{|x| > \varepsilon \sqrt{v(n)}\}} \nu(ds, dx) \;\longrightarrow\; 0 \quad \text{in } P_x\text{-probability, for all } t \text{, all } \varepsilon > 0$$

where $\nu(ds, dx)$ is the P_x-compensator of the point process of jumps of M. Then the limit process W is a continuous local martingale with respect to its own filtration, and we have

$$(5.26) \qquad (M^n, [M^n]) \;\longrightarrow\; (W, < W >) \quad , \quad (M^n, < M^n >) \;\longrightarrow\; (W, < W >)$$

(weak convergence in $D(\mathbb{R}_+, \mathbb{R} \times \mathbb{R})$, under P_x, as $n \to \infty$).

Proof : We decompose $M^n = M^{n,1} + M^{n,2}$ where $M^{n,1}$ has bounded jumps $|\Delta M^{n,1}| \leq b$ and where $M^{n,2}$ is the compensated sum of 'big' (i.e. $|\Delta M^n| > b$) jumps of M^n. Then the Lindeberg condition (3.25) implies

$$P\left(\sup_{s\leq T}|M_s^{n,2}| > \varepsilon\right) \to 0 \quad (n\to\infty) \quad \forall\, T>0$$

and thus

$$M^{n,2} \xrightarrow{w} 0 \quad , \quad M^{n,1} \xrightarrow{w} W$$

(weak convergence in $D(\mathbb{R}_+, \mathbb{R})$, under P_x, as $n\to\infty$). Then by [J-Sh 87, VI.3.26], the weak limit W is a continuous process. By [J-Sh 87, IX.1.19], since $M^{n,1}$ has bounded jumps, W is a local martingale with respect to its own filtration (let W be defined on some $(\Omega', \mathcal{A}', P')$, consider the filtration $\mathbb{F}'$ generated by W). So W has bracketts $< W >= [W]$. Again by boundedness of jumps of $M^{n,1}$, [J-Sh 87, VI.6.1] gives

$$(M^n, [M^n]) \quad \longrightarrow \quad (W, < W >)$$

(weak convergence in $D(\mathbb{R}_+, \mathbb{R}\times\mathbb{R})$, under P_x, as $n\to\infty$). In this last assertion, square bracketts can be replaced by angle bracketts

$$(M^n, < M^n >) \quad \longrightarrow \quad (W, < W >)$$

since $< M >$, $[M]$ are additive functionals of X having the same expected increment over life cycles of X: this is again the RLT combined with the argument of step 3) in the proof of 4.12 that weak convergence of increasing processes to a continuous increasing process is equivalent to convergence of finite dimensional marginals. □

5.27 Proof of theorem 3.1, 'necessary' condition : Consider $(M^n)_n$ as in 5.24. We have to prove that if M^n converges weakly in $D(\mathbb{R}_+, \mathbb{R})$ under P_x as $n\to\infty$ to some limit process $W = (W_t)_{t\geq 0}$ such that $W_0 = 0$ and $\mathcal{L}(W_1)$ is not the Dirac measure at 0, then necessarily the tails of life-cycle length distributions in the process X are regulary varying as stated in theorem 3.1. First we apply 5.24: W is continuous and a local martingale, and we have convergence in law

$$\frac{1}{v(n)} < M >_n \longrightarrow < W >_1$$

as $n\to\infty$ under P_x, where also $\mathcal{L}(< W >_1)$ is not a Dirac measure at 0. Then 5.6.A and 5.6.B apply to show the following: if $< W >_1$ is a.s. constant, then we have $\mathbf{v} \in RV_1$ and

$$\frac{1}{\mathbf{v}(n)} < M >_n \longrightarrow J$$

where $\mathbf{v}$ is given by (5.1) and $J = E(< M >_{R_2} - < M >_{R_1})$; if $< W >_1$ is not a.s. constant, then $\mathbf{v} \in RV_\alpha$ and

$$\frac{1}{\mathbf{v}(n)} < M >_n \longrightarrow J W_1^\alpha$$

for some $0 \le \alpha < 1$. It remains to exclude the case $\alpha = 0$: if the norming function $\mathbf{v}$ is slowly varying at ∞, then $< W >_\lambda$, the limit in law of $\frac{1}{\mathbf{v}(n)} < M >_{\lambda n}$, and $< W >_1$, the limit in law of $\frac{1}{\mathbf{v}(\lambda n)} < M >_{\lambda n}$, necessarily have the same law $exp(1)$; since $< W >$ is increasing, its paths must be constant on $(0,\infty)$; but $< W >$ is continuous on $[0,\infty)$ with $< W >_0 = 0$ which is a contradiction. So $0 < \alpha < 1$, and by (5.1) and (4.2)-(4.4), regular variation of the norming function $\mathbf{v}$ is translated into regular variation of tails of life-cycle length distributions of X as specified in theorem 3.1. $\square$

The 'necessary' part of theorem 3.1 in subsection 3.1 is thus proved, under assumption (H1), (H3) and (H4) for the process, and under conditions much weaker than (H5^A)+(H5^B) on the martingales under consideration. It remains to prove proposition 3.4 which gives a sufficient condition – in terms of upper bounds for the life cycle variable R_1 – for existence of weakly special functions for X and R_1. In fact, the only assumption which we need for this is (H1).

5.28 Proof of proposition 3.4 : We assume only Harris recurrence (H1) of the process X. The proof is in several steps.

1) Consider first the process $X = (X_t)_{t\ge 0}$ at jump times τ_n of an independent Poisson process with rate 1: by theorem 1.4, we have (H2), i.e. the discrete-time process $\overline{X} = (X_{\tau_n})_{n\in\mathbb{N}}$ is Harris with invariant measure μ. Revuz terms $f : E \to \mathbb{R}_+$ a *special function* for $\overline{X}$ (see [Re 75, p. 182, p. 48]) if f is $\mathcal{E}$-measurable and if

$$x \quad \to \quad E_x\left(\sum_{n=1}^{\infty}(1-h(X_{\tau_1}))\cdots(1-h(X_{\tau_{n-1}}))f(X_{\tau_n})\right)$$

is bounded in $x \in E$, for every $h \in \mathcal{U}^+$ having $\mu(h) > 0$; here $\mathcal{U}^+$ denotes the set of $\mathcal{E}$-measurable functions h on E with $0 \le h(\cdot) \le 1$. Special functions of $\overline{X}$ do exist, see [Re 75, 6.4.3 and 6.4.6]; the set of special functions forms a convex cone in $L^1(\mu)$ ([Re 75, 6.4.2]); thus in particular special functions exist which are bounded.

2) We prove that for $h \in \mathcal{U}^+$ with $\mu(h) > 0$ and $f \ge 0$ measurable, one has

$$(5.29)\quad E_x\left(\sum_{n=1}^{\infty}(1-h(X_{\tau_1}))\cdots(1-h(X_{\tau_{n-1}}))f(X_{\tau_n})\right) = E_x\left(\int_0^\infty f(X_t)e^{-\int_0^t h(X_s)\,ds}dt\right).$$

Indeed, τ_n has law $\Gamma(n,1)$, and $\left(\frac{\tau_1}{\tau_n},\ldots,\frac{\tau_{n-1}}{\tau_n}\right)$ is independent of τ_n and distributed as the order statistics of $n-1$ uniform r.v.'s on $(0,1)$; thus the summands on the l.h.s of (5.29) are

$$\begin{aligned} &E_x\left((1-h(X_{\tau_1}))\cdots(1-h(X_{\tau_{n-1}}))f(X\tau_n)\right) \\ &= E_x\left(\int_0^\infty dt\, e^{-t}\, f(X_t)\, \int_0^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{n-2}}^t dt_{n-1}\, e^{\sum_{i=1}^{n-1} g(X_{t_i})}\right) \end{aligned}$$

with $g := \log(1-h)$, for $n \in I\!N$. Fix some t and define functions $S_m(\cdot) = S_m^t(\cdot)$ on $[0,t]$ by

$$S_0(r) \equiv 1, \quad S_1(r) := \int_r^t dr' e^{g(X_{r'})}, \quad S_m(r) := \int_r^t dt_1 \int_{t_1}^t dt_2 \ldots \int_{t_{m-1}}^t dt_m\, e^{\sum_{i=1}^{m} g(X_{t_i})}, \quad m \geq 2.$$

Note that $S_m(r) \leq \frac{t^m}{m!}$ and that $S_m(r) = \int_r^t dr' e^{g(X_{r'})} S_{m-1}(r')$ for $m \geq 1$. Defining $S(\cdot) = S^t(\cdot) := \sum_{m\geq 0} S_m(\cdot)$ on $[0,t]$, we have $\frac{d}{dr}S(r) = -e^{g(X_r)}S(r)$ and $S^t(t) = 1$, thus

$$S^t(r) \;=\; \sum_{m\geq 0} S_m(r) \;=\; e^{\int_r^t (1-h)(X_{r'})dr'}, \quad 0 \leq r \leq t.$$

As a consequence, we have written the l.h.s of (5.29) as

$$E_x\left(\int_0^\infty dt\, e^{-t} f(X_t) \sum_{n=1}^\infty S_{n-1}^t(0)\right) \quad = \quad E_x\left(\int_0^\infty dt\, f(X_t)\, e^{-\int_0^t h(X_s)\, ds}\right)$$

which is the assertion.

3) We give an interpretation of the r.h.s of (5.29) in terms of position-dependent killing of the strong Markov process $X = (X_t)_{t\geq 0}$ at rate $h \in \mathcal{U}^+$ with $\mu(h) > 0$: given that X has not been killed up to time r, it will be killed in a small time interval $(r, r+\epsilon]$ with probability $\epsilon\, h(X_r) + o(\epsilon)$. First, for h bounded away from 0 and for f bounded, partial integration

$$E_x\left(\int_0^\infty f(X_t) e^{-\int_0^t h(X_s)\, ds} dt\right) = E_x\left(\int_0^\infty dr\, h(X_r) e^{-\int_0^r h(X_s)\, ds} \int_0^r f(X_v)\, dv\right),$$

allows to write the r.h.s of (5.29) as

$$E_x\left(\int_0^{\widehat{T}_h} f(X_t)\, dt\right)$$

where $\widehat{T}_h$ is the killing time, defined on an extension of the stochastic basis $(\Omega, \mathcal{A}, I\!F)$. Second, stochastic ordering of $\widehat{T}_{h_n}$ as $h_n \downarrow h$ and monotone convergence show

$$E_x\left(\int_0^\infty f(X_t) e^{-\int_0^t h(X_s)\, ds} dt\right) = E_x\left(\int_0^{\widehat{T}_h} f(X_t)\, dt\right)$$

for arbitrary $h \in \mathcal{U}^+$ with $\mu(h) > 0$ and $f \geq 0$; note that $\widehat{T}_h < \infty$ P_x-a.s for all $x \in E$ since $\mu(h) > 0$.

4) Write $\widehat{T}_B$ if $h = 1_B$, for $B \in \mathcal{E}$ with $\mu(B) > 0$, and let S_B denote the first entry time of the discrete chain $\overline{X}$ to B: then (5.29) yields

$$E_x\left(\sum_{n=1}^{S_B} f(X_{\tau_n})\right) = E_x\left(\int_0^{\widehat{T}_B} f(X_t)\,dt\right), \quad x \in E. \tag{5.30}$$

If f is a special function of $\overline{X}$ as in 1), the expressions in (5.29), for $h \in \mathcal{U}^+$ with $\mu(h) > 0$, and in (5.30), for $B \in \mathcal{E}$ with $\mu(B) > 0$, are bounded functions of $x \in E$. From now on, we will omit the reference to $\overline{X}$ and speak for short during this proof of special functions.

Consider a first entry time T_B to B

$$T_B := \inf\{t > 0 : \int_0^t 1_B(X_s)ds > 0\} \ \leq \ \widehat{T}_B;$$

then T_B is a $\mathbb{F}$-stopping time, and by construction, between T_B and $\widehat{T}_B$, the process X has to spend an independent exponential time in the set B. In particular, for $B = E$, $S := \widehat{T}_E$ is an independent exponential time. Comparison with (5.30) shows: if f is special, then

$$E_x\left(\int_0^{T_B} f(X_t)\,dt\right), \quad E_x\left(\int_0^{\widehat{T}_B} f(X_t)\,dt\right), \quad E_x\left(\int_0^{S} f(X_t)\,dt\right) \tag{5.31}$$

($B \in \mathcal{E}$ with $\mu(B) > 0$) are bounded functions of $x \in E$.

