

Percolation thresholds, critical exponents, and scaling functions on planar random lattices and their duals

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The bond-percolation process is studied on periodic planar random lattices and their duals. The thresholds and critical exponents of the percolation transition are determined. The scaling functions of the percolating probability, the existence probability of the appearance of percolating clusters, and the mean cluster size are also calculated. The simulation result of the percolation threshold is $p_c = 0.3333 \pm 0.0001$ for planar random lattices, and 0.6670 ± 0.0001 for the duals of planar random lattices. We conjecture that the exact value of p_c is $1/3$ for a planar random lattice and $2/3$ for the dual of a planar random lattice. By taking possible errors into account, the results of our critical exponents agree with the values given by the universality hypothesis. By properly adjusting the metric factors on random lattices and their duals, we demonstrate explicitly that the idea of a universal scaling function with nonuniversal metric factors in the finite-size scaling theory can be extended to random lattices and their duals for the existence probability, the percolating probability, and the mean cluster size. [S1063-651X(99)00112-9]

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I. INTRODUCTION

The percolation process [1] as defined on a variety of lattices has been studied very extensively. Recent research activities focused on finding new universal quantities and checking the finite-size scaling including corrections to scalings [2–17]. However, most of the studies have been restricted to the use of regular lattices with fixed coordination numbers. For example Yonezawa *et al.* [18] used lattices that have mixed values of coordination numbers but still have a regular pattern to release the condition imposed by regular lattices in studying. These lattices include dice, Penrose tiling, and the dual lattice of Penrose. Their results are consistent with the assertion from the universality hypothesis which states that the critical exponents are independent of the details of the lattice structure but depend on the dimension of the lattice [1,18–20]. Based on the same motivation, we consider the case of planar random lattices and their duals that preserve the rotational symmetry effectively.

Random lattices were first introduced by Dirichlet and Voronoi [21] for use in condensed matter physics, and they were also employed by Christ, Friedberg, and Lee [22,23] to formulate other types of lattice field theories. We briefly describe the construction and properties of these lattices, and refer to Refs. [20–22] for the details. Planar random lattices and their duals are defined on a rectangular area with periodic boundary conditions. To construct such lattices, we consider N sites generated by a random number generator and distributed in a rectangle of area A with periodic boundary condition. A random lattice is constructed in a way that for each of the N sites, only nearby sites are connected to it by links, and the finite area A is divided into nonoverlapping 2-simplices with vertices on the N random sites. We adopt the algorithm given by Christ, Friedberg, and Lee [22,23] to connect a site to nearby ones. Consider an arbitrary group of three lattice sites. A circle is drawn such that these three sites

lie on a circle. If there is no lattice site inside this circle, these three sites are connected by links and form a 2-simplex. This procedure, in principle, is performed repeatedly on all possible groups of three lattice sites. Note that 0-simplices are sites, 1-simplices are links, and 2-simplices are triangles. The dual of a random lattice is constructed in a way that the finite area A is partitioned into N nonoverlapping planar convex polyhedra, and there is a one to one correspondence between polyhedra and random sites. Let i be one of the N random sites. A point p in A is said to belong to i if i is the nearest site to p . The corresponding polyhedron of site i consists of all the points belonging to i . Vertices of N polyhedra are sites of the dual lattice, and they are the duals of 2-simplices on the original random lattice. An interfacial line between two polyhedra is a link between two dual sites, and it is the dual of a 1-simplex. Examples of a periodic planar random lattice with its dual on a rectangular area is shown in Fig. 1. For a large random lattice, the average coordination number for a site is 6, and a site on the average is shared by six triangles. There are local fluctuations around these average values, and these fluctuations can always be described by a γ function with appropriately chosen parameters. For the dual of a planar random lattice, the coordination number for a dual site is always 3, and a site is always shared by three polyhedra. Rotational symmetry is preserved effectively in random lattices because there is no preferred direction. To ensure that the condition of homogeneity is met, for a measurement we average the results from a number of random lattices.

In this paper we study the bond-percolation process on periodic planar random lattices and their duals by numerical simulations. For a bond-percolation model, the link, which connects two nearest neighboring sites, has two states, occupied and unoccupied, and a bond exists for an occupied link. If the neighboring links are occupied, the corresponding bonds belong to the same cluster, and the sites connected by these bonds also are in the same cluster. We use the random bond process to generate configurations as follows. For a

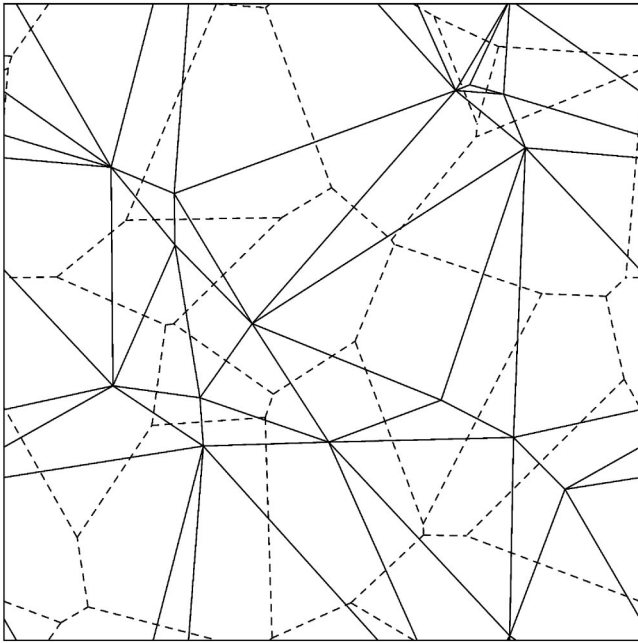


FIG. 1. Example of random lattices (solid lines) with their duals (dashed lines) on a 2D rectangular area with a periodic boundary condition.

given occupied probability p , we associate a random number r , $0 < r < 1$, with a link for all links on the lattice. If the associated random number r of a link is less than p , this link is occupied, and a bond exists. Such a bond distribution on the lattice is a configuration of the bond-percolation model. For a small value of p , only finite clusters appear in configurations. When the value of p increases, the size of a finite cluster will increase, and eventually a percolating cluster appears when p reaches a critical value p_c which is called the percolation threshold. To ensure that the number of configurations used to calculate the quantities of the model is sufficiently large, we calculate the ratio of bond numbers to link numbers for a configuration, average the ratios over all configurations, and require that the average should be equal to the given occupied probability p . Then we use these configurations to calculate the existence probability, the percolating probability, and the mean cluster size on planar random lattices and their duals. After averaging the results over a number of random lattices, we have the final values of these geometric quantities. Using these values, we can calculate the critical thresholds, critical exponents, and scaling functions.

This paper is organized as follows: In Sec. II, we propose an efficient method, which can be extended straightforwardly to higher dimensions, to identify the percolating clusters. In Sec. III, we present the simulation results of the percolating probabilities, the existence probabilities, the mean cluster size distributions, and the mean cluster size for periodic square lattices, and periodic planar random lattices and their duals. In Sec. IV, we use the finite-size scaling theory to determine the percolation thresholds and extract the critical exponents from the simulation results, and the resultant values are reported. In Sec. V, by choosing the values of the nonuniversal metric factors on random lattices and their duals properly, we show that universal scaling functions of regular lattices can be extended to random lattices for the

existence probability, the percolating probability, and the mean cluster size. Finally the summary of the results and the conclusion are given in Sec. VI.

