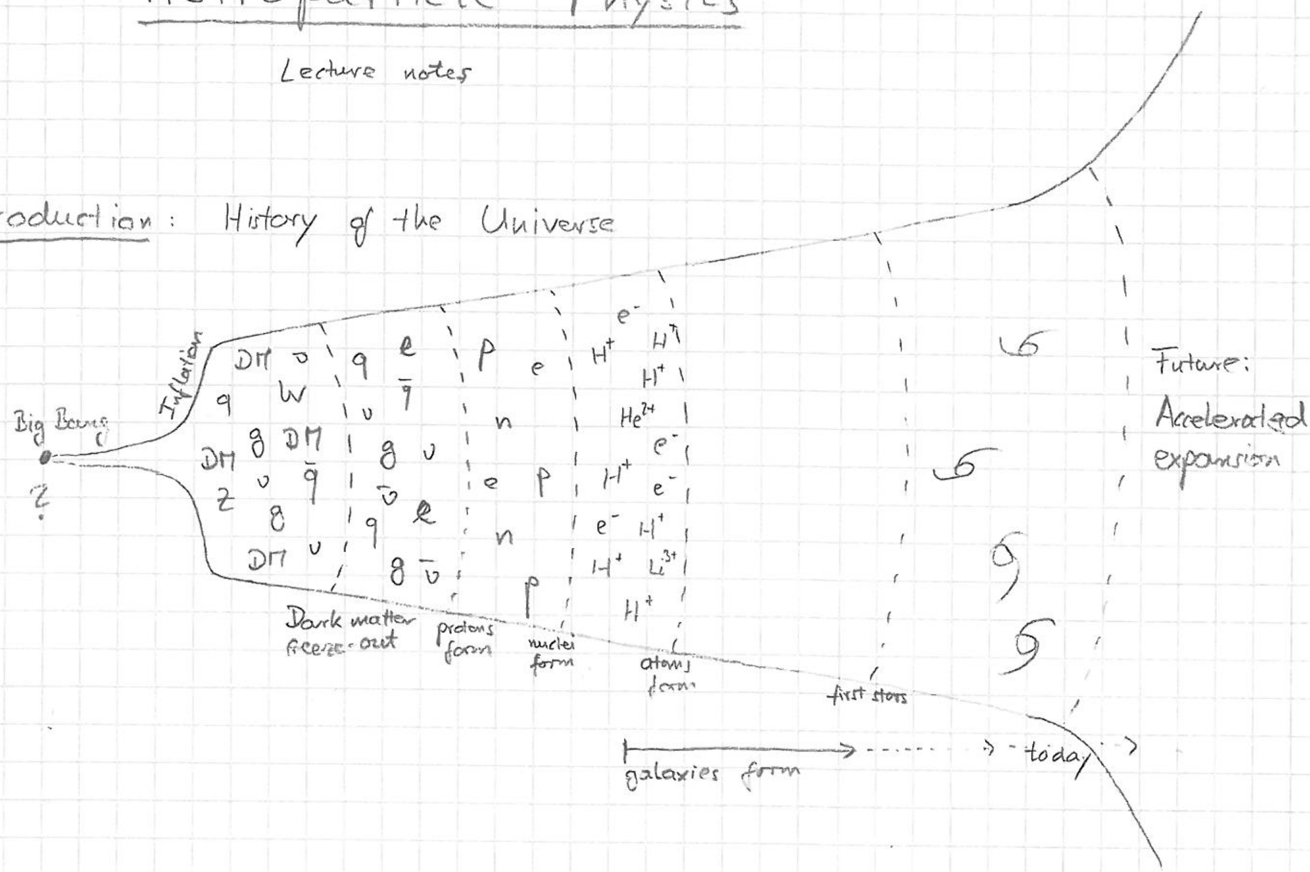
Astroparticle Physics

Lecture notes

1. Introduction: History of the Universe



E [GeV]	10^{2}	10^{-1}	10^{-4}	$3 \cdot 10^{-10}$	10^{-12}	2.3×10^{-13}
T [K]	10^{15}	10^{12}	10^{9}	3000	15	2.73
t [s, yrs]	10^{-10} s	10^{-5} s	10^{2} s	3×10^{5} yrs	10^{9} yrs	14×10^{9} yrs

Natural units

$$\hbar = c = 1$$

$$\Rightarrow \hbar \cdot c = 6.58 \cdot 10^{-16}\, eV \cdot s \times 3 \cdot 10^{8}\, \frac{m}{s} = 197 \cdot 10^{-9}\, eV \cdot m$$

$$\Rightarrow \boxed{1\, m = 5.066 \cdot 10^{6}\, eV^{-1}}$$

$$6.58 \cdot 10^{-16}\, eVs = 1 \quad \Rightarrow \boxed{1\, s = 1.52 \cdot 10^{15}\, eV^{-1}}$$

Literature

E. Kolb, M. Turner: The Early Universe
Westview Press, 1994, ISBN 0-201-62674-8
(Textbook on cosmology, relatively old, but still ~~to~~ standard book)

M. Peskin, D. Schroeder: An Introduction to Quantum Field Theory
Westview Press, 1995, ISBN 0-201-50397-2
(Practitioner's introduction to QFT, Feynman calculus, and the Standard Model)

R. Wald: General Relativity
The University of Chicago Press, 1984, ISBN 0-226-87033-2
(Standard textbook on GR)

[conventions: Lindner, Kolb-Turner (Wald: different metric)]

2. Big bang theory

Einstein's equations:

Ricci tensor, Ricci scalar, cosmological constant $\simeq 0$, metric, energy-momentum tensor

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}}_{\text{geometry}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{matter}}$$

Metric: $(g_{\mu\nu}) = \underbrace{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}}_{=\eta_{\mu\nu} \text{ flat metric}} + \underbrace{\delta g_{\mu\nu}}_{\text{curvature}}$

(Minkowski metric)
"normal coordinates"

Riemann curvature tensor

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha} \right)$$

$$\Leftrightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} \underbrace{R_{\mu\alpha\nu\beta} x^\alpha x^\beta + \dots}_{\text{for small } x}$$

Ricci tensor:

$$R_{\mu\nu} \equiv R_{\mu\alpha\nu}{}^{\alpha} \quad \left[\text{"} \hat{=} \Delta g_{\mu\nu} \text{"} \right]$$