5) Consider now a recurrent atom $A \in \mathcal{E}$ for X and a life cycle decomposition $(R_n)_n$ as in 1.9.A+B such that R_1 has the form specified in proposition 3.4:

$$R_1 \ \leq \ S_0 + (\max_{1\leq j\leq l} T_{B_j}) \circ \vartheta_{S_0}\,, \quad S_0 \ \leq \ \max_{1\leq i\leq k} \widehat{T}_{h_i} \tag{5.32}$$

where $B_j \in \mathcal{E}$ have positive invariant measure $\mu(B_j) > 0$, $1 \leq j \leq l$, and where h_i are $\mathcal{E}$-measurable, $[0,1]$-valued, with $\mu(h_i) > 0$. By the strong Markov property, for f special and bounded,

$$x \longrightarrow E_x\left(\int_0^{R_1} f(X_t)\,dt\right) \ \leq \ E_x\left(\int_0^{S_0} f(X_t)\,dt + \sum_{j=1}^{l} E_{X_{S_0}}\left(\int_0^{\widehat{T}_{B_j}} f(X_v)\,dv\right)\right) \ :$$

using (5.31), this is is again a bounded function in $x \in E$. Thus we have proved that for R_1 meeting (5.32), special functions for X are weakly special for X and R_1. This is the assertion of proposition 3.4. $\square$

6 Nummelin splitting in discrete time

The results of subsection 3.1 were formulated under the assumption that a Harris process $X = (X_t)_{t\geq 0}$ has a recurrent atom A such that suitably defined exit times $(R_n)_n$ from this atom decompose the trajectory of X into a sequence of i.i.d life cycles. Unfortunately, many interesting processes X do not possess such recurrent atoms.

Nummelin ([Num 78]) showed that discrete-time Harris chains can be embedded as first component into a a two-dimensional Harris chain (the 'split' chain) where the second component introduces a recurrent atom of positive mass. As a preparation to section 7, we retrace the approach of Nummelin in case of discrete time.

In this section, we consider a Markov chain $Y = (Y_n)_{n\in I\!N_0}$ taking values in a Polish space $(E, \mathcal{E})$, with one-step transition kernel $P(x, dy)$, and assume that Y is Harris with invariant measure μ. Nummelin used the following minorization assumption(s) (M_k), $k \in I\!N$.

6.1 Minorization assumption (M_k) **:** There is some $\mathcal{E}$-measurable function $h : E \to [0, 1]$ with $\mu(h) > 0$ and some probability measure ν on $(E, \mathcal{E})$ such that

$$P_k(x, A) \geq h(x)\nu(A) \quad \forall\, x \in E, \quad \forall\, A \in \mathcal{E} \ .$$

For our purposes, the discrete time chain Y of interest will be $\overline{X} = (X_{\tau_n})_n$, i.e. the continuous-time Harris process $X = (X_t)_{t\geq 0}$ evaluated after independent exponential waiting times; hence $P(x, dy)$ will be the potential kernel $U^1(x, dy)$ and μ the invariant measure of X, cf. proof of theorem 1.2 and theorem 1.4. The main result of the this section is proposition 6.7: it states that in case $P(x, dy) = U^1(x, dy)$, the minorization assumption (M_1) is automatically satisfied.

Under (M_1), Nummelin splitting transforms the state space $(E, \mathcal{E})$, measures λ on $(E, \mathcal{E})$, transition probabilities $P(\cdot,\cdot)$ on $(E,\mathcal{E})$, ..., as follows. For points $x \in E$ and sets $A \in \mathcal{E}$, split

$$\begin{array}{ccl} & & x_0 := (x,0) \in E_0 := E \times \{0\} \\ & \nearrow & \\ E \ni x & & \\ & \searrow & \\ & & x_1 := (x,1) \in E_1 := E \times \{1\} \end{array}$$

$$\begin{array}{ccl} & & A_0 := A \times \{0\} \subset E_0 \\ & \nearrow & \\ \mathcal{E} \ni A & & \\ & \searrow & \\ & & A_1 := A \times \{1\} \subset E_1 \ . \end{array}$$

We write

$$E^* := E_0 \,\dot{\cup}\, E_1, \quad \mathcal{E}^* = \sigma(A_0, A_1 : A \in \mathcal{E})$$

and identify sets $A \in \mathcal{E}$ with their pre-image under the projection from E^* to E:

$$\mathcal{E} \ni A \quad \longleftrightarrow \quad A \times \{0,1\} \in \mathcal{E}^*.$$

By (M_1) with h as there, a σ-finite measure λ on $(E,\mathcal{E})$ splits according to

$$\begin{array}{cclcl} & & \int 1_A(x)h(x)\lambda(dx) & =: & \lambda^*(A_1) \\ & \nearrow & & & \\ \lambda(A) & & & & \\ & \searrow & & & \\ & & \int 1_A(x)(1-h)(x)\lambda(dx) & =: & \lambda^*(A_0) \ , \end{array}$$

$A \in \mathcal{E}$; this defines a σ-finite measure λ^* on $(E^*,\mathcal{E}^*)$ such that

$$\lambda^*(A \times \{0,1\}) = \lambda(A), \quad A \in \mathcal{E}.$$

Identifying $A \in \mathcal{E}$ with $A \times \{0,1\} \in \mathcal{E}^*$ as above, we write again λ for the restriction of λ^* to the sub-σ-field $\{A \times \{0,1\} : A \in \mathcal{E}\}$ of $\mathcal{E}^*$. Extending $\mathcal{E}$-measurable $f : E \to I\!R$ to $(E^*,\mathcal{E}^*)$ via

$$f(x_0) := f(x) =: f(x_1), \quad x \in E,$$

we may consider integrals

$$\int_E f\, d\lambda = \int_{E^*} f\, d\lambda$$

without distinction on either $(E,\mathcal{E})$ or $(E^*,\mathcal{E}^*)$.

Next, one uses (M_1) and the kernel $h \otimes \nu(x, dy) := h(x)\nu(dy)$ to transform the transition kernel $P(\cdot,\cdot)$ on $(E, \mathcal{E})$. The aim is to define a transition probability $P^*(\cdot,\cdot)$ on $(E^*, \mathcal{E}^*)$ such that

i) the original Markov chain $Y = (Y_n)_{n \in \mathbb{N}_0}$ evolving under P is embedded as first component into a new chain $Y^* = (Y_n^*)_{n \in \mathbb{N}_0}$ on $(E^*, \mathcal{E}^*)$ evolving under $P^*(\cdot,\cdot)$: we then write $Y_n^* = (Y_n, \varepsilon_n)$;

ii) transitions away from points $x_1 = (x, 1) \in E_1$ do no longer keep trace of the first component x of points in E_1: thus $E_1 \in \mathcal{E}^*$ will become an atom for Y^*.

To get i) and ii), one has to solve

$$
\begin{array}{ccccc}
E_0 & & E & & E_1 \\[2ex]
\lambda^*(dx_0) = (1-h)(x)\lambda(dx) & \longleftarrow & \lambda(dx) & \longrightarrow & h(x)\lambda(dx) = \lambda^*(dx_1) \\[2ex]
P^*(x_0, dy) & \searrow & & \swarrow & P^*(x_1, dy) := \nu(dy) \\[2ex]
\ldots & \longleftarrow & \lambda(dx)P(x, dy) & \longrightarrow & \ldots
\end{array}
$$

which – noticing that $\nu(dy) = \frac{1}{h(x)}(h \otimes \nu)(x, dy)$ whenever $h(x) > 0$ – leads to a kernel $P^*(\cdot,\cdot)$ defined for points x^* in $E^* = E \times \{0, 1\}$ by

$$
P^*(x_i, dy) = \begin{cases} \frac{1}{1-h(x)}(P - h \otimes \nu)(x, dy) & \text{if } i = 0 \text{ and } h(x) < 1 \\ \nu(dy) & \text{else .} \end{cases}
$$

So far, we have defined $P^*(x^*, dy)$ as a transition probability from $(E^*, \mathcal{E}^*)$ to $(E, \mathcal{E})$: it remains to split all measures $P^*(x^*, dy)$, $x^* \in E^*$, according to the above rule (from y to $y_0 = (y, 0)$ with probability $1-h(y)$, and to $y_1 = (y, 1)$ with probability $h(y)$) to define the desired transition kernel $P^*(x^*, dy^*)$ on $(E^*, \mathcal{E}^*)$.

Resuming this discussion, we obtain

6.2 Proposition : Consider a discrete time chain $Y = (Y_n)_{n \in \mathbb{N}_0}$ with one-step transition kernel $P(\cdot,\cdot)$ on $(E, \mathcal{E})$ satisfying (M_1), with arbitrary initial distribution λ. Consider a chain $(Y_n^*)_n$ on $(E^*, \mathcal{E}^*)$ with one-step transition kernel $P^*(\cdot,\cdot)$ as defined above, and with starting law λ^*.

(i) For arbitrary $N \geq 1$ and $A_n \in \mathcal{E}$, $0 \leq n \leq N$, we have

$$
P_\lambda(Y_n \in A_n,\, 0 \leq n \leq N) = P_{\lambda^*}(Y_n^* \in A_n \times \{0,1\},\, 1 \leq n \leq N)
$$

thus the first component of Y^* is equal in law to the original chain Y. (Moreover, we may construct Y jointly with Y^* such that Y *is* the first component of $Y^* = (Y_n, \varepsilon_n)_n$, the second component $(\varepsilon_n)_n$ taking values in $\{0,1\}$.)

(ii) If Y is Harris with invariant measure μ, then Y^* is Harris with invariant measure μ^*:

$$\mu^*(A_1) = \int_A h(x)\mu(dx)\,, \quad \mu^*(A_0) = \int_A (1-h)(x)\mu(dx)\,, \quad A \in \mathcal{E}\,.$$

(iii) E_1 is an atom for Y^* having $\mu^*(E_1) = \int_E h(x)\mu(dx) > 0$.

To apply 6.2, we have to be able to check the minorization condition (M_1).

6.3 Remark : In some cases one has explicit densities $p(\cdot,\cdot)$, $\mathcal{E}\otimes\mathcal{E}$-measurable,

$$P(x,dy) = p(x,y)m(dy), \quad x,y \in E$$

with respect to some σ-finite measure m on $(E,\mathcal{E})$ which is equivalent to the invariant measure μ, and one can specify some set $C \in \mathcal{E}$ having

$$\begin{cases} \inf_{(x,y)\in C\times C} p(x,y) > 0 \\ m(C) > 0 \quad \text{(w.l.o.g also } m(C) < 1) \end{cases}$$

(thus C will be visited infinitely often, and also transitions $C \to C$ will occur infinitly often): then for $x \in E$, $A \in \mathcal{E}$

$$\begin{aligned} P(x,A) &\geq P(x, A\cap C) = \int 1_{A\cap C}(y)\; p(x,y)m(dy) \\ &\geq 1_C(x)\left[\inf_{(x,y)\in C\times C} p(x,y)\right] m(A\cap C) \\ &\geq h(x)\,\nu(A) \;=\; (h\otimes\nu)(x,A) \end{aligned}$$

where the function h and the probability measure ν are given in terms of the set C alone

$$\begin{cases} h := 1_C\left(\inf_{(x,y)\in C\times C} p(x,y)\wedge 1\right) m(C) \;=\; \alpha\, 1_C \\ \nu := m(\cdot\cap C)/m(C) \end{cases}$$

for some $\alpha \in (0,1)$. In this case, the minorization condition (M_1) holds in a very particular form, with h and ν determined from C. □

This leads to the following sharper form of minorization conditions (M_k), $k \geq 1$:

6.4 Minorization assumption $(\widetilde{M_k})$ **:** There is some set $C \in \mathcal{E}$ with $\mu(C) > 0$, some probability measure ν on $(E, \mathcal{E})$ equivalent to $\mu(\cdot \cap C)$, some function h of form $\alpha \, 1_C$, $\alpha \in (0,1)$, such that

$$P_k(x, A) \geq h(x)\nu(A) \quad \forall\, x \in E, \quad \forall\, A \in \mathcal{E} \ .$$

We quote the following result from Revuz [Rev 75].