II. PERCOLATING CLUSTERS ON RANDOM LATTICES

A percolating cluster is defined as a cluster in which the bonds can extend from one side to the opposite side of a lattice. In this section we discuss the method we used in distinguishing a percolating cluster in a periodic planar random lattice and its dual. This method can be extended to higher dimensions in a very straightforward way.

For a periodic square lattice, the condition for a cluster to be horizontally (vertically) percolated is that it contains at least one lattice site for each column (row). For a cluster fulfilling this condition, by shifting the lattice with some rows or columns we are able to find a path in it to go through the lattice from the left (top) to the right (bottom). The periodic boundary condition here does not add any complexity in distinguishing percolating clusters from finite clusters.

Periodic random lattices are similar to square lattices in that a horizontal (vertical) percolating cluster is formed when the bonds contained in the cluster can be extended from the boundary sites in the left (top) to those in the right (bottom). However, due to the periodic boundary condition, it is more difficult to identify a percolating cluster on a random lattice. To see this, we view the boundary of a random lattice as a window, and we define the bonds crossing this window as boundary bonds and all the sites connected by the boundary bonds as boundary sites. Then a percolating cluster contains at least two boundary sites, with one on the opposite side of the other. However, because of the periodic boundary condition, the boundary window can be shifted arbitrarily, and for each shift we have new boundary bonds and boundary sites from the new location of the square window. Because of the random distribution of lattice sites, we have to move the square window very slowly over the whole lattice in order to distinguish all the possible percolating clusters. This method works, but we propose a more efficient method to solve this problem.

First we fix the window for a given lattice, and obtain the corresponding boundary bonds and boundary sites to this window. Then we use the Hoshen-Kopelman method [24] to give a proper cluster label to each site. Finally, to distinguish a percolating cluster, we note that the percolating paths of a percolating cluster in this fixed square window can be divided into two types. The first type are the paths which extend from the top to the bottom or from the left to the right with or without the boundary bonds, as shown in Fig. 2(a). These paths can be observed explicitly in a given fixed square window. The second type are the paths which contain at least one boundary bond, and can extend through the lattice only when the periodic boundary condition is imposed, as shown in Fig. 2(b). These paths cannot be observed explicitly because the paths are discontinuous in the fixed square window [25].

To distinguish percolating clusters from all clusters, we have to check whether or not one of these two types of paths exist. First we notice that a percolating cluster has to contain at least one boundary site on each of the two opposite sides. Then we start from one of these boundary sites, and follow

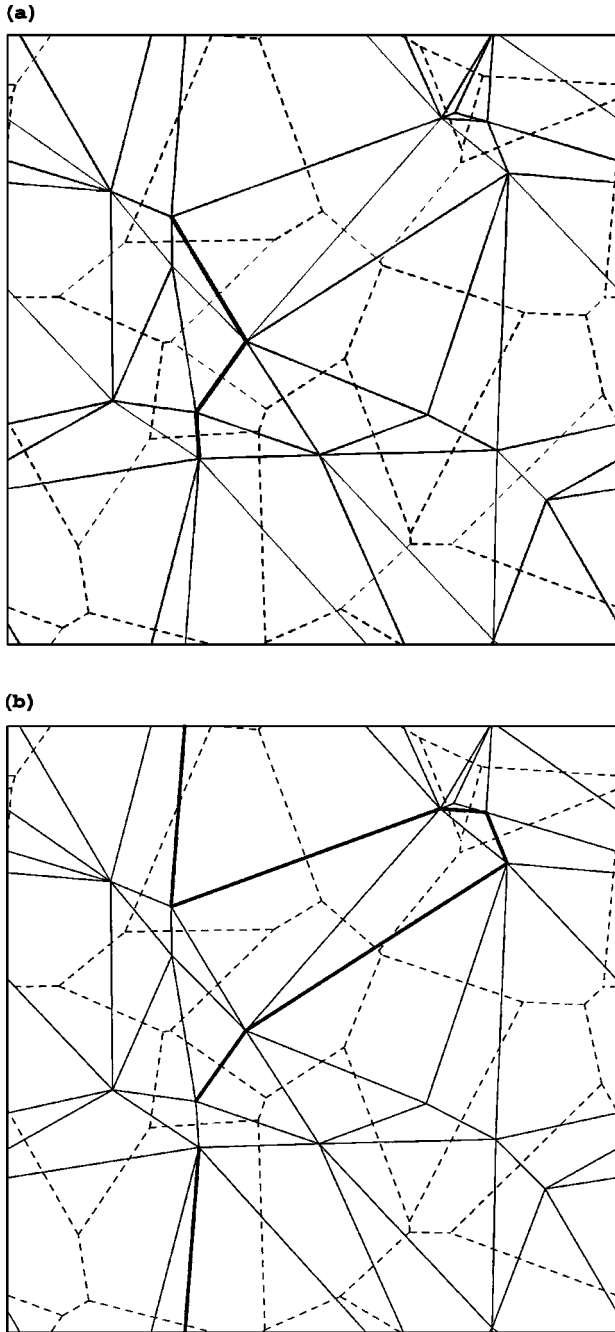


FIG. 2. Examples of (a) the first kind of percolating path without boundary bonds (bold solid lines) and (b) the second kind of percolating path with one boundary bond (bold solid lines) on a random lattice.

all possible paths to check if there is a path which extends to the other boundary site on the opposite side. When the check is positive, then this cluster is a percolating cluster with the first type of path. On the other hand, if the check is negative, then we locate the discontinuity between the two paths starting from the two boundary sites, respectively. Here the two boundary sites are connected by boundary bonds through the periodic boundary condition. For the case of a vertical cluster, this is equivalent to obtaining the lowest site at the end of the path starting from the top boundary site, and the highest site at the end of the path starting from the bottom boundary site. Such a discontinuous path is of the second type only

when there exists a site on the lattice which links to the highest site, with the vertical coordinate higher than the vertical coordinates of all the sites that link to the lowest site. For a percolating cluster containing this path, we can always move the square window to a new location, so that the two end sites of the two paths are two boundary sites located at two opposite sides of the new square window. By repeating this procedure to all boundary sites in the same cluster and by performing a similar check of horizontal clusters, we can determine all percolating clusters in a configuration defined on periodic planar random lattices.

III. GEOMETRICAL QUANTITIES

The main task of the simulations is to determine the distribution of finite and percolating clusters for configurations obtained from random bond processes. This distribution is then used to determine the geometric quantities, including the mean cluster size distribution $n_s(p)$, defined as the ratio of the average number of clusters with s bonds to the total number of bonds; the percolating probability $P(p)$, defined as the ratio of the number of bonds in the percolating cluster to the total number of bonds; the existence probability $E_p(p)$, defined as the probability of the appearance of percolating configurations; and the mean cluster size $S(p)$, defined as

$$S(p) = \sum_{s=1}^{\infty} s \left(\frac{sn_s}{\sum_{s=1}^{\infty} sn_s} \right), \quad (1)$$

where the factor, $sn_s / \sum_{s=1}^{\infty} sn_s$ is the probability that a bond belongs to a cluster containing s bonds. Note that one can use different spanning rules to define a percolating cluster [2], and the superscript X in E_p^X is used to denote the spanning rule for the existence probability E_p . We use E_p^v for the existence probability of percolating clusters of vertical crossings, E_p^h for those of horizontal crossings, E_p^l for those of simultaneous vertical and horizontal crossings, and E_p^u for those of either vertical or horizontal crossings. There is a relation among these four probabilities, $E_p^u = E_p^v + E_p^h - E_p^l$, which holds for any occupied probability. There is an additional probability used in our analyses, which is the average of E_p^v and E_p^h defined as $E_p^A = \frac{1}{2}(E_p^v + E_p^h)$.