Laplace

Ricci scalar:

$$R = R^{\mu}{}_{\mu}$$

[
$x_k = g_{kj} x^j$

$\delta_{ki} = g_{kj,i} x^j + g_{ki}$

$0 = g_{kj,il} x^j + g_{kl,i} + g_{ki,l}$

$0 \overset{\text{normal coord.}}{=} g_{km,il} + g_{kl,im} + g_{ki,lm}$

$= g_{km,il} + g_{ki,lm} + g_{kl,mi}$ (*)

Use $\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} (g_{bd,a} + g_{ad,b} - g_{ab,d})$ to express $R_{\mu\nu\varsigma}{}^{\sigma}$ in terms of the metric. This requires the identity

$R_{\mu\nu\varsigma}{}^{\sigma} = \Gamma^{\sigma}_{\mu\varsigma,\nu} - \Gamma^{\sigma}_{\nu\varsigma,\mu}$

Then use (*), together with $g_{\alpha\beta,\mu\nu} = g_{\mu\nu,\alpha\beta}$ (which can be shown using (*)).]

In cosmology, we observe:

- Universe looks homogeneous at large scales
- No preferred direction (isotropy)
- at large scales, only gravity matters

$$\Rightarrow (T^{\mu}{}_{\nu}) = \operatorname{diag}\left(\rho(t), p(t), p(t), p(t) \right)$$

energy density, pressure

Space-like components:
force $= \partial \vec{p} / \partial t$
pressure = force per area $= \partial \vec{p} / \partial^3$
Note: KG field $= \partial_\mu \phi \Rightarrow p_\mu \phi$
consider Fourier-expansion of ϕ, note $\frac{1}{\sqrt{E_k}}$ in denominator

Motivates ansatz (in flat spacetime)

$$ds^2 = dt^2 - R^2(t) \left[dx^2 + dy^2 + dz^2 \right]$$

$\Rightarrow$ (00) component of Einstein equation ($\Lambda = 0$):

[Homework: Derive!]
[Curved spacetime: Homework]

$$\boxed{\Big(\underbrace{\frac{\dot{R}}{R}}_{\text{Hubble function}}\Big)^2 + \underbrace{\frac{k}{R^2}}_{=0 \text{ in flat spacetime}} = \frac{8\pi G}{3} \rho(t) \quad \text{Friedmann equation}} \quad (1)$$

$$\boxed{\text{value today}: \frac{\dot{R}(t_0)}{R(t_0)} \equiv H_0 \ (\text{Hubble constant}) \equiv \underbrace{h_0}_{\sim 74.3 \pm 1.5} \cdot 1 \text{ km s}^{-1} \text{ Mpc}^{-1}}$$

(ii) components: $$\boxed{2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p} \qquad (2)$$

Classical "derivation" of (1):

Expansion of sphere:

$$r = R(t)\cdot r_0$$

$$v = \dot{r} = \dot{R}\cdot\frac{r}{R}$$

Energy conservation:

$$\frac{1}{2}\cancel{m}v^2 - G\frac{\cancel{m}M}{r} = \text{const} \equiv -\frac{k}{2}\cancel{m}r_0^2$$

$$\left(\frac{\dot{R}}{R}\right)^2\frac{r^2}{2} - G\cdot\frac{\frac{4}{3}\pi r^3\rho}{r} = -\frac{k}{2}r_0^2$$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8}{3}\pi G\rho$$

Caution: Correct result, but not the real situation

Behavior of ρ:

$$\frac{d}{dt}\left[a^2(1)\right] \Rightarrow 2\dot{R}\ddot{R} = \frac{8\pi G}{3}\left(\dot{\rho}R^2 + 2\rho R\dot{R}\right) \qquad (3)$$

$$\frac{(2)-(1)}{2} \Rightarrow \boxed{\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G\left(\rho + 3p\right)} \quad \text{Friedmann-Lemaître equation} \qquad (4)$$

$$(4) \text{ in } (3) \Rightarrow -\cancel{2}\dot{R}\cdot\frac{4}{\cancel{3}}\pi G\cancel{R}\left(\rho + 3p\right) = \frac{8\pi G}{\cancel{3}}\left(\dot{\rho}R^{\cancel{2}} + 2\rho\cancel{R}\dot{R}\right)$$

$$\Leftrightarrow 3\dot{R}\rho + 3\dot{R}p + R\dot{\rho} = 0$$

We write $\boxed{p(t) = \underset{\text{Equation of state parameter}}{\omega}\,\rho(t)}$

$$\hookrightarrow 3\dot{R}\rho(1+\omega) = -R\dot{\rho}$$

$$\boxed{\frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}(1+\omega)} \qquad (5)$$

$$\Leftrightarrow \frac{d}{dt}\log\rho = -3(1+\omega)\frac{d}{dt}\log R \qquad \Big| \int dt;\ \exp$$

$$\boxed{\rho = c\,R^{-3(1+\omega)}} \qquad (6)$$

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Insert into Friedmann eq. (1) for flat Universe:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G c \, R^{-3(1+\omega)}$$

$$R^{\frac{3\omega+1}{2}} \, dR = \sqrt{\frac{8}{3}\pi G c} \, dt$$

$$\frac{2}{3(1+\omega)} R^{\frac{3(1+\omega)}{2}} = \sqrt{\frac{8}{3}\pi G c} \; t$$

$$\boxed{R \sim t^{\frac{2}{3(1+\omega)}}} \qquad (7)$$

Special cases:

$\omega = \frac{1}{3}$: Relativistic gas (radiation domination)
$p = \frac{1}{3}\rho$

$\rho \sim R^{-4}$ $\qquad R \sim t^{1/2}$

$\omega = 0$: Nonrelativistic matter (matter domination)
$p = 0$

$\rho \sim R^{-3}$ $\qquad R \sim t^{2/3}$

$\omega = -1$: Vaccuum domination
$p = -\rho$

$\rho = \text{const.}$ $\qquad R \sim e^{Ht}$ (see sec. 3.3)

3. Inflation

3.1 The flatness problem

wrong on blackboard on 25.04.2013

$(1) \Rightarrow$ For $k=0$: $\quad \rho = \frac{3H^2}{8\pi G} \equiv \rho_c$ (critical density)