6.5 Proposition : ([Rev 75, p. 160]) Consider a Harris chain $Y = (Y_n)_{n \in I\!N_0}$ taking values in $(E, \mathcal{E})$, $\mathcal{E}$ countably generated, with one-step transition kernel $P(x, dy)$ and invariant measure μ. Let m denote a probability measure on $(E, \mathcal{E})$ which is equivalent to μ. Then there is a family of Lebesgue decompositions of k-step transition probabilities $P_k(x, \cdot)$ with respect to m

$$P_k(x, dy) = p_k(x, y) m(dy) + \check{P}_k(x, dy) \ , \quad x, y \in E, \quad k \geq 1$$

with the following properties: $p_k(\cdot, \cdot)$ is $\mathcal{E} \otimes \mathcal{E}$-measurable for all $k \geq 1$, and there is some set $C \in \mathcal{E}$ with $m(C) > 0$ and some integer $k \in I\!N$ such that $\inf\limits_{(x,y) \in C \times C} p_k(x, y) > 0$.

As a consequence, arguing exactly as in remark 6.3 above, we deduce from 6.5:

6.6 Proposition : Consider a Harris chain $Y = (Y_n)_{n \in I\!N_0}$ taking values in $(E, \mathcal{E})$, $\mathcal{E}$ countably generated, with one-step transition kernel $P(x, dy)$. Then there is some $k \geq 1$ such that the minorization condition $(\widetilde{M_k})$ is satisfied.

We apply this to the special situation of interest for us.

6.7 Proposition : Consider $X = (X_t)_{t \geq 0}$, a continuous-time Harris process with semigroup $(P_t(\cdot, \cdot))_{t \geq 0}$ and invariant measure μ, taking values in a Polish space $(E, \mathcal{E})$.
Then for all $0 < \alpha < \infty$, the transition kernels

$$\alpha U^\alpha(x, dy) = \int_0^\infty \alpha e^{-\alpha t} P_t(x, dy) dt$$

satisfy a minorization condition $(\widetilde{M_1})$.

Proof : Consider $\overline{X}^\alpha = (X_{\rho_n})_{n\geq 0}$ where $\rho_{n+1}-\rho_n$, $n \geq 0$, are i.i.d $exp(\alpha)$-waiting times independent of X, $\rho_0 = 0$. Then ρ_n has law $\Gamma(n,\alpha)$, and a mixture formula for Gamma laws gives

$$\sum_{n=0}^{\infty}(1-q)q^n\Gamma(n+1,\alpha) = \Gamma(1,\alpha(1-q))$$

for arbitrary $0 < q < 1$. Thus we have

$$\alpha(1-q)U^{\alpha(1-q)} = \sum_{n=0}^{\infty}(1-q)q^n \int_0^\infty \Gamma(n+1,\alpha)(dt)P_t = \sum_{n=0}^{\infty}(1-q)q^n \left(\alpha U^\alpha\right)^{n+1} .$$

Since $X = (X_t)_{t\geq 0}$ is by assumption Harris, we know from theorem 1.4 that (H2$^\alpha$) holds for arbitrary $0 < \alpha < \infty$: thus $\overline{X}^\alpha$ with one-step transition kernel αU^α is by assumption Harris. Then proposition 6.6 yields that at least one of the kernels $(\alpha U^\alpha)^n$, $n \geq 1$, satisfies a minorization condition $(\widetilde{M}_1)$. So we have a minorization condition $(\widetilde{M}_1)$ also for $\alpha' U^{\alpha'}$ where $\alpha' = \alpha(1-q)$. Since α and q were arbitrary, this proves the assertion. $\square$

7 Nummelin-like splitting for general continuous time Harris processes and proofs for subsection 3.3

The results of subsection 3.1 were formulated under the assumption that the Harris process $X = (X_t)_{t\geq 0}$ under consideration has a life cycle decomposition. This restrictive assumption will be removed now, and we will prove the general results 'without life cycles' of subsection 3.3. Touati ([Tou 88]) used Nummelin splitting to argue that for Harris processes $X = (X_t)_{t\geq 0}$ with Polish state space, life cycles may always be introduced artificially: he thus could state the main theorem of section 3 without explicit reference to concrete life cycles of X, giving by the way the result in its most general form. Using quite different arguments, we will prove this result in the present section (theorems 7.16 and 7.20 below).

The setting is the following: we consider a continuous-time strong Markov process $X = (X_t)_{t\geq 0}$ with semigroup $(P_t(\cdot,\cdot))_{t\geq 0}$, taking values in a Polish space $(E,\mathcal{E})$, and with càdlàg paths. Slightly more restrictive than in section 1, we take X as canonical process on $(\Omega, \mathcal{A}, I\!\!F)$, where Ω is the Skorohod space $D(I\!\!R_+, E)$ with canonical σ-field and with canonical filtration; we have shifts $(\vartheta_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, I\!\!F)$ (note that for results on weak convergence of stochastic processes, this choice is no restriction of generality). We do not assume more than

(H1): $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ.

By theorem 1.4, we know that (H1) implies the property
(H2): $\overline{X} = (X_{\sigma_n})_{n\geq 0}$ is Harris, with $\sigma_n - \sigma_{n-1}$ iid $exp(1)$-waiting times independent of X
and that in virtue of proposition 6.7 the following holds:
(H6): The one-step transition kernel $U^1(\cdot,\cdot)$ of $\overline{X}$ satisfies a minorization condition $(\widetilde{M}_1)$: there is some set $C \in \mathcal{E}$ with $\mu(C) > 0$, some probability measure ν on $(E,\mathcal{E})$ equivalent to $\mu(\cdot\cap C)$, and some $0 < \alpha < 1$ such that $U^1(x,dy) \geq \alpha 1_C(x)\nu(dy)$, for all $x, y \in E$.

We start with embedding X as first component into a richer Harris process $\check{X} = (\check{X}_t)_{t\geq 0}$. To $\check{X}$ we will associate processes $\widetilde{X}^m$ which - close to $\check{X}$ if m is large - can be equipped with a recurrent atom $\widetilde{A}^m$ and a life cycle decomposition $(\widetilde{R}^m_n)_n$. Then the idea is as follows: shifting additive functionals of X to $\widetilde{X}^m$ by means of ratio limit theorems, we can apply theorem 3.1

in $\widetilde{X}^m$ to prove that 'additive functionals of X converge as if X had life cycles', where norming function and limiting process are now determined from regular variation at 0 of the resolvent of X.

7.1 The process $\check{X}$: Prepare i.i.d. exponential times $\tau_n, n \geq 1$, and i.i.d. random variables U_n, V_n, $n \geq 1$, uniformly distributed on $(0,1)$, all independent and independent of X. Write $T_n := \tau_1 + \cdots + \tau_n$, $n \geq 1$, $T_0 := 0$. Define the process $\check{X}$ by

$$\check{X}_t := (X_t, N_t)\,, \quad N_t = (N_t^1, N_t^2, N_t^3) := 1_{[[0,T_1[[}(t)(z,u,v) + \sum_{n\geq 1} 1_{[[T_n,T_{n+1}[[}(t)(X_{T_n}, U_n, V_n)$$

$t \geq 0$, under initial conditions $\check{X}_0 = (x,z,u,v)$. $\check{E} := E\times E\times[0,1]\times[0,1]$ is the state space for $\check{X}$, equipped with Borel-σ-field $\check{\mathcal{E}}$.

$\check{X}$ is defined on a standard extension $(\check{\Omega}, \check{\mathcal{A}}, \check{\mathbb{F}}, (P_{\check{x}})_{\check{x}\in\check{E}})$ of the original space $(\Omega, \mathcal{A}, \mathbb{F}, (P_x)_{x\in E})$: for this extension, we take also N as canonical process on its canonical path space, the set of all right-continuous piecewise constant functions $\mathbb{R}_+ \to E\times[0,1]\times[0,1]$, with canonical σ-field and canonical filtration; without ambiguity, we write again $(\vartheta_t)_{t\geq 0}$ for the shifts on $(\check{\Omega}, \check{\mathcal{A}}, \check{\mathbb{F}})$. By construction, $\check{X}$ is then the canonical process on $(\check{\Omega}, \check{\mathcal{A}}, \check{\mathbb{F}})$, $\check{\mathbb{F}}$ the filtration generated by $\check{X}$, and the original process X appears now as first component of $\check{X}$.

Jumps in the N-component of $\check{X}$ occur at constant rate 1; note that since the T_n, $n \geq 1$, are constructed from independent exponential waiting times, they have a.s. no intersection with the countably many jump times of the original càdlàg process X. Thus at a jump time T_n, the successor state $\check{X}_{T_n}$ for $\check{X}_{T_n^-}$ is selected according to $K(\check{X}_{T_n^-}, \cdot)$, for the transition probability

$$K(\,(x,z,u,v),\, d(x',z',u',v')\,) := \epsilon_{(x,x)}(dx',dz')\mathcal{R}(du',dv')$$

on $(\check{E}, \check{\mathcal{E}})$, where ϵ denotes Dirac measure and $\mathcal{R}(du,dv) = 1_{(0,1)}(u)du1_{(0,1)}(v)dv$. Between successive jumps of the N-component, the X-component of $\check{X}$ evolves according to the semigroup of X, and the N-component remains constant. So we are pasting together in a Markovian way pieces of 'killed' strong Markov processes; it is known that this preserves the strong Markov property, hence $\check{X}$ is strongly Markov with state space $(\check{E}, \check{\mathcal{E}})$ (see [I-N-W 66 a,b], [I-N-W 68]).

We will prove now that

$$(7.1') \qquad \check{\mu}(dx,dz,du,dv) := \mu(dz)\,\mathcal{R}(du,dv)\,U^1(z,dx)$$

is the invariant measure for $\check{X}$. Since the discrete time process $(X_{T_n})_n$ is Harris with invariant measure μ and since $U^1(\cdot, A)$ gives the expected sojourn time of X in A up to an independent exponential time, (7.1') implies via 1.3 that the process $\check{X}$ is again a Harris process.

We have to show that $\check{\mu}$ defined by (7.1') is invariant for the 1-potential kernel of $\check{X}$. Write $\check{A} := A_1 \times A_2 \times A_3 \times A_4$ for arbitrary $A_1, A_2 \in \mathcal{E}$, $A_3, A_4 \in \mathcal{B}([0,1])$. Write σ for an $exp(1)$-waiting time independent of $\check{X}$. Conditioning with respect to T_1, the first jump of the N-component of $\check{X}$, one has

$$\begin{aligned} E_{(x,z,u,v)}\left(1_{\check{A}}(\check{X}_t)\right) &= \int_0^t dr\, e^{-r} \int_{E\times[0,1]^2} P_r(x,dy)\,\mathcal{R}(du',dv')\,E_{(y,y,u',v')}\left(1_{\check{A}}(\check{X}_{t-r})\right) \\ &+ \; e^{-t}\,P_t(x,A_1)\,1_{A_2}(z)\,1_{A_3\times A_4}(u,v) \end{aligned}$$

for every $t > 0$; integrating this equation w.r.t. $e^{-t}dt$, we get

$$\begin{aligned} E_{(x,z,u,v)}\left(1_{\check{A}}(\check{X}_\sigma)\right) &= \int_{E\times[0,1]^2} U^2(x,dy)\,\mathcal{R}(du',dv')\,E_{(y,y,u',v')}\left(1_{\check{A}}(\check{X}_\sigma)\right) \\ &+ \; U^2(x,A_1)\,1_{A_2}(z)\,1_{A_3\times A_4}(u,v) \end{aligned}$$

where U^2 is the 2-potential kernel of X. μ being invariant for X, we deduce from the last equation with particular initial condition $x = z$

$$\frac{1}{2}\int_{E\times[0,1]^2} \mu(dz)\,\mathcal{R}(du,dv)\,E_{(z,z,u,v)}\left(1_{\check{A}}(\check{X}_\sigma)\right) = \int_E \mu(dz)\,1_{A_2}(z)\,U^2(z,A_1)\,\mathcal{R}(A_3\times A_4)\,.$$

As a consequence of both last equations, we obtain for $\check{\mu}$ defined by (7.1')

$$\begin{aligned} \int_{\check{E}} \check{\mu}(dx,dz,du,dv)\,E_{(x,z,u,v)}\left(1_{\check{A}}(\check{X}_\sigma)\right) &= \frac{1}{2}\int_{E\times[0,1]^2} \mu(dy)\,\mathcal{R}(du',dv')\,E_{(y,y,u',v')}\left(1_{\check{A}}(\check{X}_\sigma)\right) \\ &+ \int_E \mu(dz)\,1_{A_2}(z)\,\left(U^1U^2\right)(z,A_1)\,\mathcal{R}(A_3\times A_4) \\ &= \int_E \mu(dz)\,1_{A_2}(z)\,\left(U^2 + U^1U^2\right)(z,A_1)\,\mathcal{R}(A_3\times A_4)\,. \end{aligned}$$

An obvious calculation on Gamma densities gives $\sum_{l=1}^\infty 2^{-l}\Gamma(l,2) = \Gamma(1,1)$; since the transition probability $2U^2$ involves a $\Gamma(1,2)$-waiting time, this gives $\sum_{l=1}^\infty 2^{-l}\left(2U^2\right)^l = U^1$ and thus

$$\left(U^2 + U^1U^2\right) = \frac{1}{2}\left((2U^2) + U^1(2U^2)\right) = U^1\,.$$

Hence the last integral equals $\check{\mu}\left(A_1\times A_2\times A_3\times A_4\right) = \check{\mu}\left(\check{A}\right)$ which proves (7.1').

