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PERCOLATION THRESHOLDS, CRITICAL EXPONENTS, AND SCALING FUNCTIONS ON SPHERICAL RANDOM LATTICES AND THEIR DUALS

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Bond-percolation processes are studied for random lattices on the surface of a sphere, and for their duals. The estimated threshold is 0.3326 ± 0.0005 for spherical random lattices and 0.6680 ± 0.0005 for the duals of spherical random lattices, and the exact threshold is conjectured as 1/3 for two-dimensional random lattices and 2/3 for their duals. A suitably defined spanning probability at the threshold, $E_p(p_c)$, for both spherical random lattices and their duals is 0.980 ± 0.005, which may be universal for a 2-d lattice with this spanning definition. The shift-to-width ratio of the distribution function of the threshold concentration and the universal values of the critical value of the effective coordination number can be extended from regular lattices to spherical random lattices and their duals. The results of critical exponents are consistent with the assertion from the universality hypothesis. Finite-size scaling is also examined.

Keywords: Percolation threshold; critical exponent; scaling function; spherical random lattice.

1. Introduction

Percolation processes,[1] viewed as a simple geometric transitions, have attracted much interest. Things such as new universal quantities and their finite-size corrections, universal scaling functions of geometric quantities, etc., have been studied very actively.[2–17] But most of the studies have been conducted on a variety of regular lattices with periodic or free boundary conditions. For irregular lattices, there exist only few results. Yonezawa *et al.*[18] used the lattices of dice, Penrose tilings, and the dual of Penrose tilings. The critical percolation exponents calculated in these lattices belong to the same universality class as on regular lattices. Considering that these lattices have mixed values of coordination numbers but still own a regular pattern. Recently the authors used periodic planar random lattices and

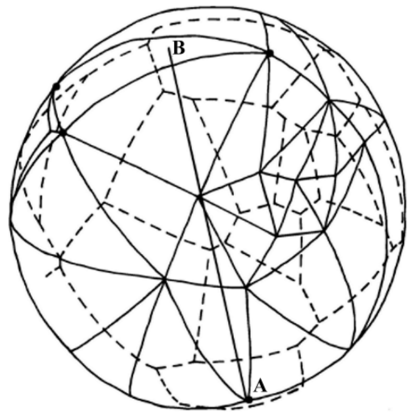

Fig. 1. An example of a random lattice (—) with its dual (---) on a 2D spherical surface.

their duals to simulate the percolation process.[19] The coordination number varies from site to site with the average number 6 on a planar random lattice, and is always 3 on the dual of a planar random lattice.[20–22] The results from these lattices show that not only the critical exponents belong to the same universality class as those from regular lattices, but also the scaling functions with the properly chosen metric factors are universal. In order to extend the above consideration to another boundary condition, in this paper we study the percolation process on spherical random lattices and their duals.

Spherical random lattices are random lattices defined on a two-dimensional spherical surface. A sample lattice is shown in Fig. 1. About the construction and the structure of such lattices, see Refs. 21 and 22. Here we briefly describe some of their properties. Consider N_0 sites randomly distributed on a sphere of area σ with unit site density. Each of the N_0 sites is connected to the nearby sites by links in a way that the finite area σ of the spherical surface is completely covered by nonoverlapping convex triangles whose vertices are on the N_0 random sites. Under this construction, the ratio of the link number to the site number in a sufficiently large sphere is $3(1-2/N_0)$, and the ratio of the triangle number to the site number is $2(1-2/N_0)$. From these ratios, we find the average coordination number for a site is $6(1-2/N_0)$, and a site on the average is shared by $6(1-2/N_0)$ triangles. There are two special features for this type of lattices: a homogeneous distribution of lattice sites and the absence of a boundary. These geometric characteristics are the main motivation for this study.