Today: $\rho_{c,0} = \rho_c(t_0) \simeq 10^{-26} \, \text{kg m}^{-3} \simeq 5$ H atoms $/m^3$

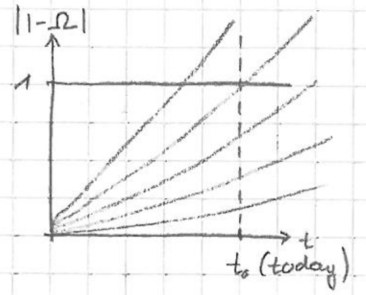
Definition: Dimensionless density

$$\Omega = \frac{\rho}{\rho_c}$$

Note: (6), (7) $\quad \frac{\rho}{\rho_c} \sim \frac{R^{-3(1+\omega)}}{\dot{R}^2/R^2} \sim \frac{t^{-3(1+\omega)\cdot\frac{2}{3(1+\omega)} + 2\frac{2}{3(1+\omega)}}}{t^{\left(\frac{2}{3(1+\omega)}-1\right)\cdot 2}} \sim t^0$ (→ self-consistency of assumption $k=0$)

But: Allow for $k \neq 0$

$$\hookrightarrow \quad 1-\Omega = -\frac{k}{\dot{R}^2} \overset{(7)}{\sim} \frac{1}{\left[t^{\frac{2}{3(1+w)}-1}\right]^2} \sim t^{\frac{2}{3}\frac{1+3w}{1+w}}$$



Extreme fine-tuning of initial value to get $1-\Omega \sim \mathcal{O}(1)$ today

Estimate: $T = 0.1$ MeV $(t \simeq 100\ s)$; $w = \frac{1}{3}$ for most of history (on a log scale)

$$\Longrightarrow \quad 1-\Omega \lesssim 10^{-15}$$

$T = 100$ GeV $(t \simeq 10^{-10} s)$; $w = \frac{1}{3}$

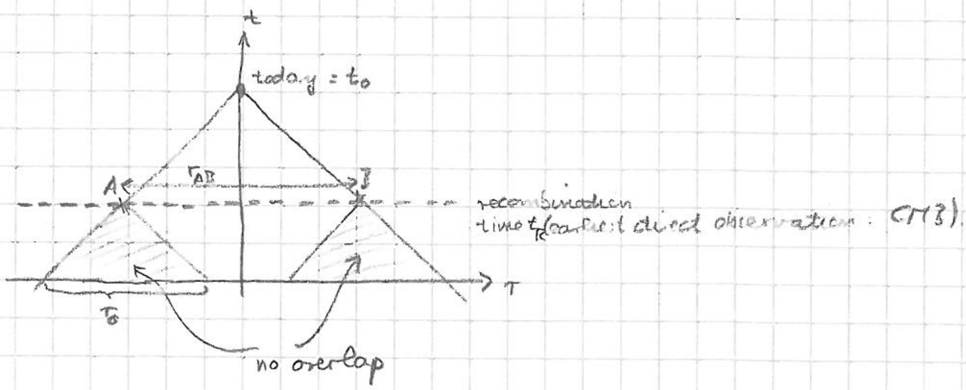
$$\Rightarrow \quad 1-\Omega \lesssim 10^{-27}$$

There is no reason why $1-\Omega$ should be so tiny

$\hookrightarrow$ $\boxed{\text{Flatness problem}}$

3.2 The horizon problem

Consider backward light cone



$$ds^2 = dt^2 - R^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \doteq 0 \quad \text{on the light cone}$$

$$\hookrightarrow \frac{1}{R(t)}dt = \frac{dr}{\sqrt{1-kr^2}}$$

In a flat Universe: $$\frac{r_{AB}}{r_0} = \frac{2\int_{t_R}^{t_0}\frac{dt}{R(t)}}{2\int_0^{t_R}\frac{dt}{R(t)}} \quad \begin{array}{l}\longleftarrow \text{matter dominated} \\ \longleftarrow \text{radiation dominated}\end{array}$$

for correct dimension; comes from matching RD $\leftrightarrow$ MD

$$\frac{r_{AB}}{r_0} \sim \frac{\int_{t_R}^{t_0} t^{-\frac{2}{3}}\,dt}{\int_0^{t_R} t^{-\frac{1}{2}}\,dt} \sim \frac{\left(t_0^{1/3} - t_R^{1/3}\right)\cdot t_R^{1/6}}{t_R^{1/2}} \gg 1$$

$\Rightarrow$ A and B were never in causal contact, still they "look the same" (large-scale homogeneity)

$\hookrightarrow$ $\boxed{\text{Horizon problem}}$

3.3 The cosmological constant Λ

Einstein equation with $\Lambda \neq 0$: $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu} + \Lambda g_{\mu\nu}$

Define $T_{\mu\nu}' = T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu}$

$$= \mathrm{diag}\left(\rho + \frac{\Lambda}{8\pi G}\,;\; p - \frac{\Lambda}{8\pi G}\,;\; p - \frac{\Lambda}{8\pi G}\,;\; p - \frac{\Lambda}{8\pi G}\right)$$

<u>Note</u>: For vacuum-dominated Universe: $\omega_\Lambda = p/\rho = -1$

Modified Friedmann equation (cf. (1))

$$\left(\frac{\dot R}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

Modified Friedmann-Lemaître equation (cf. (4))

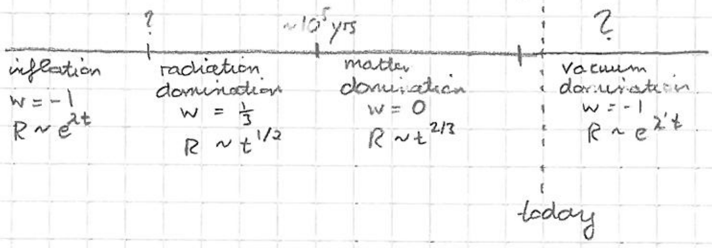
$$\frac{\ddot R}{R} = -\frac{4}{3}\pi G \underbrace{(\rho + 3p)}_{\substack{= \rho(1+3\omega_M) \\ > 0}} + \frac{\Lambda}{3}$$