Now we associate to the process $\check{X}$ of 7.1 a family of processes $\widetilde{X}^m$, close to $\check{X}$ for large m. To do this, we use (H6): whenever X_{T_n} visits the set C occurring in the minoration condition $(\widetilde{M}_1)$, $n \geq 1$, we will forget with probability 2^{-m} the fluctuation of X on the remaining interval $]]T_n, T_{n+1}[[$.

7.2 The processes $\widetilde{X}^m$, $m \geq 0$: For C of (H6) and $\check{X} = (X, N)$ of 7.1, we define

$$\widetilde{X}^m_t = \sum_{n\geq 0} 1_{[[T_n,T_{n+1}[[}(t) \left(\check{X}_t \, 1_{\{\check{X}_{T_n} \in \check{E} \setminus F_{C,m}\}} + \check{X}_{T_n} \, 1_{\{\check{X}_{T_n} \in F_{C,m}\}} \right) , \quad t \geq 0$$

with notation $F_{C,m} := E \times C \times (0, 2^{-m}) \times [0,1]$.

Viewed as $\check{I\!\!F}$-adapted process, $\widetilde{X}^m$ is a functional of $\check{X}$: $\widetilde{X}^m_t$ coincides with $\check{X}_t$ on intervals where $\check{X}_t$ visits $(E \times C \times (0, 2^{-m}) \times [0,1])^c$, and remains constant right-continuous as long as $\check{X}_t$ visits $E \times C \times (0, 2^{-m}) \times [0,1]$. The N-component of $\widetilde{X}^m$ is the N-component of $\check{X}$. In this sense, $\widetilde{X}^m$ is close to $\check{X}$ if m is large.

Consider now the (smaller) filtration $\widetilde{I\!\!F}^m$ generated by $\widetilde{X}^m$ alone. With respect to its own past $\widetilde{I\!\!F}^m$, $\widetilde{X}^m$ is again strongly Markov: jumps of the N-component occur at constant rate 1; at a jump time T_n, a successor state $\widetilde{X}^m_{T_n}$ for $\widetilde{X}^m_{T_n^-}$ is selected according to the transition probability $K(\,(x,z,u,v),\, d(x',z',u',v')\,)$ on $(\check{E}, \check{\mathcal{E}})$ given by

$$(7.2') \qquad \left[\epsilon_x + 1_C(z) 1_{(0,2^{-m})}(u) \left(U^1(z,\cdot) - \epsilon_x \right) \right] (dx') \, \epsilon_{x'}(dz') \, \mathcal{R}(du', dv') \; ;$$

between successive jumps of the N-component, the X-component of the process $\widetilde{X}^m$ evolves according to the semigroup of X whenever $\widetilde{X}^m$ is in $(E \times C \times (0, 2^{-m}) \times [0,1])^c$, and remains constant otherwise. With respect to $\widetilde{I\!\!F}^m$, $\widetilde{X}^m$ is again Harris and has invariant measure

$$(7.2'') \quad \widetilde{\mu}^m(dx, dz, du, dv) := \mu(dz) \, \mathcal{R}(du, dv) \left[U^1(z,\cdot) + 1_C(z) 1_{(0,2^{-m})}(u) \left(\epsilon_z - U^1(z,\cdot) \right) \right] (dx) \; .$$

In the sense of equality of laws of processes, i.e. of probability laws on the Skorohod space $D(I\!\!R_+, \check{E})$, we shall always switch between these two interpretations of $\widetilde{X}^m$.

7.3 An atom for $\widetilde{X}^m$: Let α, C, ν be given by (H6). The Harris process $\widetilde{X}^m$ with respect to $\widetilde{I\!\!F}^m$ admits an interpretation in terms of Nummelin splitting with recurrent atom

$$\widetilde{A}^m := E \times C \times (0, 2^{-m}) \times (0, \alpha) \; \in \check{\mathcal{E}} \, , \quad \widetilde{\mu}^m(\widetilde{A}^m) = \alpha 2^{-m} \mu(C) > 0 \, .$$

Indeed, at a jump time T_n, knowing $\widetilde{X}^m_{T_n^-}$ and thus knowing whether $\widetilde{X}^m$ was constant on $[[T_{n-1}, T_n[[$ or not (this is seen from the N-component of $\widetilde{X}^m_{T_n^-}$), we can rewrite the transition kernel $K(\cdot,\cdot)$ of (7.2') as follows:

i) on $\{N_{T_n^-} \notin C\times(0,2^{-m})\times[0,1]\}$, we select $\widetilde{X}^m_{T_n}$ according to

$$\epsilon_{X_{T_n^-}}(dx')\,\epsilon_{x'}(dz')\,\mathcal{R}(du',dv')$$

(on this event, $X_{T_n^-}$ is the first component of $\widetilde{X}^m_{T_n^-}$);

ii) on $\{N_{T_n^-} \in C\times(0,2^{-m})\times[0,\alpha)\}$, we select $\widetilde{X}^m_{T_n}$ according to

$$\nu(dx')\,\epsilon_{x'}(dz')\,\mathcal{R}(du',dv')\ ;$$

iii) on $\{N_{T_n^-} \in C\times(0,2^{-m})\times[\alpha,1]\}$, we select $\widetilde{X}^m_{T_n}$ according to

$$\frac{1}{1-\alpha}\left(U^1(X_{T_{n-1}},dx') - \alpha\nu(dx')\right)\,\epsilon_{x'}(dz')\,\mathcal{R}(du',dv')$$

(on this event, $X_{T_{n-1}}$ is the second component of $\widetilde{X}^m_{T_n^-}$) in virtue of (H6).

Note that we have applied Nummelin's splitting technique only between those jump times of the N-component of $\widetilde{X}^m$ where the process $\widetilde{X}^m$ itself remained constant.

As a consequence of 7.3, $\widetilde{X}^m$ has life cycles. In order to apply the results of subsection 3.1 to $\widetilde{X}^m$, we need (H4): we have to specify a life cycle decomposition $(\widetilde{R}^m_n)_n$ for $\widetilde{X}^m$ such that weakly special functions for $\widetilde{X}^m$ and $\widetilde{R}^m_1$ do exist.

7.4 Proposition : For the process $\widetilde{X}^m$ with recurrent atom $\widetilde{A}^m := E\times C\times(0,2^{-m})\times(0,\alpha)$, we define a life cycle decomposition $(\widetilde{R}^m_n)_n$ by

$$\widetilde{R}^m_1 := S_0 + T_{(\widetilde{A}^m)^c}\circ\vartheta_{S_0}\ ,\quad S_0 := \inf\{t : \widetilde{X}^m_t \in \widetilde{A}^m\} \tag{7.5}$$

where $T_{(\widetilde{A}^m)^c}$ is the first entry time to $(\widetilde{A}^m)^c$. Then for any special function f for X, the function $\check{f}(x,z,u,v) := f(z)$ on $(\check{E},\check{\mathcal{E}})$ is weakly special for $\widetilde{X}^m$ and $\widetilde{R}^m_1$.

Proof : 1) First note that $\widetilde{R}^m_1$ is defined as the first entry time to $(\widetilde{A}^m)^c$ following the first visit to the atom $\widetilde{A}^m$; since ν in 7.3 ii) is by (H6) concentrated on C, a visit in $\widetilde{A}^m$ during an independent exponential time leads with probability $\alpha 2^{-m}$ to another visit in $\widetilde{A}^m$ during a new independent exponential time. So having entered the atom, the sojourn time of $\widetilde{X}^m$ in $\widetilde{A}^m$ is distributed according to

$$\sum_{j\geq 0}(1-\alpha 2^{-m})(\alpha 2^{-m})^j\Gamma(j+1,1) = \Gamma(1,1-\alpha 2^{-m})\,.$$

2) Consider the original process X. In virtue of (H1)+(H2), special functions f for X do exist, cf. 5.28; w.l.o.g., we take f bounded. Put $h := \alpha 2^{-m}\, 1_C$ with α, C as in 7.3. Since $\mu(C) > 0$ by (H6), we have with notations of 5.28

$$z \to E_z\left(\int_0^{\widehat{T}_h} f(X_s)ds\right) \quad \text{is bounded on } E$$

where $\widehat{T}_h$ is a killing time for position dependent killing of X at rate h. Prepare - on an extension of $(\Omega, \mathcal{A}, I\!\!F, P)$ - a sequence $\rho_n \uparrow \infty$ such that $\rho_i - \rho_{i-1}$, $i \geq 0$ are i.i.d $\sim exp(1)$, $\rho_0 = 0$, and prepare (U_n, V_n) i.i.d $\sim \mathcal{R}(du, dv)$ for $n \geq 0$, all independent and independent of X. By (5.29) we see that

$$z \to E_z\left(\sum_{n\geq 1}\big(1-h(X_{\rho_1})\big)\,...\,\big(1-h(X_{\rho_{n-1}})\big)\, f(X_{\rho_n})\right) \quad \text{is bounded.}$$

The expectation in the last relation is equal to

$$E_z\left(\sum_{n\geq 1}\big(1-h_{C,m,\alpha}(W_1)\big)\,...\,\big(1-h_{C,m,\alpha}(W_{n-1})\big)\, f(X_{\rho_n})\,(\rho_{n+1}-\rho_n)\right) \tag{7.6}$$

with notation $h_{C,m,\alpha} := 1_{C\times(0,2^{-m})\times(0,\alpha)}$ and $W_j := (X_{\rho_j}, U_j, V_j)$: this is seen by multiplying out summands and using the independence assumptions.

3) Consider the process N arising in the construction of $\check{X}$ and $\widetilde{X}^m$. In notation of 7.1 and 7.2, the expectation (7.6) equals

$$E_{(z,u,v)}\left(\int_{T_1}^{S+T_1\circ S}(f\circ\pi_1)(N_s)ds\right)$$

where we define

$$S := \inf\{t \geq T_1 : N_t \in C\times(0,2^{-m})\times(0,\alpha)\}\,,$$

where π_1 is the projection $(z,u,v) \to z$ and where u, v are arbitrary. So we have proved in 2)+3)

$$(z,u,v) \to E_{(z,u,v)}\left(\int_{T_1}^{S+T_1\circ S}(f\circ\pi_1)(N_s)ds\right) \quad \text{is bounded on } E\times[0,1]\times[0,1]. \tag{7.7}$$

4) Consider now $\widetilde{X}^m$. With notation $\check{f}(x,z,u,v) := f(z)$, (7.7) reads

$$\check{x} \to E_{\check{x}}\left(\int_{T_1}^{S+T_1\circ S}\check{f}(\widetilde{X}^m_s)ds\right) \quad \text{is bounded on } \check{E}\ .$$

Noticing that $\check{f}$ is bounded and that T_1 has law $exp(1)$, we may replace the interval of integration by $[[0, S + T_1 \circ S[[$. By construction of the atom $\widetilde{A}^m$ in 7.3, the first entry time S_0 of $\widetilde{X}^m$ to $\widetilde{A}^m$ equals the first entry time of N to $C{\times}(0, 2^{-m}){\times}(0, \alpha)$: thus we have $S_0 \le S$. By step 1), the interval $[[S_0, S_0{+}T_{(\widetilde{A}^m)^c}{\circ}\vartheta_{S_0}[[$ has length distributed according to $\Gamma(1, 1{-}\alpha 2^{-m})$. All this together with the strong Markov property allows to deduce

$$\check{x} \to E_{\check{x}}\left(\int_0^{\widetilde{R}_1^m} \check{f}(\widetilde{X}_s^m)ds\right) \quad \text{is bounded on } \check{E} \tag{7.8}$$

with $\widetilde{R}_1^m$ defined by (7.5). We have proved that for every bounded special function f of X, $\check{f}(x, z, u, v) := f(z)$ is weakly special for $\widetilde{X}^m$ and $\widetilde{R}_1^m$. □

Remark : As a consequence of 7.2, 7.3 and 7.4, we know that assumptions (H1)+(H3)+(H4) hold for the process $\widetilde{X}^m$ with atom $\widetilde{A}^m$ and with life cycles defined by (7.5). So all results of subsection 3.1 (or of sections 4+5) can be applied to $\widetilde{X}^m$.