To study percolation processes on spherical random lattices and their duals, we first calculate the percolating probabilities (defined in Sec. 2), the spanning probabilities, the mean cluster size distributions, and the mean cluster sizes in a bond percolation model. The configurations used in calculating these quantities are generated by a random bond process as follows. For a given occupation probability p, we randomly assign a number r, $0 < r < 1$ to all links on the lattice. If the number

r of a link is less than p, this link is occupied and a bond exists. The minimum number of configurations used in the calculations is determined by the condition that the average of the ratios of bond number to link number over the configurations should be equal to the given value of p. From these geometric quantities we determine the percolation thresholds and the critical exponents by mainly using the techniques of the finite-size scaling theory. Finally we examine the scaling functions of these geometric quantities using the obtained thresholds and critical exponents.

This paper is organized as follows. In Sec. 2, we report the simulation results of various geometric quantities. The estimated values of the percolation thresholds and critical exponents from the simulation results are given in Sec. 3. In Sec. 4, using the estimated values of the percolation thresholds and critical exponents, we investigate if the simulation data of the geometric quantities of finite systems are consistent with the predictions from the finite-size scaling theory. Finally in Sec. 5 we summarize the results.

2. Geometric Quantities

In the bond-percolation model studied in this work, the occupation probability is set to be the same for all the links in a lattice. We define a percolating cluster as a cluster in which the following holds: There exists at least one site, say A, of the cluster with the property that one of the three corners of the triangle containing the "antipode" B (see Fig. 1) in the cluster. Using the distributions of clusters in the lattice of size L, we then calculate the mean cluster size distribution $n_s(p, L)$, which is defined as the ratio of the average number of clusters with s bonds to the total number of bonds. The percolating probability $P(p, L)$ defined as the ratio of the number of bonds in the percolating cluster to the total number of bonds. The spanning probability $E_p(p, L)$ defined as the probability of the appearance of percolating clusters, and the mean cluster size $S(p, L)$ defined as:

$$S(p, L) = \sum_{s=1}^{\infty} s \left(\frac{sn_s}{\sum_{s=1}^{\infty} sn_s} \right), \tag{1}$$

where the quantity, $sn_s / \sum_{s=1}^{\infty} sn_s$, is the probability that an occupied site belongs to a cluster containing the s sites.

In the simulation, we use spherical random lattices of unit density and their duals on spheres of sizes $L = 80$, 100, 120 and 160. Here the size L is defined as the square root of the spherical area σ with the total number of sites $N = \sigma = L^2$. We took 60 occupation probabilities around the critical percolation threshold with $\Delta p = 0.002$, and generated 10^5 configurations for each. This number of configurations was doubled at the occupation probability around the peaks of $S(p, L)$. The results of P, E_p, and S for spherical random lattices and their duals of different sizes are shown in Figs. 2–4. These results show that due to the larger coordination number percolating clusters form more easily than on the duals.

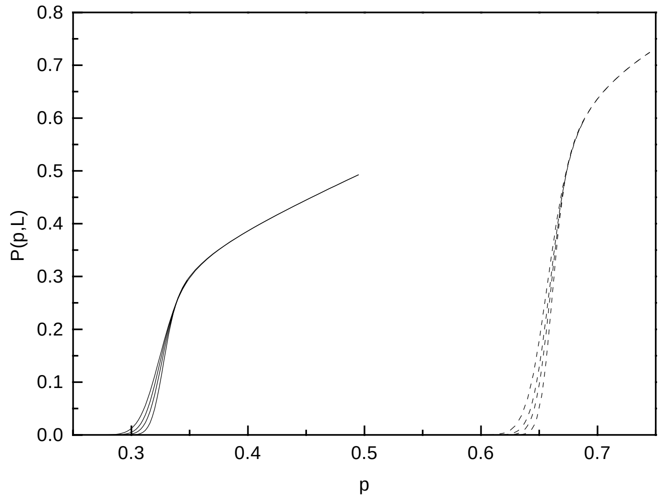

Fig. 2. The percolating probabilities $P(p, L)$ for spherical random lattices (—) and their duals (- - -). The curves from left to right are for linear dimensions $L = 80$, 100, 120, and 160.