- $\Lambda \ll 4\pi G \rho (1 + 3\omega_M)$ $\Longrightarrow$ as before, Λ negligible
- $\Lambda = 4\pi G \rho (1 + 3\omega_M)$ $\Longrightarrow$ no acceleration / deceleration
- $\Lambda \gg 4\pi G \rho (1 + 3\omega_M)$ $\Longrightarrow$ $\frac{\ddot R}{R} = \frac{\Lambda}{3}$; $\Lambda < 0$: oscillatory solution

  $\Lambda > 0$: $\boxed{R = R_0\, e^{\sqrt{\Lambda/3}\, t}}$

  Inflation

Assume an epoch of inflation shortly after the Big Bang



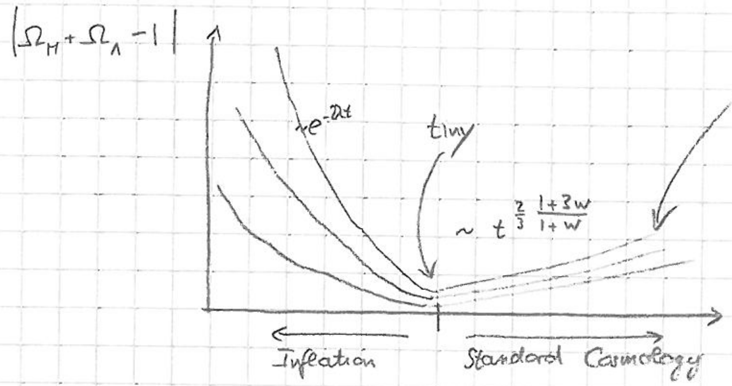
- <u>Flatness problem solved:</u>

Friedmann equation:

$$1 = \underbrace{-\frac{k}{R^2 \underbrace{H^2}_{=\dot{R}^2/R^2}}}_{\equiv \Omega_k} + \underbrace{\frac{8\pi G}{3H^2}\rho}_{\substack{\equiv \Omega_M \\ = \rho/\rho_c}} + \underbrace{\frac{\Lambda}{3H^2}}_{\equiv \Omega_\Lambda}$$

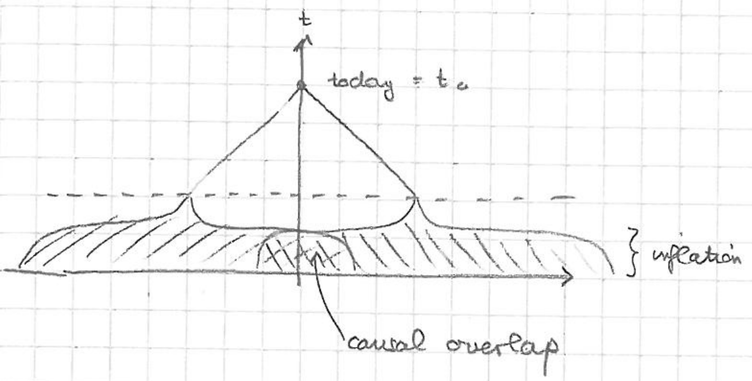
Inflation — $R \sim e^{\lambda t} \Rightarrow \Omega_k \sim e^{-2\lambda t} \to 0$

$\hookrightarrow \Omega_M + \Omega_\Lambda \to 1$



$$R \sim t^{\frac{2}{3(1+w)}}$$
$$\dot{R} \sim t^{\frac{2}{3(1+w)}-1} \sim t^{\frac{-1-3w}{3(1+w)}}$$

- <u>Horizon problem solved</u>



3.4 Λ in QFT

Particles correspond to excitations of fields $A_\mu(x)$, $\psi(x)$, $\phi(x)$

e.g. scalar field: $\mathcal{L}(\phi(x), \partial_\mu \phi(x)) = (\partial_\mu \phi)^+ (\partial_\mu \phi) - V(\phi^+\phi)$

scalar potential $V(\phi^+\phi) = m^2 \phi^+\phi + \frac{\lambda}{2}(\phi^+\phi)^2$

Action $S = \int d^4x\, \mathcal{L}$

Require $\delta S = 0 \Rightarrow \int d^4x \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \underbrace{\delta(\partial_\mu \phi)}_{=\partial_\mu \delta\phi} + \frac{\delta \mathcal{L}}{\delta \phi} \delta\phi \right] = 0$ | Partial integration fields $\to 0$ at boundaries

$$\Leftrightarrow \int d^4x \left[-\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} + \frac{\delta \mathcal{L}}{\delta \phi} \right] \delta\phi = 0$$

Requires $\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \frac{\delta \mathcal{L}}{\delta \phi}$ (Euler Lagrange eq.) (ELE)

Scalar field: $\left[\partial_\mu \partial^\mu + m^2 + \lambda \phi^+\phi\right]\phi^+ = 0$

<u>Question</u>: How to combine with gravity?
Is there a Lagrangian which leads to Einstein's equation via the ELE?

Yes! $\mathcal{L} \equiv \mathcal{L}_{GR}(g_{\mu\nu}) + \mathcal{L}_{Matter}(g_{\mu\nu}, A_\mu, \psi, \phi, \dots)$

$$\mathcal{L}_{GR} = -\frac{1}{16\pi G} \underbrace{\sqrt{g}}_{g \equiv -\det g_{\mu\nu}} (R + 2\Lambda)$$

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Reference: Wald & Wikipedia

$$\delta\sqrt{g} = \frac{1}{2}\frac{1}{\sqrt{-g}} \cdot (-g)\, g^{\mu\nu} \delta g_{\mu\nu}$$