However, we have to reformulate the conditions on life cycle length distributions in $\widetilde{X}^m$ (which is an artificial object) into conditions formulated for the original process X. After two preliminary results, this will be done in theorem 7.14.

7.9 Lemma : For the life cycle decomposition defined for $\widetilde{X}^m$ by (7.5), put

$$\widetilde{\mathbf{v}}^m(t) := \left(1 - E\left(e^{-\frac{1}{t}(\widetilde{R}_2^m - \widetilde{R}_1^m)}\right)\right)^{-1}, \quad t > 0\,.$$

a) For $0 < \alpha \le 1$ and $\widetilde{l}^m(\cdot)$ varying slowly at ∞, the following assertions i) - iv) are equivalent:

i) $$\widetilde{r}^m(t) := \int_0^t P(\widetilde{R}_2^m - \widetilde{R}_1^m > x)\, dx \quad \sim \quad \frac{1}{\Gamma(2-\alpha)}\, t^{1-\alpha}\, \widetilde{l}^m(t) \quad \text{as} \quad t \to \infty$$

(in case $0 < \alpha < 1$, this is equivalent to $P(\widetilde{R}_2^m - \widetilde{R}_1^m > x) \sim \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha}\, \widetilde{l}^m(x)$ as $t \to \infty$);

ii) $$\widetilde{\mathbf{v}}^m(t) \sim t^\alpha \frac{1}{\widetilde{l}^m(t)} \quad \text{as} \quad t \to \infty\,;$$

iii) for every bounded special function f of X, defining $\check{f}(x, z, u, v) := f(z)$, the resolvant of $\widetilde{X}^m$ satisfies

$$\left(\widetilde{R}_{1/t}^m \check{f}\right)(\check{x}) \quad \sim \quad t^\alpha \frac{1}{\widetilde{l}^m(t)}\, E\left(\int_{\widetilde{R}_1^m}^{\widetilde{R}_2^m} \check{f}(\widetilde{X}_s^m)ds\right) \quad \text{as} \quad t \to \infty$$

for every $\check{x} \in \check{E}$;

iv) for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, for N^1 the second component of $\widetilde{X}^m$ (i.e the first component of N),

$$E_{\check{x}} \left(\int_0^\infty e^{-\frac{1}{t}s}\, g(N^1_s)\, ds \right) \quad \sim \quad t^\alpha \, \frac{1}{\widetilde{l}^m(t)}\, E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} g(N^1_s)\, ds \right) \quad \text{as} \quad t \to \infty$$

for $\widetilde{\mu}^m$-almost all $\check{x} \in \check{E}$.

b) Under $P_{(z,z,\cdot,\cdot)}$, the law of $N^1 = \sum_{n\ge 0} 1_{[[T_n,T_{n+1}[[} X_{T_n}$ does not depend on m. There is some constant $\widetilde{c}^m$ such that

$$E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} g(N^1_s)\, ds \right) \quad = \quad \widetilde{c}^m\, \mu(g) \tag{7.10}$$

for all g in iv), and there is a slowly varying function $l(\cdot)$ not depending on m such that $\widetilde{l}^m(\cdot)$ in a) can be replaced by

$$\widetilde{l}^m(t) \;=\; \widetilde{c}^m\, l(t)\,, \quad t \ge 0\,, \quad \text{for arbitrary} \quad m\,. \tag{7.11}$$

Proof : 1) The equivalence of i) and ii) is (5.1) together with (4.2)-(4.4), the equivalence of ii) and iii) is the decomposition of the resolvant in the proof of lemma 5.3, all this applied to the process $\widetilde{X}^m$ (the assumptions (H1)+(H3)+(H4) relative to $\widetilde{X}^m$ which we need here have been checked, and $\check{f}$ is weakly special for $\widetilde{X}^m$ and $\widetilde{R}^m_1$ if f is special for X). By definition of $\check{f}$, iii) can be written in terms of N^1:

iv') for every bounded special function f of X and for all $\check{x} \in \check{E}$,

$$E_{\check{x}} \left(\int_0^\infty e^{-\frac{1}{t}s}\, f(N^1_s)\, ds \right) \quad \sim \quad t^\alpha \, \frac{1}{\widetilde{l}^m(t)}\, E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} f(N^1_s)\, ds \right), \quad t \to \infty\,.$$

So we have proved a) with iv') in place of iv).

2) Under $P_{(z,z,\cdot,\cdot)}$, the notation $N^1 = \sum_{n\ge 0} 1_{[[T_n,T_{n+1}[[} X_{T_n}$ is unambiguous, and the l.h.s of iv') is the resolvant of a Markov step process with $exp(1)$-holding times in all states and with jump heigth governed by the potential kernel $U^1(\cdot,\cdot)$ of X. So in this case, there is asymptotically as $t \to \infty$ no dependence on m in the r.h.s of iv'), and there is a function $l(\cdot)$ varying slowly at ∞, not dependent on m, such that

$$\widetilde{l}^m(t)\, \frac{1}{E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} f(N^1_s)\, ds \right)} \quad \sim \quad l(t)\, \frac{1}{\mu(f)}\,, \quad t \to \infty\,. \tag{$*$}$$

Proposition 1.10 applied to additive functionals $\int_0^t g(N^1_s)\, ds$ of $\widetilde{X}^m$ - with invariant measure $\widetilde{\mu}^m$ whose image under the projection $(x,z,u,v) \to z$ equals μ - shows that there is a constant $\widetilde{c}^m$

with property (7.10), so (*) implies (7.11).

3) In order to complete the proof of the lemma, note that assertion iv') is equivalent to iv) in a) by the ratio limit theorem 1.8 for resolvants in $\widetilde{X}^m$. $\square$

7.12 Lemma : We have $\widetilde{c}^m = 2^m / [\mu(C)\alpha(1-\alpha 2^{-m})]$ in (7.10)+(7.11).

Proof : Consider g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, and an arbitrary $\check{\mathcal{E}}$-measurable nonnegative function $\check{g}$ with $0 < \widetilde{\mu}^m(\check{g}) < \infty$. Proposition 1.10 applied to $\int_0^t g(N^1_s)\, ds$ and $\int_0^t \check{g}(\widetilde{X}^m_s)\, ds$ - additive functionals of $\widetilde{X}^m$ with invariant measure $\widetilde{\mu}^m$ - shows that (7.10) can be extended to

$$(7.13) \qquad E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} \check{g}(\widetilde{X}^m_s)\, ds\right) \quad = \quad \widetilde{c}^m\, \widetilde{\mu}^m(\check{g})\,.$$

Consider the counting process

$$\widetilde{\beta}^m_t \quad := \quad \sum_{n\geq 1} 1_{[[\widetilde{R}^m_n,\infty[[}(t)\,, \quad t\geq 0$$

associated to the life cycle decomposition $(\widetilde{R}^m_n)_n$ in $\widetilde{X}^m$. By (7.5), the $(\widetilde{R}^m_n)_{n\geq 1}$ are passage times from the atom $\widetilde{A}^m = E\times C\times(0,2^{-m})\times(0,\alpha)$ to its complement. By 7.3 ii), the measure ν being concentrated on the set C, the atom $\widetilde{A}^m$ can only be left by a change from $(0,2^{-m})\times(0,\alpha)$ to $((0,2^{-m})\times(0,\alpha))^c$ in the two last components of $\widetilde{X}^m$. So the $\widetilde{\mathbb{F}}^m$-compensator of the counting process $\widetilde{\beta}^m$ is

$$\int_0^t (1-\alpha 2^{-m})\, 1_{\widetilde{A}^m}(\widetilde{X}^m_s)\, ds$$

and (7.13) gives

$$1 = E\left(\widetilde{\beta}^m_{\widetilde{R}^m_2} - \widetilde{\beta}^m_{\widetilde{R}^m_1}\right) = (1-\alpha 2^{-m})\, E\left(\int_{\widetilde{R}^m_1}^{\widetilde{R}^m_2} 1_{\widetilde{A}^m}(\widetilde{X}^m_s)\, ds\right) = (1-\alpha 2^{-m})\, \widetilde{c}^m\, \widetilde{\mu}^m(\widetilde{A}^m)$$

and the assertion follows from $\widetilde{\mu}^m(\widetilde{A}^m) = \alpha 2^{-m}\mu(C)$. $\square$

We deduce from 7.9 and 7.12 that regular variation at ∞ of tails of life cycle length distributions in $\widetilde{X}^m$ can be expressed in terms of regular variation at 0 of the resolvant of the original process X.

7.14 Theorem : Consider $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞. Then for arbitrary m, for life cycle decompositions $(\widetilde{R}^m_n)_n$ of $\widetilde{X}^m$ given by (7.5) and constants $\widetilde{c}^m$ given in 7.12, the following assertions i) - iii) are equivalent:

i) $$\widetilde{r}^m(t) = \int_0^t P(\widetilde{R}_2^m - \widetilde{R}_1^m > x)\, dx \quad \sim \quad \frac{1}{\Gamma(2-\alpha)}\, t^{1-\alpha}\, \widetilde{c}^m\, l(t) \quad \text{as} \quad t \to \infty$$

(in case $0 < \alpha < 1$, this is equivalent to $P(\widetilde{R}_2^m - \widetilde{R}_1^m > x) \sim \frac{1}{\Gamma(1-\alpha)}\, x^{-\alpha}\, \widetilde{c}^m\, l(x)$ as $t \to \infty$);

ii) $$\widetilde{\mathbf{v}}^m(t) \;=\; \left(1 - E\left(e^{-\frac{1}{t}(\widetilde{R}_2^m - \widetilde{R}_1^m)}\right)\right)^{-1} \sim\; t^\alpha\, \frac{1}{\widetilde{c}^m\, l(t)} \quad \text{as} \quad t \to \infty\,;$$

iii) for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, one has regular variation at 0 of resolvants in the original process X

(7.15) $$\left(R_{1/t}g\right)(x) = E_x\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right) \quad \sim \quad t^\alpha\, \frac{1}{l(t)}\, \mu(g), \quad t \to \infty$$

for μ-almost all $x \in E$ (the exceptional set depending on g).

Proof : Note that i) and ii) above rephrase assertions i) and ii) of 7.9 a). Note also that the resolvant (7.15) in X can be rewritten as a resolvant in the process $\check{X} = (X, N)$ of 7.1, of form

$$E_{(x,z,u,v)}\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right)$$

where z, u, v are arbitrary. Fix some bounded special function f for X. Consider

(+) $$t \to E_{\check{x}}\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right)\,, \quad t \to E_{\check{x}}\left(\int_0^\infty e^{-\frac{1}{t}s}\, f(N_s^1)\, ds\right)$$

as resolvants in $\check{X}$ with invariant measure $\check{\mu}$ on $(\check{E}, \check{\mathcal{E}})$. Since N-components coincide in $\check{X}$ and $\widetilde{X}^m$, the second expression in (+) is also a resolvent in $\widetilde{X}^m$. Thus according to 7.9 a) iii) together with (7.10)+(7.11), regular variation

$$E_{\check{x}}\left(\int_0^\infty e^{-\frac{1}{t}s}\, f(N_s^1)\, ds\right) \quad \sim \quad t^\alpha\, \frac{1}{l(t)}\, \mu(f), \quad t \to \infty$$

for all $\check{x} \in \check{E}$ is equivalent to i) and ii). It remains to apply the RLT 1.8 to the resolvants (+) in $\check{X}$ and to note that μ is image of $\check{\mu}$ under projections $(x, z, u, v) \to x$ and $(x, z, u, v) \to z$. $\square$

7.16 Theorem : a) For $0 < \alpha \le 1$ and $l(\cdot)$ varying slowly at ∞, the following i) and ii) are equivalent:

i) for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, one has regular variation at 0 of resolvants in X

$$\left(R_{1/t}g\right)(x) = E_x\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right) \quad \sim \quad t^\alpha\, \frac{1}{l(t)}\, \mu(g), \quad t \to \infty$$

for μ-almost all $x \in E$ (the exceptional set depending on g);

ii) for every additive functional A of X with $0 < E_\mu(A_1) < \infty$, one has weak convergence

$$\frac{(A_{tn})_{t\geq 0}}{n^\alpha / \, l(n)} \quad \to \quad E_\mu(A_1)\, W^\alpha$$

(in $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$, under P_x for all $x \in E$) where W^α is the Mittag-Leffler process of index α.

b) The cases in a) are the only ones where weak convergence of $\dfrac{(A_{tn})_{t\geq 0}}{v(n)}$ to a continuous non-decreasing limit process W (with $W_0 = 0$ and $\mathcal{L}(W_1)$ not degenerate at 0) is available for some norming function v.