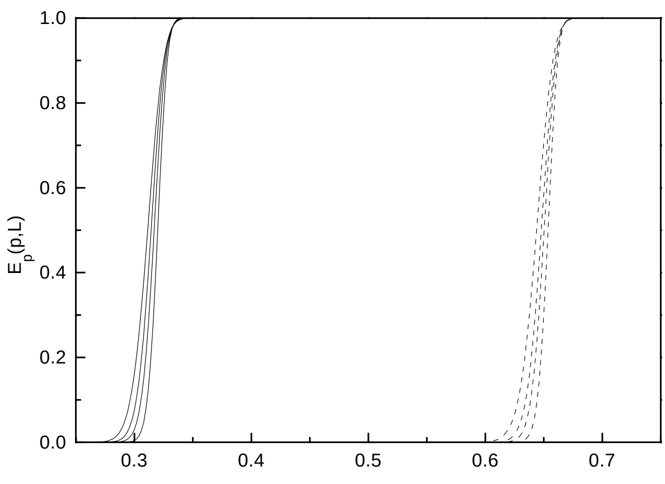

Fig. 3. The spanning probabilities $E_p(p, L)$ for spherical random lattices (—) and their duals (- - -). The curves from left to right are for linear dimensions $L = 80$, 100, 120, and 160.

3. Percolation Thresholds and Critical Exponents

We use the finite-size scaling theory[23,24] to determine the percolation thresholds, and to extract the critical exponents from the simulation results. The method we used here is the same as we used in planar random lattices, and in the following we briefly describe this method. For the estimation of percolation thresholds we notice that the spanning probability, $E_p(p)$, is a step function for an infinite system, and for a finite large system the derivative, dE_p/dp, has a sharp peak close

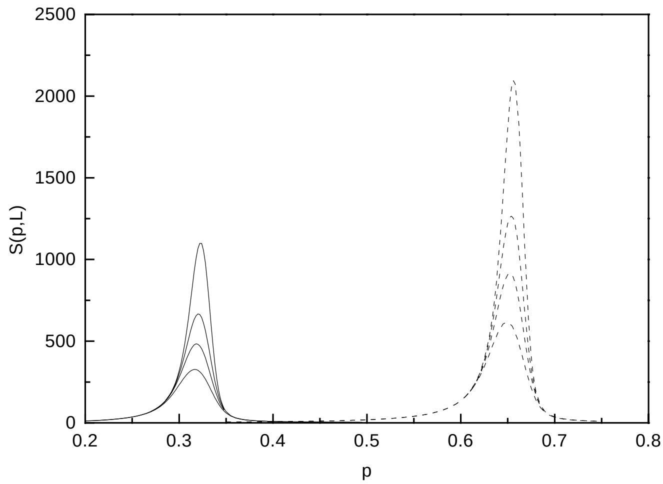

Fig. 4. The mean cluster size $S(p, L)$ for spherical random lattices (—) and their duals (---). The peaks from up to down curves are for linear dimensions $L = 160$, 120, 100, and 80.

to $p_c(L)$,[25–27] and the spanning probability changes from close zero towards close to one in a transition region $\Delta(L)$. Here $p_c(L)$ is the average threshold for a finite system of size L, and the width $\Delta(L)$ is defined as the deviation

$$\Delta^2(L) = \langle p^2 \rangle - \langle p \rangle^2 \,. \tag{2}$$

We adopt standard algorithms[16,27] to generate a fixed sequence of random numbers for each configuration and do the binomial search in p to find the value of p_c, at which the percolating cluster appears for the first time. We start from the initial guess $p = 1/2$, and check if the percolating cluster exists. If there exists at least one percolating cluster in the system, we decrease the value of p by 2^{-n-1}, where n is the number of iteration. If there is no percolating cluster, we increase the value of p by 2^{-n-1}. After we repeat the process over 14 times, the accuracy of the estimation is around 10^{-4}. According to the finite-size scaling theory, we have

$$|p_c(L) - p_c| \propto L^{-1/\nu} \,, \tag{3}$$

where p_c is the threshold of the infinite system. Finite size scaling predicts that the width also scales as