$\left[\frac{\partial(\det A)}{\partial \alpha} = \det A \, \mathrm{tr}\left(A^{-1} \frac{\partial A}{\partial \alpha}\right) \right]$

$$\delta\mathcal{L}_{GR} = -\frac{1}{16\pi G}\left[\delta\sqrt{-g}\,(R + 2\Lambda) + \sqrt{-g}\,\delta(R_{\mu\nu} g^{\mu\nu}) \right]$$

$$\overset{\text{see next page}}{=} -\frac{1}{16\pi G}\Big[(R + 2\Lambda)\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} - \sqrt{-g}\, R^{\mu\nu}\delta g_{\mu\nu}$$

$$+ \sqrt{-g}\, \underline{g^{\mu\nu}\delta R_{\mu\nu}} \Big]$$

can be shown to yield total covariant derivative

To compute $\delta R^{\mu\nu}$ use $R_{\mu\nu\rho}{}^{\sigma} \overset{\text{Wald 3.4.4}}{=} \Gamma^\sigma_{\mu\rho,\nu} - \Gamma^\sigma_{\nu\rho,\mu} + \Gamma^\alpha_{\mu\rho}\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\sigma_{\alpha\mu}$

and definition of covariant derivative:

$$\nabla_\mu t^\nu \overset{\text{Wald 3.1.15}}{=} \partial_\mu t^\nu + \Gamma^\nu_{\mu\rho} t^\rho$$

$$\nabla_\mu t_\nu = \partial_\mu t_\nu - \Gamma^\rho_{\mu\nu} t_\rho$$

$$\begin{aligned}
\nabla_\mu t_\nu &= g_{\nu\rho}\nabla_\mu t^\rho \\
&= g_{\nu\rho}\partial_\mu t^\rho + \frac{1}{2} g_{\nu\rho}\Gamma^\rho_{\mu\alpha} t^\alpha \\
&= \partial_\mu t_\nu - g_{\nu\rho,\mu} t^\rho + \frac{1}{2} g_\nu{}^\sigma (g_{\alpha\sigma,\mu} + g_{\mu\sigma,\alpha} - g_{\mu\alpha,\sigma}) t^\alpha \\
&= \partial_\mu t_\nu + \frac{1}{2}\Big(-2g_{\nu\alpha,\mu} + g_{\nu\alpha,\mu} - g_{\alpha\sigma,\mu}\underbrace{g_\nu{}^\sigma{}_{,\mu}}_{=0} \\
&\qquad + g_{\nu\mu,\alpha} - g_{\mu\alpha,\nu}\Big) g^{\alpha\beta} t_\beta \\
&= \partial_\mu t_\nu - \frac{1}{2} g^{\alpha\beta}(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha}) t_\beta \\
&= \partial_\mu t_\nu - \Gamma^\beta_{\mu\nu} t_\beta
\end{aligned}$$

$$\hookrightarrow \delta R_{\mu\nu\rho}{}^{\sigma} = \delta(\partial_\nu \Gamma^\sigma{}_{\mu\rho}) - \delta(\partial_\mu \Gamma^\sigma{}_{\nu\rho}) + \delta(\Gamma^\alpha{}_{\mu\rho})\Gamma^\sigma{}_{\alpha\nu}$$

$$+ \Gamma^\alpha{}_{\mu\rho}\,\delta\Gamma^\sigma{}_{\alpha\nu} - \delta\Gamma^\alpha{}_{\nu\rho}\cdot\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\alpha{}_{\nu\rho}\,\delta\Gamma^\sigma{}_{\alpha\mu}$$

Note: $\Gamma^\sigma{}_{\mu\rho}$ is not a tensor, but $\delta\Gamma^\sigma{}_{\mu\rho}$ is because it can be written as (for some vector field t^ρ)

Note: $\Gamma^\alpha_{\mu\rho} = \Gamma^\alpha_{\rho\mu}$

$$\delta(\Gamma^\sigma{}_{\mu\rho}\, t^\rho) = \delta(\nabla_\mu t^\sigma) - \partial_\mu \delta t^\sigma$$

$$(\delta\Gamma^\sigma_{\mu\rho})\, t^\rho + \Gamma^\sigma_{\mu\rho}\delta t^\rho = \delta(\nabla_\mu t^\sigma) - \partial_\mu \delta t^\sigma$$

$$(\delta\Gamma^\sigma_{\mu\rho})\, t^\rho = \underbrace{\delta(\nabla_\mu t^\sigma)}_{\text{covariant}} - \underbrace{\nabla_\mu\, \delta t^\sigma}_{\text{covariant}}$$

$\Rightarrow$ Can form covariant derivative

$$\nabla_\nu \delta\Gamma^\sigma{}_{\mu\rho} = \partial_\nu\, \delta\Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\alpha}\delta\Gamma^\alpha{}_{\mu\rho}$$

$$- \Gamma^\alpha{}_{\nu\mu}\delta\Gamma^\sigma{}_{\alpha\rho} - \Gamma^\alpha{}_{\nu\rho}\delta\Gamma^\sigma{}_{\mu\alpha}$$

$$\Rightarrow \delta R_{\mu\nu\rho}{}^{\sigma} = \nabla_\nu \delta\Gamma^\sigma{}_{\mu\rho} + \cancel{\Gamma^\alpha{}_{\nu\mu}\,\delta\Gamma^\sigma{}_{\alpha\rho}} - \nabla_\mu\delta\Gamma^\sigma{}_{\nu\rho} - \cancel{\Gamma^\alpha{}_{\mu\nu}\,\delta\Gamma^\sigma{}_{\alpha\rho}}$$

$$\Rightarrow \delta R_{\mu\rho} = \delta R_{\mu\alpha\rho}{}^{\alpha} = \nabla_\alpha\, \delta\Gamma^\alpha{}_{\mu\rho} - \nabla_\mu \delta\Gamma^\alpha{}_{\alpha\rho}$$

$$\Rightarrow \delta R = \delta(R_{\mu\rho}\, g^{\mu\rho}) = (\nabla_\alpha\, \delta\Gamma^\alpha{}_{\mu\rho})g^{\mu\rho} - (\nabla_\mu\, \delta\Gamma^\alpha{}_{\alpha\rho})g^{\mu\rho}$$

$g^{\mu\rho}$ can be pulled into ∇ since $\nabla_\alpha g_{\mu\nu} = 0$

$$+ \underbrace{R_{\mu\rho}\;\delta g^{\mu\rho}}_{= -R^{\mu\rho}\,\delta g_{\mu\rho}}$$

In the action: $\int d^4x \overset{\text{can be pulled into}}{\sqrt{g}}\nabla_\alpha\, t^\alpha = 0$ (boundary term)

need derivative with lower index here, i.e. works only for $\delta R_{\mu\nu}$, not $\delta R^{\mu\nu}$

$$\Rightarrow \delta\mathcal{L}_{GR} = -\frac{1}{16\pi G}\left[(R + 2\Lambda)\frac{g^{\mu\nu}}{2} - R^{\mu\nu}\right]\sqrt{-g}\;\delta g_{\mu\nu}$$

$$\rightarrow \mathcal{L}_{Matter} = \sqrt{-g}\;\tilde{\mathcal{L}} \quad \overset{\text{scalar}}{\underset{\text{field}}{=}} \sqrt{-g}\left[(\nabla_\mu\phi)^\dagger(\nabla^\mu\phi) - V(\phi^\dagger\phi)\right]$$

$$\rightarrow \delta\mathcal{L}_{Matter} = \tilde{\mathcal{L}} \cdot \frac{1}{2}\sqrt{-g}\; g^{\mu\nu}\,\delta g_{\mu\nu} + \sqrt{-g}\,\frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}$$ + field variation (neglected here)