Proof : 1) The additive functional A of X is also an additive functional of $\check{X} = (X, N)$. The RLT in $\check{X}$ with invariant measure $\check{\mu}$ shows

$$\frac{A_t}{\int_0^t g(N_s^1)ds} \quad \to \quad \frac{E_\mu(A_1)}{\mu(g)} \quad \text{as } t \to \infty,\ P_{\check{x}}\text{-a.s. for all } \check{x} \in \check{E}$$

where $g \geq 0$ is any fixed $\mathcal{E}$-measurable function with $0 < \mu(g) < \infty$. We can also view $\int_0^t g(N_s^1)ds$ as additive functional of $\widetilde{X}^m$ since N-components in $\check{X}$ or $\widetilde{X}^m$ are the same, and compare it via ratio limits in $\widetilde{X}^m$ with invariant measure $\widetilde{\mu}^m$ to the counting process $\widetilde{\beta}^m = \sum_{n\geq 1} 1_{[[\widetilde{R}_n^m, \infty[[}$, or to the compensator

$$(1 - \alpha 2^{-m}) \int_0^t 1_{\widetilde{A}^m}(\widetilde{X}_s^m)\, ds$$

of $\widetilde{\beta}^m$ relative to $\widetilde{\mathbb{F}}^m$ (see proof of 7.12). Thus

$$\frac{\int_0^t g(N_s^1)ds}{\widetilde{\beta}_t^m} \quad \to \quad \frac{\mu(g)}{(1-\alpha 2^{-m})\, \alpha 2^{-m}\, \mu(C)} \;=\; \widetilde{c}^m\, \mu(g)\,, \quad t \to \infty$$

$P_{\check{x}}$-a.s. for all $\check{x} \in \check{E}$, where $\widetilde{c}^m$ is given in 7.12. So it remains to consider weak convergence of the counting process $\widetilde{\beta}^m$ associated to the life cycle decomposition $(\widetilde{R}_n^m)_n$ of $\widetilde{X}^m$.

2) Assume regular variation of the resolvant of X at 0 as in a) i), and thus by theorem 7.14 regular variation of $\widetilde{X}^m$-life cycle length distributions as in 7.14 a) i) together with regular variation of the function $\widetilde{\mathbf{v}}^m(\cdot)$ in 7.14 a) ii). For this setting, it has been proved in section 4 (see in particular (4.16) with $v \sim \widetilde{\mathbf{v}}^m$ and with $a(\cdot)$ an asymptotic inverse to $v(\cdot)$) that

$$\frac{1}{\widetilde{\mathbf{v}}^m(n)} \left(\widetilde{\beta}_{tn}^m\right)_{t\geq 0} \quad \to \quad W^\alpha$$

(weakly in $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$, under P_x for all $x \in E$), or using the above ratio limits

$$\frac{1}{\widetilde{\mathbf{v}}^m(n)} \,(A_{tn})_{t\geq 0} \quad \to \quad E_\mu(A_1)\, \widetilde{c}^m\, W^\alpha\,.$$

By the structure of $\widetilde{\mathbf{v}}^m$ in 7.14, the $\widetilde{c}^m$ cancels, and we have assertion a) ii) of the theorem.

3) Assume now that one has weak convergence of rescaled and suitably normed additive functionals $\frac{1}{v(n)}\left(\widetilde{\beta}^m_{tn}\right)_{t\geq 0}$ of $\widetilde{X}^m$ as $n\to\infty$ to a continuous limit process W. Then in virtue of theorems 5.6.A+B, we have necessarily regular variation of $\widetilde{\mathbf{v}}^m$ at ∞ with some index $0<\alpha\leq 1$ (see also remark 5.7), in which case we are back in step 2) - so no other types of limits can arise under weak convergence - and have by theorem 7.14 regular variation at 0 of the resolvant of X as in a) i). So the proof of theorem 7.16 is completed. $\square$

Now we consider martingales $M\in\mathcal{M}^{2,\mathrm{loc}}(P_x,I\!F)$ meeting

(H5^A): M has the property

$$\forall\, y\,,\forall\, s,t:\quad M_{t+s}-M_t=M_s\circ\vartheta_t\quad P_y\text{-a.s.}\,,$$

$\langle M\rangle$ and $[M]$ are additive functionals of X, and $E_\mu\left(\langle M\rangle_1\right)<\infty$.

By construction of $\check{X}$ in 7.1, we can lift the processes M, $\langle M\rangle$, $[M]$ to $(\check{\Omega},\check{\mathcal{A}},\check{I\!F})$: then M is in $\mathcal{M}^{2,\mathrm{loc}}$ with respect to $\check{I\!F}$ and to laws $\check{P}:=P_{(x,z,u,v)}$ with arbitrary z,u,v, with predictable quadratic variation and quadratic variation as before. (H5^A) will remain true with respect to shifts $(\vartheta)_{t\geq 0}$ on $(\check{\Omega},\check{\mathcal{A}},\check{I\!F})$, with laws $P_{(y,z',u',v')}$ replacing P_y, and with $E_{\check{\mu}}$ replacing E_μ.
(Note that $I\!F$-stopping times become $\check{I\!F}$-stopping times; to see that martingale properties relative to $(P_x,I\!F)$ do carry over to $(\check{P},\check{I\!F})$ as asserted, consider a $(P_x,I\!F)$-martingale M', $s<t$, and sets $\check{F}\in\check{\mathcal{F}}_s$ of form

$$\check{F}=\{\,\check{X}_{s_i}\in\check{A}^i:=A^i_1\times A^i_2\times A^i_3\times A^i_4,\;0\leq i\leq l\,\}\,,\quad 0=s_0<s_1<...<s_l=s\,,\quad l\in I\!N$$

with $A^i_1,A^i_2\in\mathcal{E}$, $A^i_3,A^i_4\in\mathcal{B}([0,1])$. Then for every n

$$\begin{aligned}&E_{(x,z,u,v)}\left(1_{\check{F}\cap\{T_{n+1}>s\}}(M'_t-M'_s)\right)\\&=E_{(x,z,u,v)}\left(\int_0^\infty ...\int_0^\infty dt_1e^{-t_1}...dt_{n+1}e^{-t_{n+1}}\,1_{\{t_1+...+t_{n+1}>s\}}\,\check{G}^{t_1,\dots,t_{n+1}}_{z,u,v}\,(M'_t-M'_s)\right)\end{aligned}$$

where $\check{G}^{t_1,\dots,t_{n+1}}_{z,u,v}$ is given by

$$\prod_{i=0}^{l}\left(1_{\{s_i<t_1\}}1_{\check{A}^i}(X_{s_i},z,u,v)+\sum_{k=1}^{n}1_{\{t_0+...+t_k\leq s_i<t_0+...+t_{k+1}\}}\,1_{\check{A}^i}(X_{s_i},X_{t_0+...+t_k},U_k,V_k)\right)$$

with notations of 7.1 and $t_0=0$. By the independence assumptions in 7.1, the above integral is

$$\int_0^\infty ...\int_0^\infty dt_1e^{-t_1}...dt_{n+1}e^{-t_{n+1}}\,1_{\{t_1+...+t_{n+1}>s\}}\,\gamma^{t_1,\dots,t_{n+1}}_{z,u,v}\,E_x\left(G^{t_1,\dots,t_{n+1}}(M'_t-M'_s)\right)$$

with $G^{t_1,\dots,t_{n+1}}$ the indicator function of an event in $\sigma\left(X_{s_i},\, 0\le i\le l,\, X_{(t_0+\dots+t_k)\wedge s},\, 0\le k\le n\right)$, thus in $\mathcal{F}_s$, and for suitable $\gamma^{t_1,\dots,t_{n+1}}_{z,u,v}\in[0,1]$. Thus the last integral equals 0. As $n\to\infty$, we have $E_{\check P}\left(1_{\check F}(M'_t-M'_s)\right)=0$ for $\check P=P_{(x,z,u,v)}$. Since $\check{I\!\!F}$ is the filtration generated by $\check X$, one deduces $E_{\check P}\left(1_{\check F'}(M'_t-M'_s)\right)=0$ for arbitrary events $\check F'\in\check{\mathcal{F}}_s$.)

7.17 Lemma : From martingales $M\in\mathcal{M}^{2,\mathrm{loc}}(P_x,I\!\!F)$ meeting (H5^A), and such that in addition $\langle M\rangle$ is a locally bounded process, consider

$$M^m_t=\sum_{n\ge0}1_{\{\check X_{T_n}\notin F_{C,m}\}}\left(M^{T_{n+1}}-M^{T_n}\right)_t\,,\quad t\ge0$$

defined on $(\check\Omega,\check{\mathcal{A}})$, where $F_{C,m}=E\times C\times(0,2^{-m})\times[0,1]$ is the set occurring in 7.2.

Then the following holds, for arbitrary $m\ge1$.

a) The process M^m is $\widetilde{I\!\!F}^m$-adapted.

b) M^m belongs to $\mathcal{M}^{2,\mathrm{loc}}(\widetilde{I\!\!F}^m,\check P)$ with $\check P=P_{(x,z,u,v)}$ for arbitrary z,u,v.

c) Write $\langle M^m\rangle$, $[M^m]$ for angle and square brackett of M^m with respect to $(\widetilde{I\!\!F}^m,\check P)$. On $(\check\Omega,\check{\mathcal{A}},\widetilde{I\!\!F}^m,(\vartheta_t)_{t\ge0},(P_{\check y})_{\check y\in\check E})$, with lifecycles for $\widetilde X^m$ defined by (7.5) and invariant measure $\widetilde\mu^m$ given by (7.2"), the processes M^m, $\langle M^m\rangle$, $[M^m]$ satisfy all conditions (H5^A) + (H5^B); one has

$$\widetilde c^m\,E_{\widetilde\mu^m}\left(\langle M^m\rangle_1\right)=E\left(\langle M^m\rangle_{\widetilde R^m_2}-\langle M^m\rangle_{\widetilde R^m_1}\right)<\infty \tag{7.18}$$

with $\widetilde c^m$ as in 7.12.

Proof : a) To see that the process M^m on $(\check\Omega,\check{\mathcal{A}})$ is $\widetilde{I\!\!F}^m$-adapted, we shall prove

$$\sigma(n,t):=(t-T_n)\vee0\ \text{ is an }\left(\widetilde{\mathcal{F}}^m_{T_n+u}\right)_{u\ge0}\text{-stopping time} \tag{+}$$

$$\text{the process }\left(1_{\{\check X_{T_n}\notin F_{C,m}\}}\left(M^{T_{n+1}}-M^{T_n}\right)_{T_n+u}\right)_{u\ge0}\text{ is }\left(\widetilde{\mathcal{F}}^m_{T_n+u}\right)_{u\ge0}\text{-adapted} \tag{++}$$

for $n\in I\!\!N_0$, $t\ge0$. Combining (+) and (++) yields

$$1_{\{\check X_{T_n}\notin F_{C,m}\}}\left(M^{T_{n+1}}-M^{T_n}\right)_{T_n+\sigma(n,t)}\quad\text{is }\widetilde{\mathcal{F}}^m_{T_n+\sigma(n,t)}\text{-measurable}\,.$$

Now $T_n+\sigma(n,t)=T_n\vee t$ is an $\widetilde{I\!\!F}^m$-stopping time which equals t on $\{T_n\le t\}=\{T_n\vee t\le t\}$. Thus by definition of $\widetilde{\mathcal{F}}^m_{T_n\vee t}$

$$\{T_n\le t\}\cap\{\check X_{T_n}\notin F_{C,m}\}\cap\{\left(M^{T_{n+1}}-M^{T_n}\right)_t\in A\}\ \in\ \widetilde{\mathcal{F}}^m_t$$

for sets $A \in \mathcal{B}(I\!R)$: thus

$$M^m = \sum_{n=0}^{\infty} 1_{\{\check{X}_{T_n} \notin F_{C,m}\}} \, 1_{[[T_n,\infty[[} \left(M^{T_{n+1}} - M^{T_n} \right)$$

is $\widetilde{I\!F}^m$-adapted which is a). We show (+) and (++).