$$\Delta(L) \propto L^{-1/\nu} \,. \tag{4}$$

Thus we first use the scaling law Eq. (4) to determine the critical exponent ν, and then using the result for ν we determine the percolation threshold p_c for the infinite system from Eq. (3). Once ν and p_c are determined, we use the scaling laws

$$P(p, L) \propto L^{-\beta/\nu} \,, \tag{5}$$

$$S(p, L) \propto L^{\gamma/\nu} \,, \tag{6}$$

$$n_s(p_c, L) \propto s^{-\tau} \,, \tag{7}$$

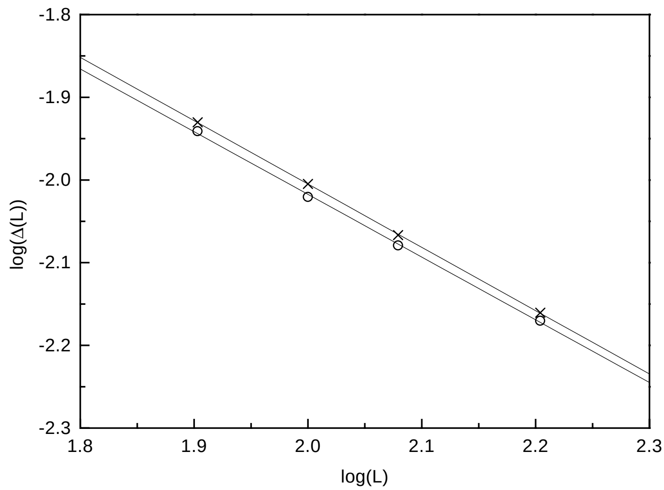

Fig. 5. The logarithms of the deviation $\Delta(L)$, $\log(\Delta(L))$, versus $\log L$ for spherical random lattices and their duals.

and

$$s_{\mathrm{perc}}(p_c, L) \propto L^D \,, \tag{8}$$

to determine the critical exponents D, τ, β, and γ. Here $s_{\mathrm{perc}}(p_c, L)$ is the mean size of a percolating cluster at the percolation threshold p_c for the lattice of size L.

Using the above method, we obtain the effective percolation thresholds $p_c(L)$ and the width $\Delta(L)$. Then from Eq. (4) we obtain the critical exponent ν by using the least-square fit to find the slope of the line $\log \Delta(L)$ versus $\log L$ as shown in Fig. 5. The estimated value of ν is 1.3188 for spherical random lattices and 1.3139 for the duals. The deviations of these values from the exact value for regular lattices, which is $\nu = 4/3$, are only about 2%. We then take the value $\nu = 4/3$, and we use Eq. (3) to obtain the percolation threshold p_c by extrapolating as shown in Fig. 6. The thresholds we obtain are 0.3326(5) for spherical random lattices and 0.6680(5) for the duals.

We check the values of the thresholds with two known properties. One is the dual relation. For an infinite planar lattice L_∞ and its dual L_∞^d, there is a relation between the two corresponding thresholds,[18,28,29]

$$p_c(L_\infty) + p_c(L_\infty^d) = 1 \,. \tag{9}$$

Our threshold results agree with this relation up to the third digit after the decimal point. The other known property is an empirical approximate relationship for the thresholds. They seem to occur at a critical value η_c of *the effective coordination number*, which is defined as the product of the threshold p_c and the coordination number z. This critical value seems to be given as[18,30]:

$$\eta_c \equiv z p_c = \frac{d}{d-1} \,, \tag{10}$$

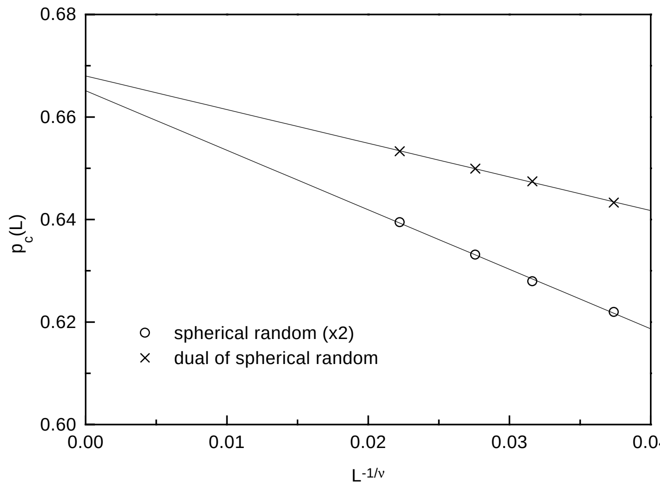

Fig. 6. Thresholds p_c versus $L^{-1/\nu}$ for spherical random lattices and their duals. Notice that $2p_c$ is shown for the spherical lattices, in order to reduce the scale of the y-axis.

which is 2 for $2D$. For spherical random lattices, the coordination number varies from site to site, and we take the average value 6 to obtain the result $\eta_c = 1.9956$. For their duals, the coordination number is a constant, 3, and the corresponding η_c is 2.0004. They all agree with the value 2 very well. We notice that we had obtained previously[19] $p_c = 0.3333(1)$ for planar random lattices and $p_c = 0.6670(1)$ for the duals of planar random lattices. From these threshold results and the above two properties, we conjecture that the exact value of p_c is $1/3$ for planar or spherical random lattices and $2/3$ for their duals. There is another universal quantity

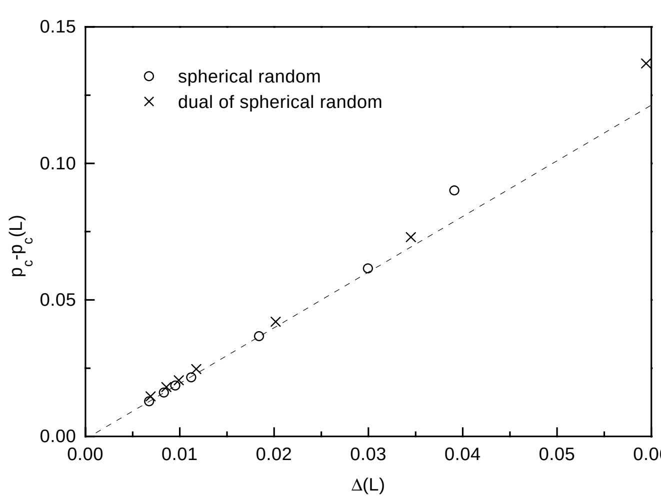

Fig. 7. The shift $(p_c - p_c(L))$ versus the width $\Delta(L)$ for spherical random lattices and their duals.

pointed out by Gropengiesser and Stauffer,[9] and it is the shift-to-width ratio $(p_c - p_c(L))/\Delta(L)$, which is about 2 for two-dimensional finite, but large systems. We plot $(p_c - p_c(L))$ versus $\Delta(L)$ curve in Fig. 7 for spherical random lattices and their duals of sizes $L = 10$, 20, 40, 80, 100, 120, and 160, and the result indicates that the data are consistent with the relation, $(p_c - p_c(L)) \simeq 2\Delta(L)$. Recently the value of $E_p(p)$ at $p = p_c$ has been investigated by many researchers, and it is known that this value depends on the aspect ratio and the boundary condition. In our previous study, we showed that the universal value of $E_p(p_c)$ for a specified aspect ratio