To compare the results, we first perform simulations on regular square lattices defined on a square of sizes $L=80, 100, 120, \text{ and } 160$. Here the size L is defined by relation $N=L^2$, and N is the total number of sites. We take 60 occupied probabilities around the critical percolation threshold for every 0.002, and use the random bond process to generate 10^5 configurations for an occupied probability. Then we perform simulations on random lattices in the case when all the links on a random lattice have the same occupied probability. The lattices we used are periodic planar random lattices of unit density with their duals defined on a square of sizes $L=80, 100, 120, \text{ and } 160$. Here we still use the random bond process to generate 10^5 configurations for an occupied probability, and increase the number of configurations around the peaks of the mean cluster size $S(p)$. Note that in order to ensure that the obtained results are the same as the results

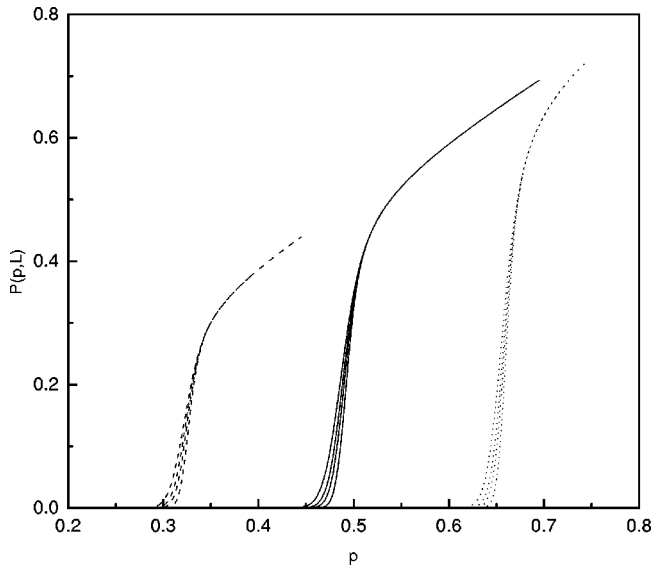


FIG. 3. The percolating probabilities $P(p, L)$ for periodic square lattices (solid curves), periodic planar random lattices (dashed curves), and the duals of periodic planar random lattices (dotted curves). Below the intersection, the curves from left to right are for the sizes $L=80, 100, 120,$ and 160 .

from a homogeneous lattice, we keep averaging the results from random lattices of the same size, but use different site distributions until the condition $E_p^v = E_p^h$ is satisfied. We take the average over 6–10 random lattices depending on the size of the lattice.

The results of the percolating probabilities, the existence probabilities, and the mean cluster size for different types of lattices of different sizes are shown in Figs. 3–5. By comparing the results shown in Figs. 3 and 4, we can see that for a given occupied probability the existence probability of the dual of a planar random lattice is less than that of a regular

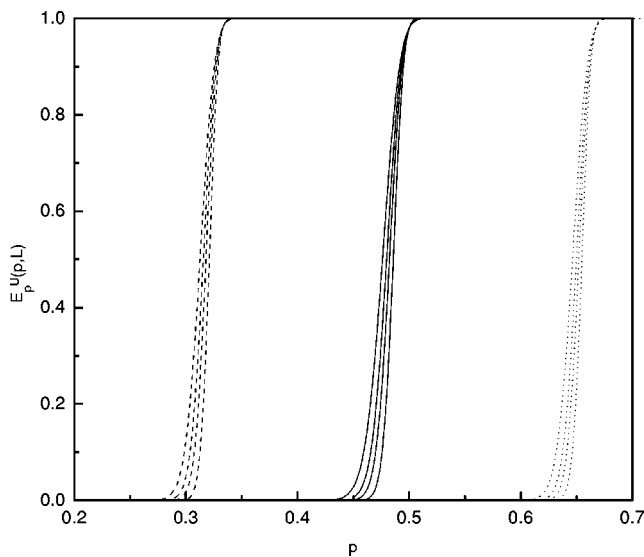


FIG. 4. The existence probabilities $E_p^U(p, L)$ for periodic square lattices (solid curves), periodic planar random lattices (dashed curves), and the duals of periodic planar random lattices (dotted curves). Below the intersection, the curves from left to right are for the sizes $L=80, 100, 120,$ and 160 .

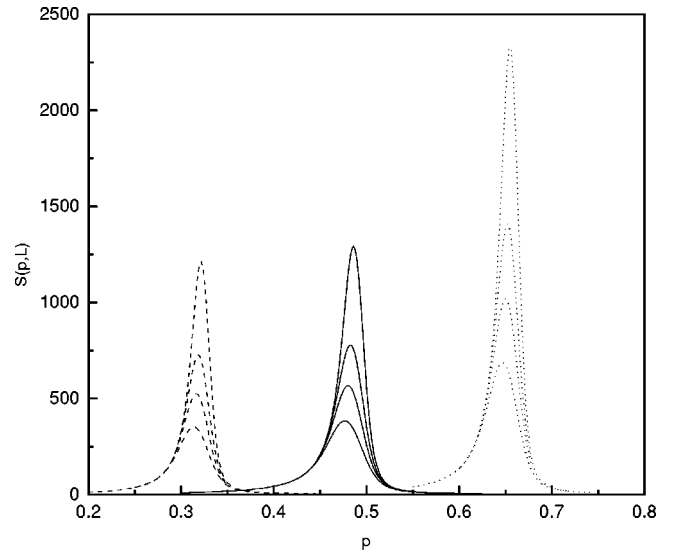


FIG. 5. The mean cluster size $S(p, L)$ for periodic square lattices (solid curves), periodic planar random lattices (dashed curves), and the duals of periodic planar random lattices (dotted curves). The curves of the peaks from up to down curves are for linear dimensions $L=160, 120, 100,$ and 80 .

lattice, and the existence probability of the latter is less than that of a planar random lattice. The percolating probability also has the same feature. From this feature we may conclude that the coordination number of a lattice plays an essential role in the occurrence of percolation, and homogeneity enhances the occurrence of percolations.

IV. PERCOLATION THRESHOLDS AND CRITICAL EXPONENTS

There are many ways of determining the percolation thresholds and the critical exponents from the simulation results given in Sec. IV. Here we mainly use the finite-size scaling theory [26,27] due to the lack of analytic results for the percolation model on planar random lattices and their duals. In this section, we first describe the method we use, and then we present the results. Error analyses and discussions of the results are also given.