Since T_n is an $\widetilde{I\!F}^m$-stopping time, $\sigma(n,t)$ is $\widetilde{\mathcal{F}}^m_{T_n^-}$-measurable and nonnegative, hence (+) is obvious since $\{\sigma(n,t) \le v\} \in \widetilde{\mathcal{F}}^m_{T_n^-} \subset \widetilde{\mathcal{F}}^m_{T_n+v}$, $v \ge 0$. To see (++), note first that by (H5^A)

$$\left[1_{\{\check{X}_{T_n} \notin F_{C,m}\}} \left(M^{T_{n+1}} - M^{T_n} \right)_{T_n+u} \right](\omega) = \left[1_{\{\check{X}_0 \notin F_{C,m}\}} \, M^{T_1}_u \right] (\vartheta_{T_n}(\omega)) \, .$$

Now M is $\check{I\!F}$-adapted; stopped at time T_1, the process M^{T_1} is $\left(\check{\mathcal{F}}_{u\wedge T_1}\right)_{u\ge 0}$-adapted, thus

$$(*) \qquad \{\check{X}_0 \notin F_{C,m}\} \cap \{M^{T_1}_u \in A\} \;\in\; \check{\mathcal{F}}_{T_1\wedge u} \, , \quad A \in \mathcal{B}(I\!R) \, , \quad u \ge 0 \, .$$

Since $\left(\check{\Omega}, \check{\mathcal{A}}, \check{I\!F}\right)$ is the canonical path space for $\check{X}$, see 7.1 - 7.2, we have by construction of $\widetilde{X}^m$

(**) the σ-fields $\check{\mathcal{F}}_{T_1\wedge u}$ and $\widetilde{\mathcal{F}}^m_{T_1\wedge u}$ coincide in restriction to $\{\check{X}_0 \notin F_{C,m}\}$.

By (*) and (**), the process $1_{\{\check{X}_0 \notin F_{C,m}\}} \, M^{T_1}$ is in particular $\widetilde{I\!F}^m$-adapted. Then (H5^A) shows that

$$\left[1_{\{\check{X}_{T_n} \notin F_{C,m}\}} \left(M^{T_{n+1}} - M^{T_n} \right)_{T_n+u} \right](\omega) = \left[1_{\{\check{X}_0 \notin F_{C,m}\}} \, M^{T_1}_u \right] (\vartheta_{T_n}(\omega))$$

is $\widetilde{\mathcal{F}}^m_{T_n+u}$-measurable, for all $n \in I\!N_0$, $u \ge 0$: this is (++). So assertion a) is proved.

b) By assumption, M and thus M^m belong to $\mathcal{M}^{2,\mathrm{loc}}(\check{I\!F}, \check{P})$. Since $\widetilde{I\!F}^m$ is smaller than $\check{I\!F}$, $(\check{I\!F}, \check{P})$-martingales which are $\widetilde{I\!F}^m$-adapted will be $(\widetilde{I\!F}^m, \check{P})$-martingales. By a), M^m is $\widetilde{I\!F}^m$-adapted. It remains to show that there are localizing sequences $(\widetilde{\rho}^m_l)_{l\ge 1}$ for M^m which are $\widetilde{I\!F}^m$-stopping times: then M^m will belong to $\mathcal{M}^{2,\mathrm{loc}}(\widetilde{I\!F}^m, \check{P})$.

We consider first the particular case where the process $\langle M \rangle$ is continuous. Then

$$Y^m := \sum_{n=0}^{\infty} 1_{\{\check{X}_{T_n} \notin F_{C,m}\}} \left(\langle M \rangle^{T_{n+1}} - \langle M \rangle^{T_n} \right)$$

is continuous and nondecreasing, and one proves exactly as in a) that Y^m is $\widetilde{I\!F}^m$-adapted. So

$$\widetilde{\rho}^m_l := \inf\{t > 0 : Y^m_t > l\} \, , \quad l \ge 1$$

is a sequence of $\widetilde{I\!F}^m$-stopping times increasing to ∞ such that

$$(M^m)^{(\widetilde{\rho}^m_l)} \text{ is in } \mathcal{M}^2(\widetilde{I\!F}^m, \check{P}) \text{ with angle brackett } (Y^m)^{(\widetilde{\rho}^m_l)}, \; l \ge 1.$$

Now we consider the case of a locally bounded process $\langle M\rangle$. Then there is a sequence $(\lambda_l)_{l\geq 1}$ of $\check{I\!\!F}$-stopping times increasing to ∞ and a sequence of constants $(C_l)_{l\geq 1}$ such that

$$\langle M\rangle^{(\lambda_l)} \leq C_l \ \text{ on } I\!\!R_+ \times \Omega, \text{ for every } l \geq 1.$$

We restrict λ_l to the event $\{\check{X}_0 \notin F_{C,m}\}$ on which we observe $\widetilde{X}^m$ up to time T_1, by construction of $\widetilde{X}^m$ in 7.2, and define

$$\widetilde{\lambda}_l^m := (\lambda_l)_{\{\check{X}_0\notin F_{C,m}\}} \wedge T_1\,, \quad l \geq 1\,.$$

By construction we have

$$\widetilde{\lambda}_l^m \leq T_1 \quad \forall\, l\,, \qquad \widetilde{\lambda}_l^m \uparrow \ \text{as } l \to \infty\,, \text{ and } \{\widetilde{\lambda}_l^m < T_1\} \downarrow \emptyset \quad \text{as } l \to \infty\,.$$

Let us prove that

$$\widetilde{\lambda}_l^m \text{ are } \widetilde{I\!\!F}^m\text{-stopping times, } l \geq 1.$$

Since $\widetilde{\lambda}_l^m$ has been constructed as $\check{I\!\!F}$-stopping time, one has

$$\{\widetilde{\lambda}_l^m \leq v\} = \{\widetilde{\lambda}_l^m \leq v \wedge T_1\} \in \check{\mathcal{F}}_{v\wedge T_1}\,, \quad v \geq 0\,.$$

By (**) above, the σ-fields $\check{\mathcal{F}}_{v\wedge T_1}$ and $\widetilde{\mathcal{F}}^m_{v\wedge T_1}$ coincide in restriction to $\{\check{X}_0 \notin F_{C,m}\}$, thus

$$\{\check{X}_0 \notin F_{C,m}\} \cap \{\widetilde{\lambda}_l^m \leq v\} \in \widetilde{\mathcal{F}}^m_{v\wedge T_1}\,, \quad v \geq 0\,.$$

Since T_1 and $v \wedge T_1$ are $\widetilde{I\!\!F}^m$-stopping times,

$$\{\check{X}_0 \in F_{C,m}\} \cap \{\widetilde{\lambda}_l^m \leq v\} = \{\check{X}_0 \in F_{C,m}\} \cap \{T_1 \leq T_1 \wedge v\} \in \widetilde{\mathcal{F}}^m_{v\wedge T_1}\,, \quad v \geq 0\,.$$

Both assertions together prove that $\widetilde{\lambda}_l^m$ are $\widetilde{I\!\!F}^m$-stopping times, $l \geq 1$. Since $(T_n)_n$ are $\widetilde{I\!\!F}^m$-stopping times, also

$$\Lambda_l^{n,m} := T_n + \widetilde{\lambda}_l^m \circ (\vartheta_{T_n})\,, \quad l \geq 1\,, n \in I\!\!N_0$$

are $\widetilde{I\!\!F}^m$-stopping times. The sequence $\left(\Lambda_l^{n,m}\right)_{l\geq 1}$ has the properties

$$T_n \leq \Lambda_l^{n,m} \leq T_{n+1} \quad \forall\, l\,, \qquad \Lambda_l^{n,m} \uparrow \ \text{as } l \to \infty\,, \text{ and } \{\Lambda_l^{n,m} < T_{n+1}\} \downarrow \emptyset \quad \text{as } l \to \infty$$

$$1_{\{\check{X}_{T_n}\notin F_{C,m}\}} \left(\langle M\rangle^{T_{n+1}} - \langle M\rangle^{T_n}\right)_{\Lambda_l^{n,m}} = \left[1_{\{\check{X}_0\notin F_{C,m}\}} \langle M\rangle^{T_1}_{\widetilde{\lambda}_l^m}\right] \circ (\vartheta_{T_n}) \leq C_l\,, \quad l \geq 1$$

where we have used (H5^A). Let us define for $l \geq 1$

$$\widetilde{\rho}_l^m := \left(\Lambda_l^{0,m}\right)_{\{\Lambda_l^{0,m} < T_1\}} \wedge \ldots \wedge \left(\Lambda_l^{l-1,m}\right)_{\{\Lambda_l^{l-1,m} < T_l\}} \wedge T_l\,.$$

Then $\left(\widetilde{\rho}_l^m\right)_{l\geq 1}$ is an increasing sequence of $\widetilde{\mathbb{F}}^m$-stopping times. Since $\{\Lambda_l^{n,m} < T_{n+1}\} \downarrow \emptyset$ as $l\to\infty$ for every n fixed, the sequence increases to ∞ as $l\to\infty$, and meets by construction

$$(Y^m)^{\left(\widetilde{\rho}_l^m\right)} \leq \sum_{n=0}^{l-1} 1_{\{\check{X}_{T_n}\notin F_{C,m}\}} \left(\langle M\rangle^{T_{n+1}} - \langle M\rangle^{T_n}\right)_{\Lambda_l^{n,m}} \leq l\cdot C_l$$

on $\mathbb{R}_+\times\Omega$, for every $l\geq 1$. Thus we have a sequence of $\widetilde{\mathbb{F}}^m$-stopping times increasing to ∞ such that

$$(M^m)^{\left(\widetilde{\rho}_l^m\right)} \text{ is in } \mathcal{M}^2(\widetilde{\mathbb{F}}^m,\check{P}) \text{ with angle brackett } (Y^m)^{\left(\widetilde{\rho}_l^m\right)},\ l\geq 1.$$

This proves b).

c) By assumption we have (H5^A) for $M\in\mathcal{M}^{2,\mathrm{loc}}(\check{\mathbb{F}},\check{P})$: the processes $\langle M\rangle$, $[M]$ are additive functionals of $\check{X}$, and M satisfies

$$\forall\,\check{y}\,,\ \forall\, s,t\ :\quad M_{t+s}-M_t = M_s\circ\vartheta_{T_n}\quad P_{\check{y}}\text{-a.s.}$$

These properties carry over to M^m, $\langle M^m\rangle$ with respect $\widetilde{X}^m$ since

$$dM_s^m = 1_{(C\times(0,2^{-m})\times[0,1])^c}(N_{s^-})\,dM_s\,,\quad d\langle M^m\rangle_s = 1_{(C\times(0,2^{-m})\times[0,1])^c}(N_{s^-})\,d\langle M\rangle_s$$

depend only on the trajectory of $\widetilde{X}^m$, by construction in 7.2; for the quadratic variation $[M^m]$, use approximation by sums of quadratic increments over time partitions with mesh tending to 0. (7.18) is obtained from the ratio limit theorem together with (7.13) or (7.10). This shows that the processes M^m, $\langle M^m\rangle$, $[M^m]$ on $(\check{\Omega},\check{\mathcal{A}},\widetilde{\mathbb{F}}^m,(\vartheta_t)_{t\geq0},(P_{\check{y}})_{\check{y}\in\check{E}})$ satisfy assumption (H5^A). We check (H5^B). With lifecycles for $\widetilde{X}^m$ defined by (7.5) and invariant measure $\widetilde{\mu}^m$ given by (7.2”), note that every $\widetilde{R}_n^m$, $n\geq1$, is a passage time from $\widetilde{A}^m$ to $\left(\widetilde{A}^m\right)^c$: since M^m is constant before time $\widetilde{R}_n^m$ and since M is càdlàg, the paths of M^m are continuous at $\widetilde{R}_n^m$. Hence $(M^m)_{\widetilde{R}_n^m}$ is measurable with respect to $\widetilde{\mathcal{F}}^m_{(\widetilde{R}_n^m)^-}$, which is (*) of (H5^B). □