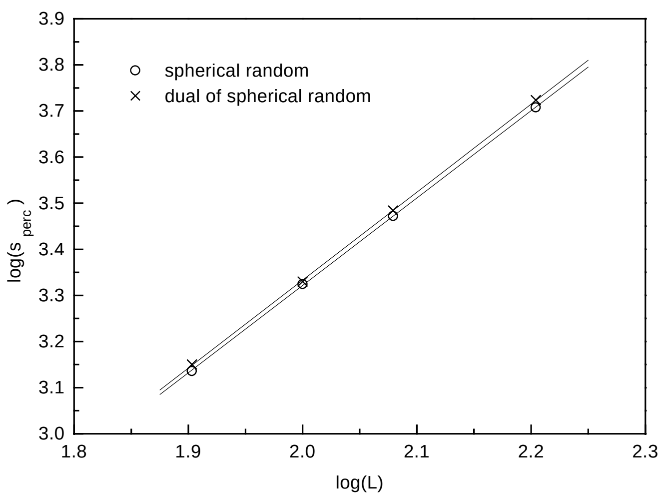

Fig. 8. Logarithm of the average size of a percolating cluster, $\log s_{\rm perc}$, as a function of $\log L$ for spherical random lattices and their duals.

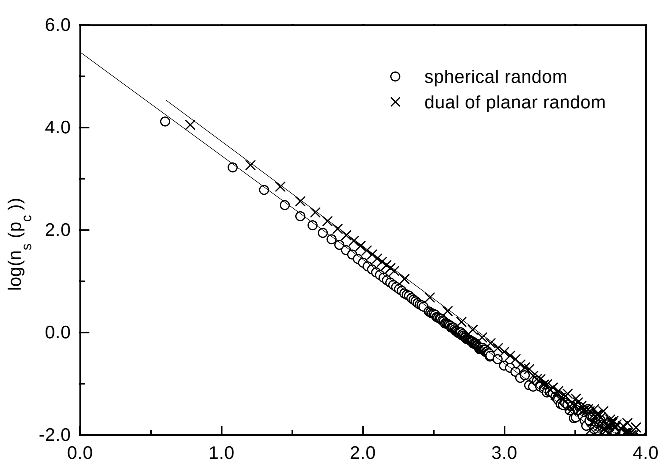

Fig. 9. The logarithm of the number of clusters composed of s bonds, $\log n_s$, as a function of $\log L$ for spherical random lattices and their duals.

obtained from periodic regular lattices can be extended to periodic planar random lattices and their duals. For spherical random lattices and their duals, there are no boundary condition and aspect ratios, and from the results shown in Fig. 3 the $E_p(p_c)$ value for both spherical random lattices and their duals is 0.980(5), which may be an universal value for our definition of spanning.

We further use Eqs. (5)–(8) to obtain the critical exponents D, τ, β, and γ. We show the line $\log s_{\text{perc}}$ versus $\log L$ in Fig. 8, the line $\log n_s(p_c)$ versus $\log s$ in Fig. 9, the line $\log P$ versus $\log L$ in Fig. 10, and the line $\log S$ versus $\log L$ in Fig. 11. We

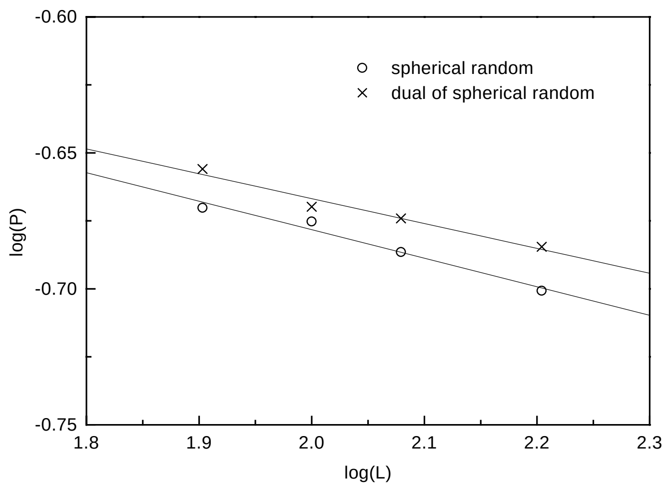

Fig. 10. The logarithm of the percolating probability, $\log P$, as a function of $\log L$ for spherical random lattices and their duals.