Note: Taking variations in $\nabla_\mu\phi$ will double-count compared to the case where we vary $g_{\mu\nu}$

$$\delta(\mathcal{L}_{GR} + \mathcal{L}_{Matter}) = -\frac{1}{16\pi G}\left[(R + 2\Lambda)\frac{g^{\mu\nu}}{2} - R^{\mu\nu}\right] + \frac{1}{2}\tilde{\mathcal{L}}\, g^{\mu\nu} + \frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}$$

$$\Rightarrow \quad R^{\mu\nu} - \frac{1}{2}R\, g^{\mu\nu} = 16\pi G\Big[\underbrace{-\frac{1}{2}\tilde{\mathcal{L}}\, g^{\mu\nu} - \frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}}_{\equiv \frac{T^{\mu\nu}}{2}} + \frac{\Lambda}{16\pi G} g^{\mu\nu}\Big]$$

$$= 8\pi G\, T^{\mu\nu} + \Lambda g^{\mu\nu} \quad \text{... Einstein's equation}$$

Assume scalar field and note $\nabla_\mu \phi = \partial_\mu \phi$

$$\tilde{\mathcal{L}} = (\partial_\mu\phi)^+(\partial_\nu\phi)\, g^{\mu\nu} - V(\phi^+\phi)$$

Note: The fundamental field is $\partial_\mu\phi$, not $\partial^\mu\phi$ → have to include the $g^{\mu\nu}$ to pull the index up

$$\hookrightarrow T^{\mu\nu} = -\tilde{\mathcal{L}}\, g^{\mu\nu} - 2\frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}$$

$$= -(\partial_\rho\phi)^+(\partial^\rho\phi)\, g^{\mu\nu} + V g^{\mu\nu} + 2\frac{\delta\tilde{\mathcal{L}}}{\delta g^{\alpha\beta}}\, g^{\mu\alpha} g^{\nu\beta} \qquad \left[\delta A^{-1} = -A^{-1}\delta A\, A^{-1}\right.$$

Approx const g?

$$= -(\partial_\rho\phi)^+(\partial^\rho\phi)\, g^{\mu\nu} + 2(\partial^\mu\phi)^+(\partial^\nu\phi) + V(\phi^+\phi)\, g^{\mu\nu}$$

$$\rho = T^0{}_0 = T^{00} = (\partial_0\phi)^+(\partial_0\phi) + \frac{1}{R^2}(\underbrace{\vec{\nabla}}_{=\partial_i}\phi)^+(\vec{\nabla}\phi) + V(\phi^+\phi)$$

$$p = T^i{}_i = R^2 T^{ii} = R^2\left[\frac{1}{R^2}(\partial_0\phi)^+(\partial_0\phi) - \frac{1}{R^4}(\vec{\nabla}\phi)^+(\vec{\nabla}\phi) + 2\frac{1}{R^4}(\partial_i\phi)^+(\partial_i\phi) - \frac{1}{R^2}V(\phi^+\phi)\right]$$

(no summation)

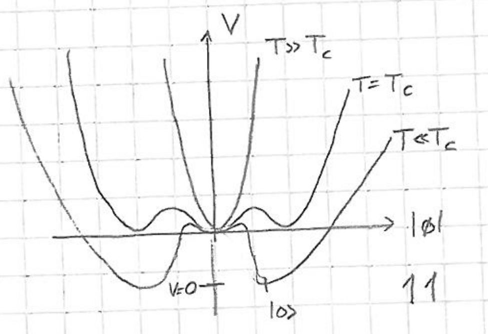
For nearly constant ϕ $(\partial^0\phi \simeq 0;\ \partial^i\phi \simeq 0)$

$$\hookrightarrow \rho = V(\phi^+\phi) = -p \quad \Rightarrow \quad \boxed{w = \frac{p}{\rho} = -1}$$

Behaves like cosmological constant!

Typical scalar potential:

$$V = -\mu^2\phi^+\phi + \lambda(\phi^+\phi)^2 + \underbrace{\lambda(\phi^+\phi)^2 \ln\phi^+\phi}_{\text{small quantum correction}} + \underbrace{\overset{\text{temperature}}{T^2}\,\phi^+\phi}_{T\neq 0 \text{ correction}} + \dots$$



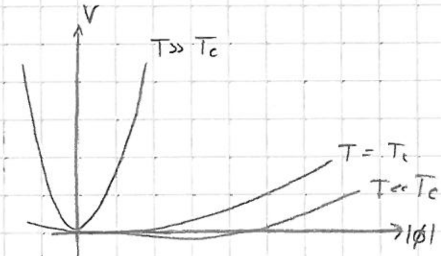
Phase transition at $T = T_c$

("Old inflation": Alan Guth, 1981)

- $V \gg 0$ → Universe inflates → at $T < T_c$, tunnelling into true vacuum begins → when tunnelling ends, inflation is over

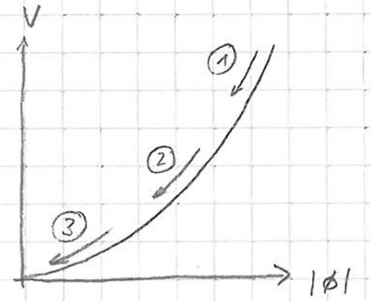
<u>Problems:</u>
- Outside "true vacuum bubbles", inflation continues → may never end in parts of the Universe
- Universe almost empty after inflation
  → need mechanism to <u>reheat</u>
- Properties of different bubbles different (e.g. curvature)
  → expect a lot of inhomogeneities on large scales

- "New inflation" (Andrei Linde, 1982)