7.19 Lemma : We have in (7.18)

$$E_{\widetilde{\mu}^m}\left(\langle M^m\rangle_1\right) = E_{\check{\mu}}\left(\langle M^m\rangle_1\right)\,,\quad \lim_{m\to\infty} E_{\check{\mu}}\left(\langle M^m\rangle_1\right) = E_{\check{\mu}}\left(\langle M\rangle_1\right) = E_{\mu}\left(\langle M\rangle_1\right)\,.$$

Proof : For m fixed, choose a function $\check{g}$ nonnegative, $\check{\mathcal{E}}$-measurable, $0<\check{\mu}(\check{g})<\infty$ such that $\check{g}$ equals 0 on $E\times C\times(0,2^{-m})\times[0,1]$: then $\check{\mu}(\check{g})=\widetilde{\mu}^m(\check{g})$, and $\int_0^t \check{g}(\check{X}_s)ds = \int_0^t \check{g}(\widetilde{X}_s^m)ds$. We apply the RLT to $\langle M^m\rangle_t$ and $\int_0^t\check{g}(\widetilde{X}_s^m)ds$ as $\widetilde{\mathbb{F}}^m$-additive functionals, and to $\langle M^m\rangle_t$ and $\int_0^t\check{g}(\check{X}_s)ds$ as

$\check{\mathbb{F}}$-additive functionals. Since $\check{\mu}(\check{g}) = \widetilde{\mu}^m(\check{g})$, this gives $E_{\widetilde{\mu}^m}(\langle M^m\rangle_1) = E_{\check{\mu}}(\langle M^m\rangle_1)$. As $m \to \infty$, the second assertion follows by dominated convergence since

$$E_{\check{\mu}}(\langle M\rangle_1 - \langle M^m\rangle_1) = E_{\check{\mu}}\left(\int_0^1 1_{E\times C\times(0,2^{-m})\times[0,1]}(\check{X}_s)d\langle M\rangle_s\right).$$

□

7.20 Theorem : Consider $0 < \alpha \leq 1$ and $l(\cdot)$ varying slowly at ∞. Assume that condition (7.15) holds: for every g nonnegative $\mathcal{E}$-measurable with $0 < \mu(g) < \infty$, one has regular variation at 0 of resolvants in X

$$\left(R_{1/t}g\right)(x) = E_x\left(\int_0^\infty e^{-\frac{1}{t}s}\, g(X_s)\, ds\right) \quad \sim \quad t^\alpha \frac{1}{l(t)}\,\mu(g), \quad t \to \infty$$

for μ-almost all $x \in E$ (the exceptional set depending on g).

Then for local martingales $M \in \mathcal{M}^{2,\mathrm{loc}}(P_x, \mathbb{F})$ meeting (H5^A) and such that $\langle M\rangle$ is locally bounded:

a) for every m fixed, we have weak convergence in $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$ under $\check{P}$

$$\frac{1}{\sqrt{n^\alpha/l(n)}}\,(M^m_{tn})_{t\geq 0} \quad \to \quad \left(E_{\check{\mu}}(\langle M^m\rangle_1)\right)^{1/2} B(W^\alpha)$$

where $B(W^\alpha)$ is Brownian motion time-changed by an independent Mittag Leffler process;

b) we have weak convergence in $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$ under $\check{P}$

$$\frac{1}{\sqrt{n^\alpha/l(n)}}\,(M_{tn})_{t\geq 0} \quad \to \quad \left(E_{\check{\mu}}(\langle M\rangle_1)\right)^{1/2} B(W^\alpha);$$

c) we have weak convergence in $D(\mathbb{R}_+, \mathbb{R})$ as $n \to \infty$ under P_x

$$\frac{1}{\sqrt{n^\alpha/l(n)}}\,(M_{tn})_{t\geq 0} \quad \to \quad \left(E_{\mu}(\langle M\rangle_1)\right)^{1/2} B(W^\alpha).$$

Proof : By lemma 7.17, for every $(P_x, \mathbb{F})$-local martingale M meeting (H5^A) and such that $\langle M\rangle$ is a locally bounded process, M^m defined in 7.17 is an $(\check{P}, \widetilde{\mathbb{F}}^m)$-local martingale on $(\check{\Omega}, \check{\mathcal{A}})$, and meets assumptions (H5^A)+(H5^B) with respect to $\widetilde{X}^m$, $\widetilde{\mathbb{F}}^m$ and with respect to the life cycles $(\widetilde{R}^m_n)_n$ defined in (7.5). By the remark preceding lemma 7.9, we know that all assumptions needed in section 4 are met for $\widetilde{X}^m$ and M^m.

Combining 7.14+(7.18)+7.19 with theorem 4.12 for M^m, we get a).

It remains to prove b). By definition of M^m and by Lenglart's inequality ([J-Sh 87, p. 35]),

$$\check{P}\left(\sup_{0\leq t\leq t_0} \frac{1}{\sqrt{n^\alpha/l(n)}}\,|M_{tn} - M^m_{tn}| > \sqrt{\varepsilon}\right)$$

(for arbitrary m, $t_0 < \infty$ and $\varepsilon, \eta > 0$) is bounded by

$$\frac{\eta}{\varepsilon} + \check{P}\left(\frac{1}{n^\alpha / l(n)} \int_0^{t_0 n} 1_{E \times C \times (0, 2^{-m}) \times [0,1]}(\check{X}_s) d\langle M \rangle_s > \eta \right)$$

where the last expression decreases to 0 as m tends to ∞. Thus, for $t_0 < \infty$ and $\varepsilon > 0$ there is some $m_0 = m_0(t_0, \varepsilon)$ such that

$$\lim_{n\to\infty} \left(\sup_{m \geq m_0} \check{P}\left(\sup_{0 \leq t \leq t_0} \frac{1}{\sqrt{n^\alpha / l(n)}} |M_{tn} - M_{tn}^m| > \sqrt{\varepsilon} \right) \right) < \varepsilon$$

where we have used theorem 7.16. Let G be nonnegative, uniformly continuous and bounded on the canonical path space $D(I\!R_+, I\!R)$ of $\frac{1}{\sqrt{n^\alpha / l(n)}}\, (M_{tn})_{t \geq 0}$. Then for every $\delta > 0$, there are constants C_1, C_2 such that for arbitrary $m \geq m_0$

$$\limsup_{n\to\infty} E_{\check{P}}\left(G\left(\frac{1}{\sqrt{n^\alpha / l(n)}}\, M_{\cdot n} \right) \right) \leq \lim_{n\to\infty} E_{\check{P}}\left(G\left(\frac{1}{\sqrt{n^\alpha / l(n)}}\, M_{\cdot n}^m \right) \right) + C_1 \delta + C_2 \varepsilon$$

$$\liminf_{n\to\infty} E_{\check{P}}\left(G\left(\frac{1}{\sqrt{n^\alpha / l(n)}}\, M_{\cdot n} \right) \right) \geq \lim_{n\to\infty} E_{\check{P}}\left(G\left(\frac{1}{\sqrt{n^\alpha / l(n)}}\, M_{\cdot n}^m \right) \right) - C_1 \delta - C_2 \varepsilon$$

(this is seen as follows: according to the definition of Skorohod distance $d(.,.)$ on $D(I\!R_+, I\!R)$, see [J-Sh 87, ch. VI], for every $\delta > 0$ there is $\rho{=}\rho(\delta) > 0$, $\varepsilon{=}\varepsilon(\rho) > 0$, $t_0{=}t_0(\rho) < \infty$ such that $\sup_{0 \leq t \leq t_0} |f(t){-}g(t)| < \sqrt{\varepsilon}$ implies first $d(f,g) < \rho$, and second $|G(f) - G(g)| < C_1 \delta$, for all $f, g \in D(I\!R_+, I\!R)$). Combining these inequalities with weak convergence

$$\frac{1}{\sqrt{n^\alpha / l(n)}}\, (M_{tn}^m)_{t \geq 0} \quad \rightarrow \quad \left(E_{\check{\mu}}\left(\langle M^m \rangle_1 \right) \right)^{1/2} B(W^\alpha)$$

according to a) and using

$$\lim_{m\to\infty} E_{\check{\mu}}\left(\langle M^m \rangle_1 \right) = E_{\check{\mu}}\left(\langle M \rangle_1 \right)$$

as shown in 7.19, we get the assertion of b). c) is a simple restatement of b). □

By theorems 7.16 and 7.20, all assertions of subsection 3.3 are proved.

Overview: assumptions (H1) - (H6)

We give a list of the assumptions used in this note and resume their connections.

$X = (X_t)_{t\geq 0}$ is a continuous-time strong Markov process with semigroup $(P_t(\cdot,\cdot))_{t\geq 0}$, taking values in a Polish space $(E,\mathcal{E})$, with càdlàg paths, living on some $(\Omega, \mathcal{A}, I\!\!F, (\vartheta_t)_{t\geq 0}, (P_x)_{x\in E})$. Only in section 7 we require that X is the canonical process on its canonical path space $D(I\!\!R_+, E)$.

The first assumption is

(H1): $X = (X_t)_{t\geq 0}$ is Harris with invariant measure μ.

This is the basic assumption used throughout the paper; (H1) is equivalent (see 1.4) to any of the following properties (H2) or (H2$^\alpha$), $0 < \alpha < \infty$:

(H2): $\overline{X} = (X_{\sigma_n})_{n\geq 0}$ is Harris, with $\sigma_n - \sigma_{n-1}$ i.i.d $exp(1)$-waiting times independent of X

(H2$^\alpha$): $\overline{X}^\alpha = (X_{\rho_n})_{n\geq 0}$ is Harris, with $\rho_n - \rho_{n-1}$ i.i.d $exp(\alpha)$-waiting times independent of X

where we put $\sigma_0 = \rho_0 = 0$, and where the invariant measure for $\overline{X}$ or $\overline{X}^\alpha$ is μ.

Via (H2)+(H2$^\alpha$) for some $\alpha > 1$, see 6.7, we have the following property (H6) which is needed for Nummelin splitting:

(H6): The one-step transition kernel $U^1(\cdot,\cdot)$ of $\overline{X}$ satisfies the minorization condition $(\widetilde{M}_1)$: there is some set $C \in \mathcal{E}$ with $\mu(C) > 0$, some probability measure ν on $(E,\mathcal{E})$ equivalent to $\mu(\cdot \cap C)$, and some $0 < \alpha < 1$ such that $U^1(x,dy) \geq \alpha 1_C(x)\nu(dy)$, for all $x, y \in E$.

A second group of assumptions is used for processes with life cycles:

(H3): X has a recurrent atom $A \in \mathcal{E}$ and a life cycle decomposition $(R_n)_{n\geq 1}$, see 1.9.A + 1.9.B.

(H4): There is some function f, bounded, nonnegative, $\mathcal{E}$-measurable, $0 < \mu(f) < \infty$, such that

$$x \quad \to \quad E_x\left(\int_0^{R_1} f(X_s)\, ds\right) \quad \text{is bounded on } E$$

(called *weakly special for* X *and* R_1).

Under *suitable* definition of the life cycle decomposition $(R_n)_n$ in (H3), (H4) will hold in virtue of the Harris property (H2), see proposition 3.4.

A third group of assumptions deals with $M \in \mathcal{M}^{2,\mathrm{loc}}(P_x, \mathbb{F})$, the class of locally square integrable local martingales w.r.t. P_x and $\mathbb{F}$, with càdlàg paths and with $M_0 = 0$:

(H5^A): M has the property

$$\forall\, y\,, \forall\, s, t: \quad M_{t+s} - M_t = M_s \circ \vartheta_t \quad P_y\text{-a.s.}\,,$$

angle brackett $\langle M \rangle$ and square brackett $[M]$ are additive functionals of X, and $E_\mu\left(\langle M \rangle_1\right) < \infty$.

Whenever we work with a life cycle decomposition $(R_n)_n$ of the process X, we need independent increments of M over life cycles of X:

(H5^B): For the life cycle decomposition $(R_n)_n$ of (H3), M satisfies either $(*)$:

$(*)$ $\qquad M_{R_n}$ is measurable with respect to $\mathcal{F}_{R_n^-}$, for all $n \geq 1$

or the following $(**)$:

$(**)$ $\qquad R_{n+1} - R_n$ and $M - M^{R_n}$ are independent of $\mathcal{F}_{R_n}$, for all $n \geq 1$.

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