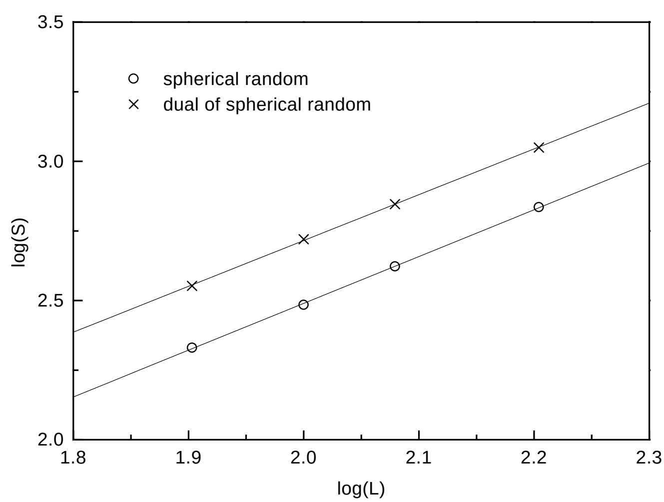

Fig. 11. The logarithm of the mean cluster size, $\log S$ versus $\log L$ for spherical random lattices and their duals.

Table 1. The simulation results of the critical exponents ν, β/ν, γ/ν, τ and D for spherical random lattices and their duals.

Lattices	ν	β/ν	γ/ν	τ	D
Spherical random	1.3188	0.1049	1.6812	2.0246	1.8951
Dual of spherical random	1.3139	0.0915	1.6459	2.0480	1.9085
Theoretical prediction	$\frac{4}{3}$	$\frac{15}{144}$	$\frac{129}{72}$	$\frac{187}{91}$	$\frac{91}{48}$

obtain exponents, D, τ, β/ν, and γ/ν, by calculating the slopes of these lines. The results of all the exponents we obtained are listed in Table 1.

The errors of our results are estimated as follows: First in finding the geometric quantities, we use the variance of block averages to get our error estimate. The deviations we obtain are less than 1% for the resultant values of the spanning probability, percolating probability, and the mean cluster size. To estimate the errors in our results of critical exponents, we applied the same method to periodic square lattices to find the critical exponents, and we take the deviations of these results from the known exact values as the estimations of the errors. For periodic square lattices, we find that the deviations for the exponents ν, D, τ, and β/ν are less than 1%, and the deviation is about 8% for γ/ν. Our results of critical exponents for spherical random lattices and their duals deviate from the exact values for regular lattices by about the same percentages. Therefore we may conclude that our results are consistent with universality.

4. Scaling Function

For a quantity X to scale as $X(t) \sim t^{-\rho}$ near the critical point $t = 0$ in the infinite system, according to the finite-size scaling theory[23,24] the quantity $X_L(t)$ in a finite system characterized by a size L should obey the general scaling law,

$$X_L(t) \sim L^{\rho/\nu} F(tL^{1/\nu}) \tag{11}$$

where $F(x)$ with $x = tL^{1/\nu}$ is called a scaling function. When finite-size scaling is valid, the scaled data, $X_L(t)/L^{\rho/\nu}$ for different values of L and t are described by a single scaling function $F(x)$. To examine this scaling form, we use the simulation results of lattice sizes $L = 80$, 100, 120 and 160 to plot E_p, $P/L^{-\beta/\nu}$ and $S/L^{-\gamma/\nu}$ as a function of $x = (p - p_c)L^{1/\nu}$ with the exponent values, $\nu = 4/3$, $\beta = 5/36$ and $\gamma = 43/18$, and the percolation threshold 0.3326 for spherical random lattices and 0.6680 for the duals of spherical random lattices. The results are shown in Figs. 12 and 13. We can see from these results that all the scaled data of E_p, P and S can be described by a single scaling function respectively.