- "Slow roll" of field into true vacuum
- Eventually, $\partial^\mu \phi$ becomes non-negligible
  ↳ oscillations around true vacuum = inflaton particles
- Inflaton decay → produces lots of ordinary particles → "reheating"

- "Chaotic inflation"



- $|\phi|$ starts at random value (large)
- EoM: (assume $\vec{\nabla}\phi \simeq 0$):

$$D_t \frac{\delta \mathcal{L}}{\delta(D_t \phi^\dagger)} - \frac{\delta \mathcal{L}}{\delta \phi^\dagger} = 0$$

$$\mathcal{L} = \sqrt{-g}\left((D^\mu \phi)^\dagger (D_\mu \phi) - V\right)$$

see lecture notes Lindner for derivation

Covariant derivative: $D_\mu \phi \equiv \partial_\mu \phi$

$$D_\mu A^\mu \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu$$

$$\Rightarrow D_t\left(\sqrt{-g}\,\partial_t \phi\right) + \sqrt{-g}\frac{dV}{d\phi^\dagger} = 0$$

$$\frac{1}{\sqrt{-g}} \partial_t \left(-g\, \partial_t \phi\right) + \sqrt{-g}\frac{dV}{d\phi^\dagger} = 0$$

$$\sqrt{-g}\,\ddot{\phi} + \frac{1}{\sqrt{-g}}\dot{\phi} \cdot 3R^2 \dot{R} + \sqrt{-g}\frac{dV}{d\phi^\dagger} = 0$$

$$\boxed{\ddot{\phi} + 3\underbrace{H}_{=\frac{\dot{R}}{R}}\dot{\phi} + \frac{dV}{d\phi^\dagger} = 0}$$

$(g_{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -R(t) & & \\ & & -R(t) & \\ & & & -R(t) \end{pmatrix}$

$\Rightarrow g = -R^3$

① Very large H
↳ $\ddot{\phi}$ negligible
$\dot{\phi}$ small $\left(|3H\dot{\phi}| \simeq \frac{dV}{d\phi^\dagger}\right)$
→ slow roll

② V decreases, H decreases
Friction term $3H\dot{\phi}$ becomes less important

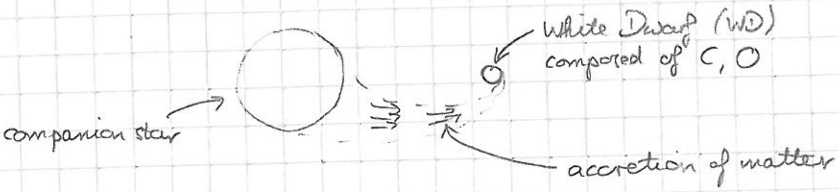
③ Friction negligible
$\ddot{\phi} + \frac{dV}{d\phi^\dagger} = 0$
→ oscillations around minimum
→ reheating

4. Dark energy

1998 : High-z supernova search team (Adam Riess, Brian Schmidt et al.)
Supernova Cosmology Project (Saul Perlmutter et al.)

Observations of Type Ia SN at high redshift

- Type Ia SN :



Binary system

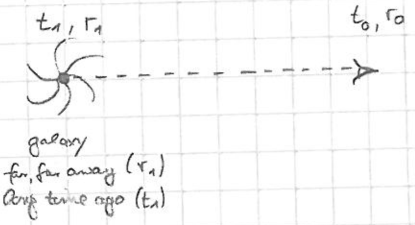
When WD mass reaches Chandrasekhar Limit $\sim 1.4\ M_\odot$
T high enough to re-ignite fusion
$\Rightarrow$ runaway fusion, thermonuclear explosion, WD destroyed

Common progenitor mass $\rightarrow$ similar E-output in all Type Ia SN

$\rightarrow$ $\boxed{\text{"Standard Candles"}}$

- Redshift

Distant source emits light



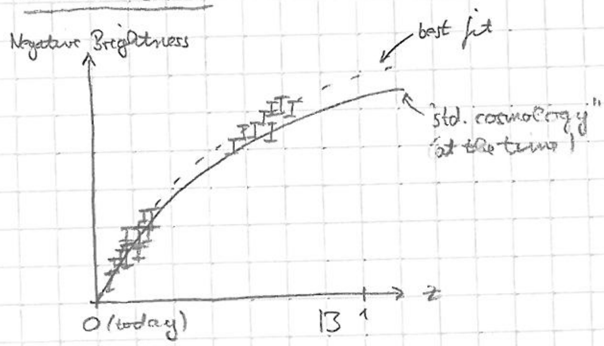
Distance between wave crests : $\lambda_1 = a(t_1)\, dr$

Today : $\lambda_0 = R(t_0)\, dr$

Here was a typo on 02.05.

Redshift : $\dfrac{\lambda_0 - \lambda_1}{\lambda_1} = \boxed{\dfrac{R(t_0)}{R(t_1)} - 1 \equiv z}$

- Observation



Two measurements of distance disagree !

Hypothesis : $a(t_1)$ was smaller than assumed

$\boxed{\Rightarrow \text{Expansion accelerating}}$

end 02.05.2013

Remember: Matter domination : $R \sim t^{2/3}$; $\dot{R} \sim t^{-\frac{1}{3}}$; $\ddot{R} \sim -t^{-\frac{4}{3}} < 0$:-(

Radiation domination : $R \sim t^{1/2}$; $\dot{R} \sim t^{-\frac{1}{2}}$; $\ddot{R} \sim -t^{-\frac{3}{2}} < 0$:-(

Vacuum domination : $R \sim e^{\lambda t}$; $\dot{R} \sim e^{\lambda t}$; $\ddot{R} \sim e^{\lambda t} > 0$:-)
(Λ or inflation)

A new form of vacuum energy : c.c. Λ or scalar potential of new field ϕ

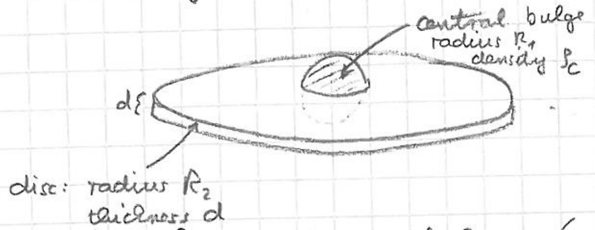
$\rightarrow$ $\boxed{\text{Dark Energy}}$

Alternative : Einstein equations wrong, GR modified

5. Dark Matter

5.1 The path to the dark side: Evidence for DM

- Jan Oorth (1932), Vera Rubin (1970): Galaxy rotation curves (v vs. r diagram)



Visible stars → estimate mass distribution → compute force on test body at distance r, mass m → compute orbital velocity v (for stable orbit)

1) Central bulge ($r \leq R_1$):

bulge density

$$F = \frac{G m M(r)}{r^2} \doteq m\frac{v^2}{r} \quad ; \quad M(r) = \frac{4}{3}\pi r^3 \rho_c$$

$$\Rightarrow \boxed{v = \sqrt{\frac{4}{3}\pi G \rho_c}\; r}$$

~~2) Disc ($R_1 \leq r \leq R_2$):~~

Disc cannot be treated as point mass!