To perform the estimations, we note the fact that the derivative dE_p^X/dp may be regarded as the distribution function of the threshold concentration p , and this distribution function takes the Gaussian form of

$$\frac{dE_p^X(p)}{dp} = \frac{1}{\sqrt{2\pi}\Delta_L^X} \exp\left[-\frac{1}{2}\left(\frac{p-p_c^X(L)}{\Delta_L^X}\right)^2\right], \quad (2)$$

where $p_c^X(L)$ is the average threshold concentration, and Δ_L^X is the standard deviation from $p_c^X(N)$ with $\Delta^2 = \langle p^2 \rangle - \langle p \rangle^2$ [1,18,29]. For a finite-size system with linear size L , it has been recognized that the system size L scales with the correlation length ξ of the bulk system [26,27]. The divergence of the correlation length ξ is described by the critical exponent ν as

$$\xi \propto |p - p_c|^{-\nu}. \quad (3)$$

Then in the finite system we have the relation

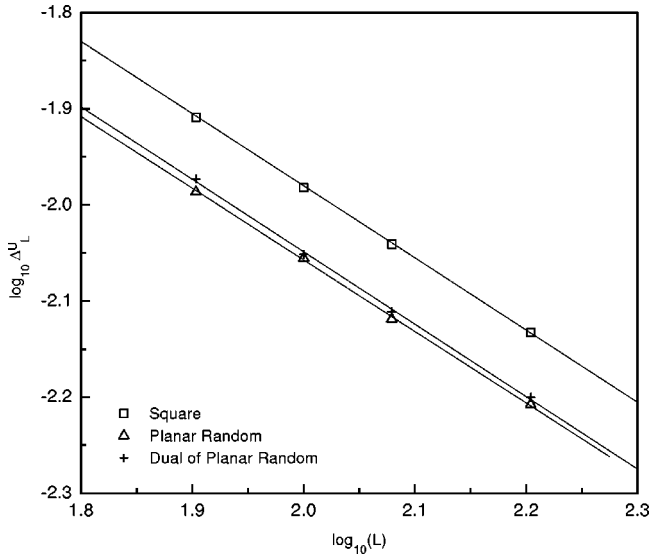


FIG. 6. The logarithms of the deviation Δ_L^X vs $\log_{10} L$ for different types of lattices with the spanning rule $X=U$.

$$|p_c^X(L) - p_c| \propto L^{-1/\nu}, \quad (4)$$

where $p_c^X(L)$ is the effective percolation threshold in the finite system. This implies that the deviation Δ_L^X is also satisfied with the scaling law

$$\Delta_L^X \propto L^{-1/\nu}. \quad (5)$$

Thus we first use this scaling law to determine the critical exponent ν , and then use Eq. (4) with the given value of ν to determine the percolation threshold p_c . Once the percolation threshold is determined, we employ the finite-size scaling relations of the percolating probability $P(p, L)$, the mean cluster size $S(p, L)$, the mean cluster size distribution $n_s(p, L)$, and the size of a percolating cluster $s_{\text{perc}}(L)$ for the finite systems of sizes L to determine the critical exponents D , τ , β , and γ . These scaling relations [1] are

$$P(p_c, L) \propto L^{-\beta/\nu}, \quad (6)$$

$$S(p_c, L) \propto L^{\gamma/\nu}, \quad (7)$$

$$n_s(p_c) \propto s^{-\tau}, \quad (8)$$

and

$$s_{\text{perc}}(L) \propto L^D. \quad (9)$$

With the above method, we first use the simulation results of $E_p^X(p, L)$ to find the derivative, $dE_p^X(p, L)/dp$. Then by fitting this derivative to the Gaussian distribution of Eq. (2), we obtain the effective percolation thresholds $p_c^X(L)$ and the deviations Δ_L^X for three different spanning rules $X=U$, A , and I . Then, from Eq. (5), we can obtain the critical exponent ν for different types of lattices by using the least-square fit. The results of $\log_{10} \Delta_L^X$ vs $\log_{10} L$ for different types of lattices are shown in Fig. 6, in which the slope of each straight line corresponds to the exponent $-1/\nu$. We can obtain three estimations of ν from three different spanning rules for a type of lattice. There are small variations among these three

TABLE I. The simulation results of the critical exponents ν , β , γ , τ , and D for square lattices, planar random lattices, and their duals with periodic boundary conditions.

Lattices	ν	β	γ	τ	D
Square	1.3304	0.1395	2.2133	2.0517	1.8954
Planar random	1.3329	0.1388	2.2208	2.0578	1.8959
Dual of planar random	1.3247	0.1320	2.2040	2.0586	1.9010
Theoretical prediction	4/3	5/36	43/18	187/91	91/48

values, and we take the average values as our estimations. The estimated values of ν are listed in Table I, and we find that the largest deviation from the exact value of regular lattices, which is $\nu = \frac{4}{3}$, is only about 1%. We then take the exact value $\nu = \frac{4}{3}$ for the three types of lattices in obtaining the critical thresholds p_c . We obtain p_c by extrapolating the effective percolation threshold $p_c^X(L)$ of finite systems with size L according to the scaling relation of Eq. (4), as shown in Fig. 7, and the results we obtain are $p_c = 0.5000 \pm 0.0001$ for a regular square lattice, $p_c = 0.3333 \pm 0.0001$ for a planar random lattice, and $p_c = 0.6670 \pm 0.0001$ for the dual of a planar random lattice.

Our values of the thresholds of planar random lattices and their duals agree with the known relations. First, for an infinite lattice L_∞ and its dual L_∞^d , there is a relation between percolation thresholds [18,30,31]:

$$p_c(L_\infty) + p_c(L_\infty^d) = 1. \quad (10)$$

Because of the self-duality of a square lattice, this equation immediately gives the value $p_c(\text{square}) = \frac{1}{2}$. Comparing with this exact result, our estimation is corrected up to the third digit after the decimal point. For planar random lattices, the corresponding value of $p_c(L_\infty) + p_c(L_\infty^d)$ is 1.0003, which agrees with the exact value again up to the third digit after the decimal point. In addition to this dual relation, there is an empirical universal quantity about the thresholds, called the

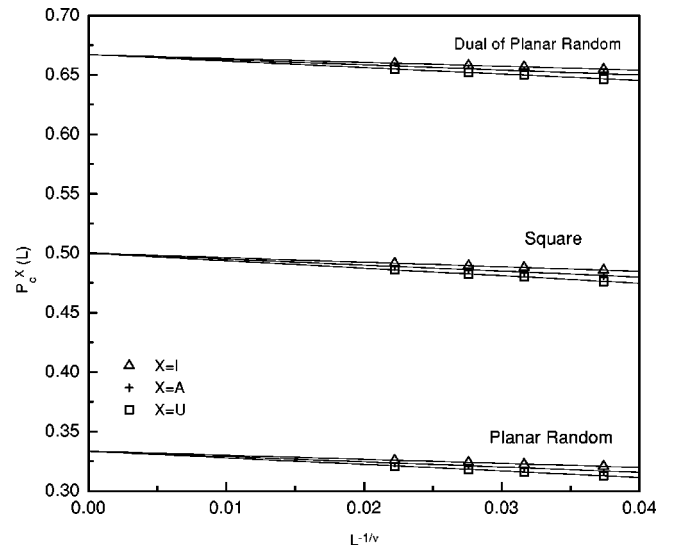


FIG. 7. The extrapolation results of three percolation thresholds p_c^U , p_c^A , and p_c^I for periodic square lattices, periodic planar random lattices, and the duals of periodic planar random lattices.