5. Summary

We have studied bond-percolation processes in spherical random lattices and their duals. By the use of the finite-size scaling theory, we estimate the percolation

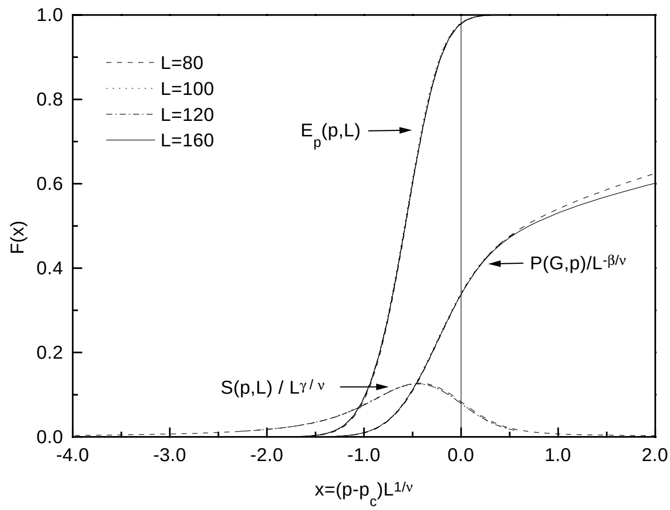

Fig. 12. The scaled results of $E_p(p, L)$, $P(p, L)/L^{-\beta/\nu}$ and $S(p, L)/L^{\gamma/\nu}$ for spherical random lattices with linear dimensions $L = 80$, 100, 120, 160 as functions of $x = (p - p_c)L^{1/\nu}$.

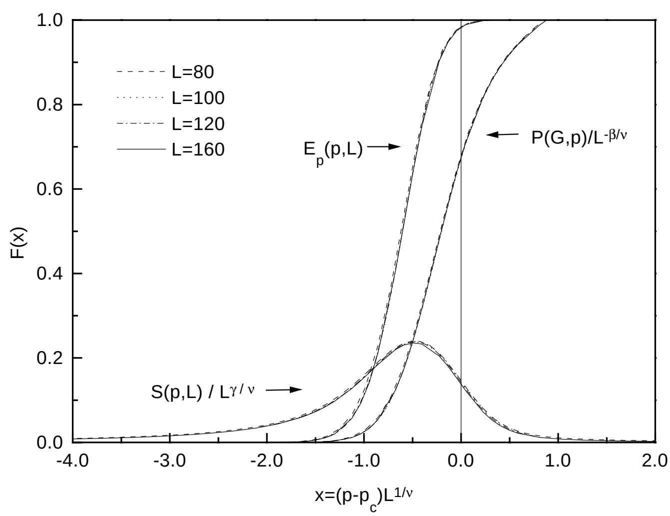

Fig. 13. The scaled results of $E_p(p, L)$, $P(p, L)/L^{-\beta/\nu}$ and $S(p, L)/L^{\gamma/\nu}$ for the duals of spherical random lattices with linear dimensions $L = 80$, 100, 120, 160 as functions of $x = (p - p_c)L^{1/\nu}$.

thresholds and the critical exponents. For the percolation thresholds the results are 0.3326 ± 0.0005 for spherical random lattices and 0.6680 ± 0.0005 for the duals of spherical random lattices. These results agree very well with the relation $p_c + p_c^d = 1$. Here p_c^d is the threshold on the dual of a lattice with threshold p_c. Also the empirical universal value of the critical value, η_c, of the effective coordination number, which is defined as the product of the threshold p_c and the coordination number, can be extended to spherical random lattices and their duals. From these results and the previous study on planar random lattices and their duals we conjecture that the exact threshold is 1/3 for two-dimensional random lattices and 2/3 for the duals of two-dimensional random lattices. We also confirm that the relation, $p_c - p_c(L) \simeq 2\Delta(L)$, holds in spherical random lattices and their duals. For spherical random lattices and their duals, there is no boundary line, and the $E_p(p_c)$ value is 0.980(5), which may be universal for a lattice defined on a two-dimensional compact space. By taking the errors into account, our results of critical exponents are consistent with the assertion of the universality hypothesis. Also the finite-size scaling of the percolating probability $P(p, L)$, the spanning probability $E_p(p, L)$, and the mean cluster size $S(p, L)$ can be described by a single scaling function respectively.

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