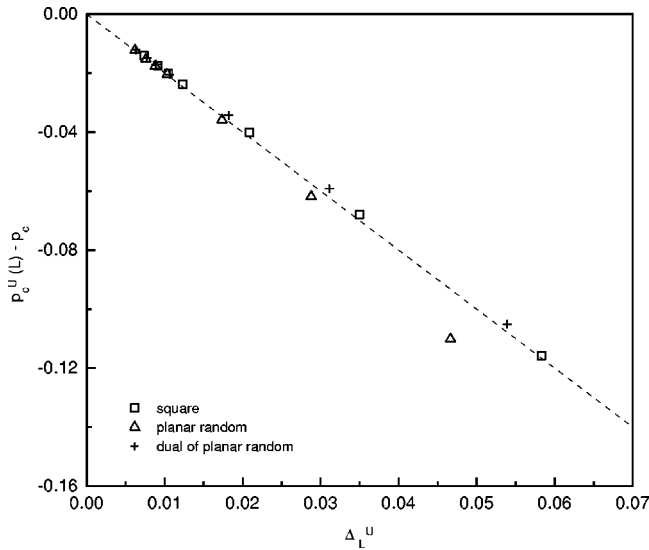


FIG. 8. The shift $(p_c^U(L) - p_c)$ vs the width Δ_L^U for different types of lattices.

critical value η_c of the effective coordination number, which is defined as the product of the threshold p_c and the coordination number z . This critical value seems to be given as [18,32]

$$\eta_c \equiv zp_c = \frac{d}{d-1}, \quad (11)$$

which is 2 for two dimensions. For planar random lattices, the coordination number varies from site to site, and we take the average value 6 to yield the result $\eta_c = 1.9998$. For the duals of random lattices, the coordination number is a constant, 3, and the corresponding η_c is 2.0010. They all agree with the value 2 very well. From the dual relation and the empirical universal value of η_c , we may conjecture that the exact value of p_c is $1/3$ for a planar random and $2/3$ for the dual of a planar random lattice. Besides η_c , there is another universal quantity pointed out by Gropengiesser and Stauffer [9], and it is the shift-to-width ratio $(p_c^X(L) - p_c)/\Delta_L^X$ of the thresholds for finite systems of enough large size. To see whether or not this universal value can be extended to random lattices, in Fig. 8 we plot $(p_c^U(L) - p_c)$ versus Δ_L^U for regular square lattices, planar random lattices, and the duals of planar random lattices with $L = 10, 20, 40, 80, 100, 120,$ and 160 . This result indicates that $(p_c^U(L) - p_c)$ is indeed proportional to Δ_L^U as the size of the lattice is increased, and, for different types of lattices, including random lattices, we have $(p_c^U(L) - p_c) \approx -2.0\Delta_L^U$. Note that this result may provide another way of determining the percolation threshold p_c without knowing the value of critical exponent ν .

To obtain the fractal dimension D and the critical exponents τ , we first plot $\log_{10} s_{\text{perc}}$ versus $\log_{10} L$ shown in Fig. 9, and $\log_{10} n_s(p_c)$ versus $\log_{10} s$ shown in Fig. 10. Then according to Eqs. (8) and (9), we can obtain exponents D and τ by calculating the slopes using the least-square fit. For the exponents β and γ , according to Eqs. (6) and (7), we first plot $\log_{10} P$ versus $\log_{10} L$ shown in Fig. 11, and $\log_{10} S$ versus $\log_{10} L$ shown in Fig. 12, and then calculate the slopes to obtain the values of β/ν and γ/ν , which give the expo-

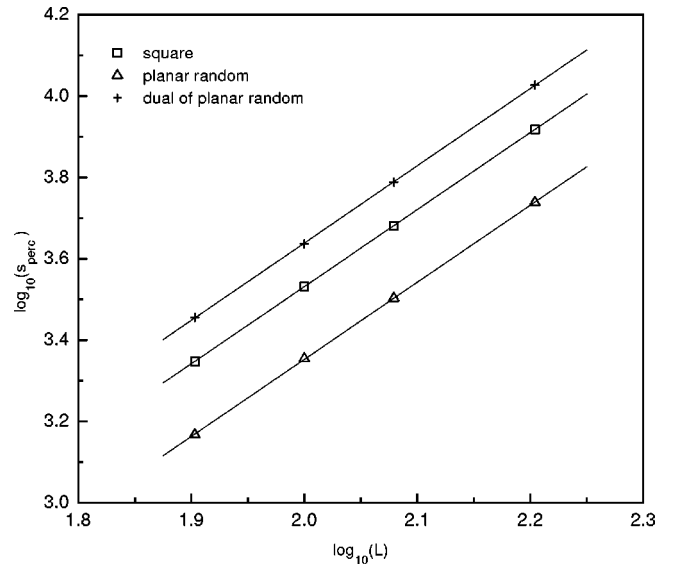


FIG. 9. The logarithm of the size of a percolating cluster, $\log_{10} s_{\text{perc}}$, as a function of $\log_{10} L$ for different types of lattices.

nents β and γ by using $\nu = \frac{4}{3}$. We list all the results in Table I along with the results obtained by Yonezawa *et al.* [18].

To give error estimations of the results of critical exponents, we note that the possible source of errors is from the numerical method and small lattice size we used, and we take the deviations of our results of square lattices from the exact values as the error estimations. These deviations are listed in Table II, and they are less than 1% for the exponents ν , D , τ , and β . A larger deviation, which is about 8%, occurs at the value of γ . If we take into account these deviations from the simulation results, then our results of the critical exponents for periodic planar random lattices and their duals deviate by less than 1% from the exact results for regular lattices, except for the value of β/ν , whose deviation is about 5%. However, there is an additional source of errors

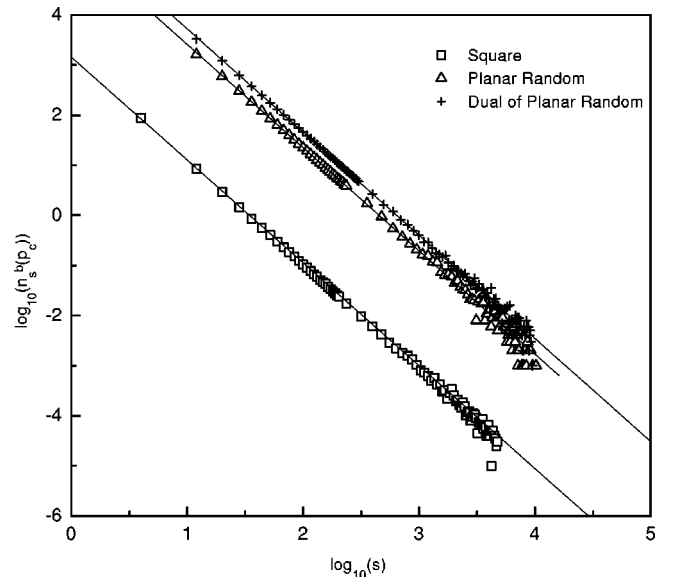


FIG. 10. The logarithm of the number of clusters composed of s bonds, $\log_{10} n_s$, as a function of $\log_{10} L$ for different types of lattices.

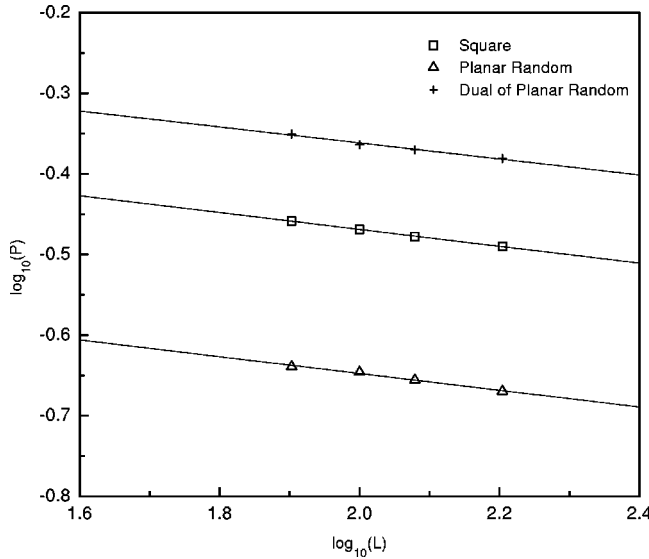


FIG. 11. The logarithm of the mean cluster size, $\log_{10} P$, as a function of $\log_{10} L$ for different types of lattices.

for random lattices. As discussed in Sec. I, there are fluctuations for the structures of random lattices, and the additional errors are due to the use of the average of the results of an insufficiently large number of sample lattices for a given site number to represent the result of a uniform random lattice. To reduce this error, we use the condition $E_p^h = E_p^v$, which holds for a homogeneous lattice, to decide whether or not the number of sample lattices for a give site number on planar random lattices is enough. If we take these possible errors into account, then we may conclude that our results for the critical exponents are consistent with the assertion from the universality hypothesis.

V. FINITE-SIZE SCALING AND SCALING FUNCTION

For a quantity X to scale as $X(t) \sim t^{-p}$ near the critical point $t=0$ in the infinite system, according to finite-size scal-

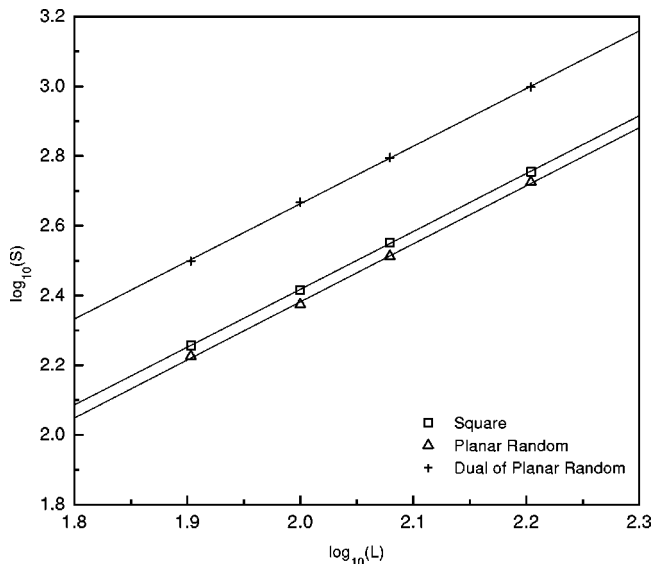


FIG. 12. The logarithm of the mean cluster size, $\log_{10} S$, as a function of $\log_{10} L$ for different types of lattices.

TABLE II. The ratios of the critical exponents from our simulations to the values of the theoretical predictions. The subscript 0 in the critical exponents denotes the value of the theoretical prediction from regular lattices. The simulation results by Yonezawa *et al.* [18].

Lattices	ν/ν_0	β/β_0	γ/γ_0	τ/τ_0	D/D_0
Square	0.998	1.004	0.926	0.998	1.000
Planar random	1.000	0.999	0.930	1.001	1.000
Dual of planar random	0.994	0.950	0.923	1.002	1.003
Square	1.01	0.95	0.92	0.96	1.00
Kagome	1.02	0.76	1.03	0.96	0.98
Dice	1.01	0.94	0.95	0.97	1.01
Penrose	1.01	0.79	1.03	0.97	0.98
Dual of Penrose	0.98	0.90	1.04	0.96	1.02

ing theory [26,27], a quantity $X_L(t)$ in a finite system characterized by a size L should obey the general scaling law

$$X_L(t) \sim L^{\rho/\nu} F(tL^{1/\nu}), \quad (12)$$

where $F(x)$, with $x = tL^{1/\nu}$, is called a scaling function. This scaling form was pushed one step further by Privman and Fisher [28] by introducing the concept of universal scaling functions and nonuniversal metric factors. They argued that the only nonuniversal factors are metrical ones relating the relevant variables of the system, and no other nonuniversal parameters enter into finite-size scaling formulas. Thus by adjusting the metric factors for different kinds of lattices, we can obtain universal scaling functions. In this section, we examine whether or not these ideals can be extended to planar random lattices and their duals.

First we use the simulation results of lattice sizes $L = 80, 100, 120, \text{ and } 160$ to plot E_p , $P/L^{-\beta/\nu}$, and $S/L^{\gamma/\nu}$ as functions of $x = (p - p_c)L^{1/\nu}$ for the three types of lattices, using the exact values of exponents $\nu = 4/3$, $\beta = 5/36$, and $\gamma = 43/18$ from regular lattices and the percolation thresholds $p_c = 0.5$ for a regular square lattice, 0.3333 for a planar random lattice, and 0.6670 for the dual of the planar random lattice; the results are shown in Figs. 13–15. We can see from these results that all the scaled data for E_p , P , and S , respectively, can be described by a single scaling function, although there is a small but detectable difference in the scaled data of S . The latter also reflects in a larger deviation of the value of γ .

To examine the ideal of a universal scaling function with nonuniversal metric factors, we first note that in the past few years the behavior of $E_p^v(G, p)$ at the critical probability $p = p_c$ for rectangular lattices of width a and height b with different aspect ratios r , $r = a/b$ and different boundary conditions has attracted researchers' interest [7–23]. There is a unique value of the aspect ratio r_0 , such that for the infinite system at p_c we have the relation

$$E_p^h(r_0, p_c) = E_p^v(r_0, p_c), \quad (13)$$

and the value of $E_p^v(r_0, p_c)$ is universal for a given boundary condition. Because of the symmetry with respect to the permutation of the two axes for a regular square lattice, one can expect that $r_0 = 1$. For other cases, it has been proposed that

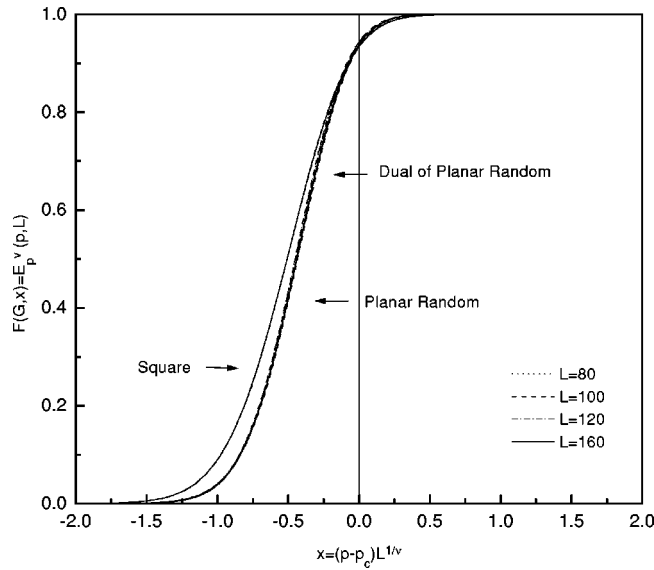


FIG. 13. The scaled results of $E_p^U(p,L)$ for different types of lattices of sizes $L=80, 100, 120,$ and 160 as functions of $x=(p-p_c)L^{1/\nu}$.

the value of r_0 is $\sqrt{3}/2$ for planar triangular lattices and $\sqrt{3}$ for honeycomb lattices [7]. Ziff [17] pointed out that there is finite-size correction to the value of r , but such a correction is not considered in this work. For planar random lattices and their duals, similar to the regular square lattice we can use a symmetry argument to conclude that $r_0=1$, which, in fact, is required in our simulations. The value of $E_p^V(r_0=1,p_c)$ is 0.93 [25] for planar random lattices and their duals with size $L=160$. By comparing with the values for other planar regular lattices with periodic boundary conditions, we can see that the universal value of $E_p^V(r_0=1,p_c)$ for periodic boundary conditions can be extended to planar random lattices and their duals.

Then we determine universal scaling functions and the values of nonuniversal metric factors by comparing the re-

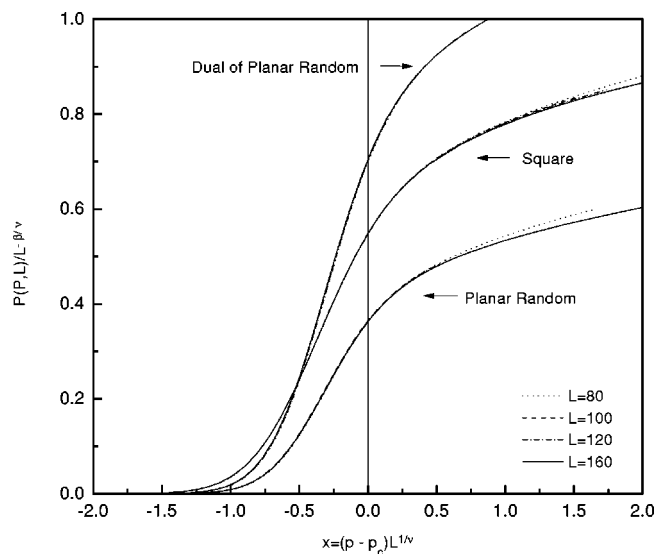


FIG. 14. The scaled results of $P(p,L)/L^{-\beta/\nu}$ for different types of lattices of sizes $L=80, 100, 120,$ and 160 as functions of $x=(p-p_c)L^{1/\nu}$.

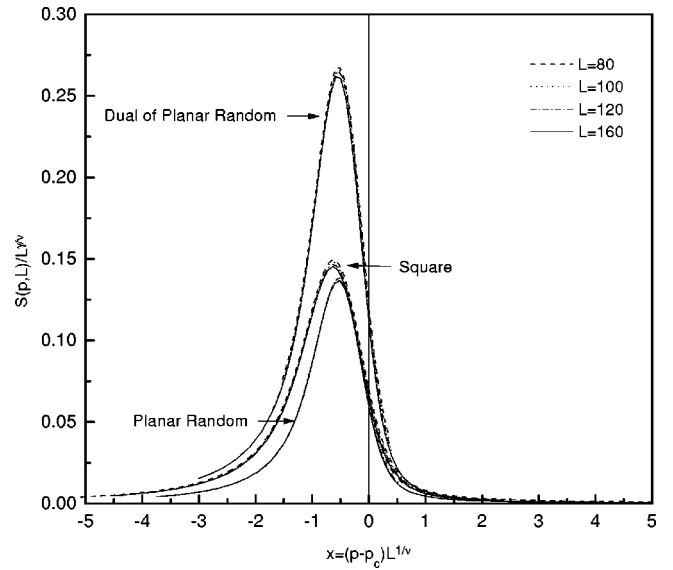


FIG. 15. The scaled results of $S(p,L)/L^{\gamma/\nu}$ for different types of lattices of sizes $L=80, 100, 120,$ and 160 as functions of $x=(p-p_c)L^{1/\nu}$.

sults of E_p^V , P , and S for random lattices with those of regular square lattices for the same size 160×160 . Following Hu *et al.* [14], we introduce three nonuniversal metric factors as follows. The first metric factor D_1 is introduced in E_p^V by the relation

$$E_p^V(p,L)=F(x), \quad (14)$$

with $x=D_1(p-p_c)L^{1/\nu}$. The other two metric factors D_2 and D_3 are introduced in P and S by the relations

$$D_3P(p,L)=L^{-\beta/\nu}S_p(z) \quad (15)$$

and

$$D_3'S(p,L)=L^{\gamma/\nu}S_S(z'), \quad (16)$$

with $z=D_2(p-p_c)L^{1/\nu}$ and $z'=D_2'(p-p_c)L^{1/\nu}$. To determine the values of D_1 , we note that E_p^V in the vicinity of p_c for an infinite lattice with an aspect ratio r_0 takes the form

$$E_p^V(p,L)=F(0)+A^V(p-p_c)L^{1/\nu}+B^V, \quad (17)$$

where B^V is the order of $[(p-p_c)L^{1/\nu}]^2$. Thus we take the value of A^V as the approximate value of D_1 , and the resultant values of D_1 for three types of lattices are given in Table III. Then we fit the data of E_p^V as a polynomial of x up to the fifth

TABLE III. The values of metric factors $D_1, D_2, D_3, D_2',$ and D_3' for square lattices, random lattices, and their duals with periodic boundary conditions.

Lattices	D_1	D_2	D_3	D_2'	D_3'
Square	1	1	1	1	1
Planar random	1.166(2)	1.163(6)	1.511(7)	1.179(1)	1.071(6)
Dual of planar random	1.177(3)	1.175(6)	0.777(5)	1.222(7)	0.571(9)

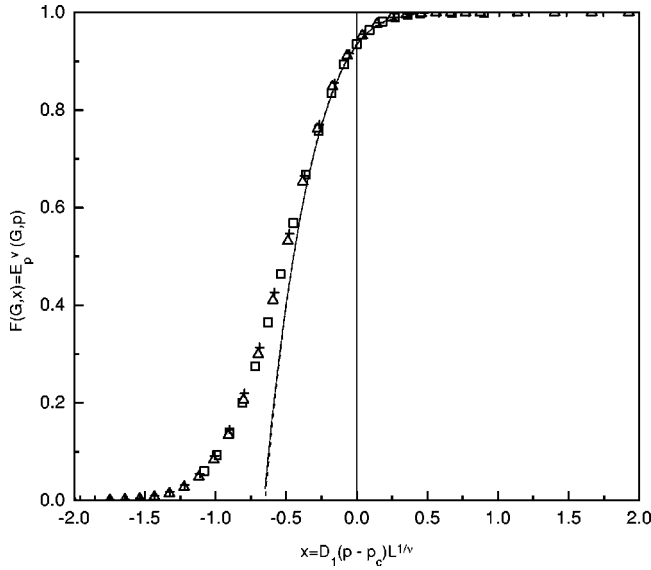


FIG. 16. The scaled results of $F(x) = E_p^v(G, p)$ as a function of $x = D_1(p - p_c)L^{1/\nu}$ for a square lattice (\square), a planar random lattice (Δ), the dual of a planar random lattice ($+$), the result of Hu (dashed curve), and the fitting (solid curve).

power for each lattice, and take the average of the fittings of the three different types of lattices as our final result. The scaling function obtained in this way is $F(x) = 0.93(6) + 0.38(5)x - 0.9(2)x^2 + 0.6(8)x^3 + 0.9(8)x^4 + \dots$, which is shown in Fig. 16 with the data. This scaling function agrees very well with the result obtained by Hu *et al.* [14], who considered square, plane triangular, and honeycomb lattices. Similar procedures are applied to the percolating probability P and the mean cluster size S as follows. We use the factor D_3 (D_3') to adjust the values of $S_p(z)$ [$S_S(z')$] such that the values of $S_p(0)$ [$S_S(0)$] are the same for the three types of lattices, and the coefficient of the linear term in the power series of $(p - p_c)L^{1/\nu}$ is used to determine the value of D_2 (D_2'). The values of D_2 (D_2') and D_3 (D_3') for different lattices are given in Table III, and the resultant scaling functions are $S_p(z) = 0.54(9) + 0.47(9)z - 0.40(9)z^2 - 0.0(0)z^3 + 0.5(0)z^4 + \dots$ and $S_S(z') = 0.06(4) - 0.15(8)z' + 0.10(7)z'^2 + 0.1(0)z'^3 - 0.0(6)z'^4 + \dots$, which are shown in Figs. 17 and 18, respectively, with the data. We notice that Hu *et al.* [14] also calculated the universal scaling function $S_p(z)$, in which the percolating probability $P(p)$ is defined in terms of the site number, and their $S_p(z)$ agrees with ours very well after rescaling an overall factor. Based on the above results, we therefore conclude that by choosing the metric factors of random lattices and their duals properly, we can extend the universal scaling functions of regular lattices to random lattices.

VI. SUMMARY AND CONCLUSION

We have extensively studied the bond-percolation process in periodic planar random lattices and their duals. Considering that the distribution functions of the derivative of the existence probability with respect to the occupied probability can be thought as the Gaussian distribution function, we estimate the percolation thresholds and the critical exponent ν by studying the peak and the standard deviation of the dis-

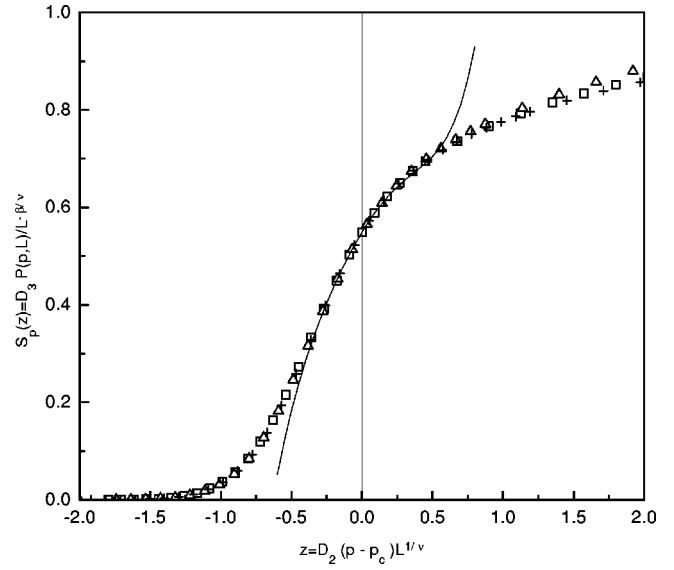


FIG. 17. The scaled results of $S_p(z) = D_3 P(G, p) / L^{-\beta/\nu}$ as a function of $z = D_2(p - p_c)L^{1/\nu}$ for a square lattice (\square), a planar random lattice (Δ), the dual of a planar random ($+$), and the fitting (solid curve).

tribution functions in finite systems. Then we apply finite-size scaling theory to the percolating probability, the mean cluster size, the mean cluster size distribution, and the size of a percolating cluster to determine the critical exponents D , τ , β , and γ . The finite-size scaling behaviors of the percolating probability, the existence probability, and the mean cluster size are also examined. The results we obtained are summarized below.

(1) From the results of the percolating probabilities $P(p, L)$, and the existence probabilities $E_p(p, L)$, we can see that the coordination number of a lattice plays an essential role in the occurrence of percolation, and homogeneity enhances the occurrence of percolations.

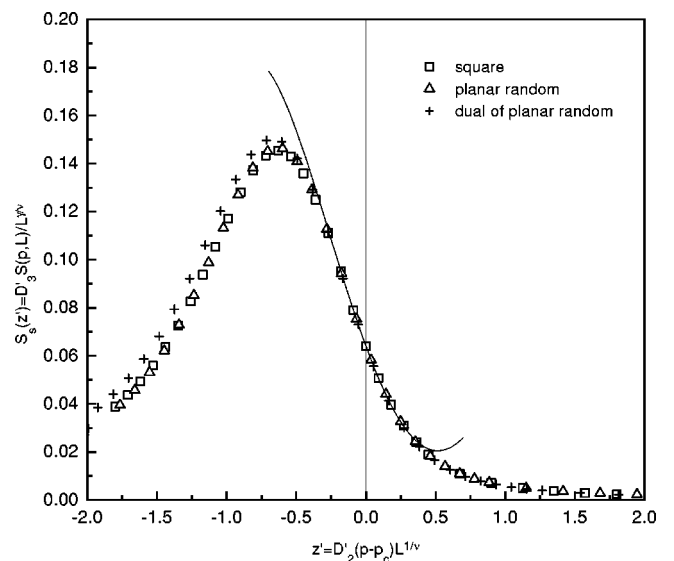


FIG. 18. The scaled results of $S_S(z') = D_3' S(G, p) / L^{-\beta/\nu}$ as a function of $z' = D_2'(p - p_c)L^{1/\nu}$ for a square lattice (\square), a planar random lattice (Δ), the dual of a planar random lattice ($+$), and the fitting (solid curve).

(2) We use the finite-size scaling $\Delta_L^X \propto L^{-1/\nu}$, with $X = U, A$, and I , to estimate the critical exponents ν of the correlation length. The results of ν of regular lattices and random lattices agree with each other.

(3) We take the theoretical prediction value $\nu = 4/3$ to estimate the percolation threshold p_c from our simulation data, and the results are $p_c = 0.3333 \pm 0.0001$ for planar random lattices, and $p_c = 0.6670 \pm 0.0001$ for the duals of the planar random lattices. Here the error bar Δp_c of the threshold is obtained from the largest values of $|p_c^A - p_c^I|$ and $|p_c^A - p_c^U|$. The shift-to-width ratio $(p_c^X(L) - p_c)/\Delta_L^X$ is a constant value -2.0 for the three different types of lattices we studied. This indicates that the universal value of the shift-to-width ratio can be extended to planar random lattices and their duals.

(4) Our percolation thresholds are consistent with the two known relations. One is the relation $p_c(L_\infty) + p_c(L_\infty^d) = 1$ for an infinite lattice L_∞ and its dual L_∞^d ; and the other is the critical value η_c of the effective coordination number $\eta_c = \bar{z}p_c$, which is a constant value 2. From these two relations, we conjecture that the exact percolation threshold is $1/3$ for planar random lattices, and $2/3$ for the duals of planar random lattices.

(5) Using the finite-size scaling theory we determine the

exponents, D , τ , β , and γ . By taking the possible errors into account, our results strongly indicate that bond-percolation processes on planar random lattices and their duals have the same critical exponents as the processes on regular lattices.

(6) By choosing five nonuniversal factors D_1, D_2, D_3, D'_2 , and D'_3 properly for planar random lattices and their duals, we show that the scaling functions of the percolating probability $P(p, L)$, the existence probability $E_p^V(p, L)$, and the mean cluster size $S(p, L)$ for planar regular lattices can be extended to planar random lattices and their duals.

These results strongly indicate that in the domain of the critical phenomena, except for providing the continuous rotation symmetry, random lattices do not change the critical phenomena of the system.

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