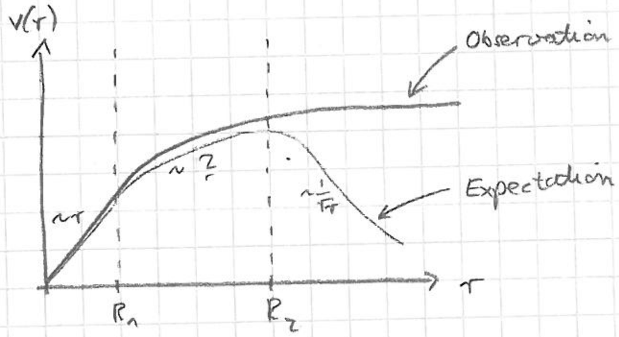
$$\cancel{M(r) = \frac{4}{3}\pi R_1^3 \rho_c + \pi r^2 d \rho_D}$$

$$\cancel{\Rightarrow v = \sqrt{\frac{G}{r}\cdot\left(\frac{4}{3}\pi R_1^3 \rho_c + \pi r^2 d \rho_D\right)} \xrightarrow{r \text{ large}} \boxed{\sqrt{\pi G d \rho_D}\cdot\sqrt{r}}}$$

3) Outside the disc ($r \gg R_2$)

$$M(r) = \frac{4}{3}\pi R_1^3 \rho_c + \pi R_2^2 d \rho_D \equiv M_0$$

$$v = \sqrt{G\cdot M_0}\cdot\frac{1}{\sqrt{r}}$$



Observation:
Flat rotation curves
→ Invisible (dark) matter

$$v(r) \sim \text{const} \sim \sqrt{\frac{G M(r)}{r}}$$

$\hookrightarrow M(r) \sim r$

Spherical DM halo: $M(r) = 4\pi\int_0^r \rho(r) r^2 dr$

$$\hookrightarrow \boxed{\rho(r) \sim \frac{1}{r^2}}$$

~~DM disc~~: $\cancel{M(r) = 2\pi d \int_0^r \rho(r) r\, dr}$

$$\cancel{\hookrightarrow \boxed{\rho(r) \sim \frac{1}{r}}}$$

- Fritz Zwicky, 1933: Dynamics of galaxy clusters

Virial theorem: $\boxed{\sum E_{kin} = -\frac{1}{2}\sum E_{pot}}$

sum over all galaxies in the cluster

from Doppler shift

from brightness + $V = \frac{GMm}{r}$

Observation: $\boxed{\sum E_{kin} \cong -\frac{1}{2}\cdot 170 \sum E_{pot}}$

↳ More gravitational pull than explained by luminous matter

- The cosmic microwave background (CMB)

[show WMAP or Planck CMB skymap]

At $t_{rec} \simeq 300\,000$ yrs: e^- and p^+ recombine to H ⇒ Universe becomes transparent

↳ Photons emitted at t_{rec} observable today

$$T(t_{rec}) \cong 3000K \xrightarrow{\text{redshift } z \simeq 1100} \boxed{T(t_0) = 2.73 \text{ K}}$$

Tiny fluctuations of T from primordial quantum fluctuations; strongest at eigenfrequencies of the plasma

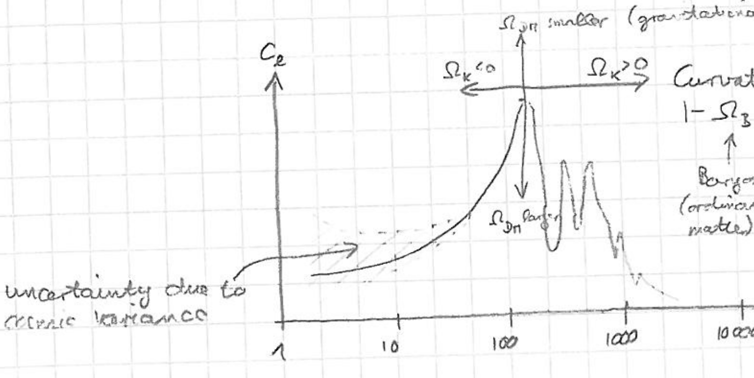
↳ CMB Power spectrum

spherical harmonic $\left[\sim e^{im\phi} P_\ell^m(\cos\theta)\right.$ associated Legendre function

$$T(\underbrace{\theta, \phi}_{\text{galactic coordinates}}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

$$a_{\ell m} = \int d(\cos\theta)\, d\phi \; Y^*_{\ell m}(\theta, \phi)\, T(\theta, \phi)$$

$$C_\ell = \frac{1}{2\ell+1}\sum_m |a_{\ell m}|^2$$

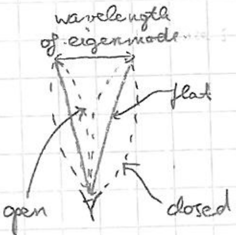


Curvature
$$1 - \Omega_B - \Omega_{DM} - \Omega_\Lambda \equiv \Omega_k = -\frac{k}{R^2}$$

Baryons (ordinary matter)

Massless matter

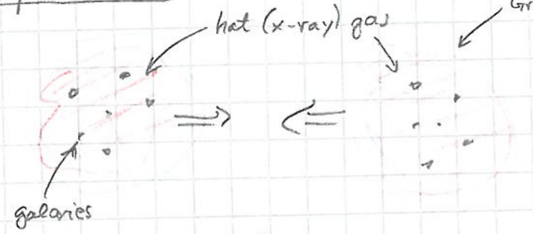
cosmological constant

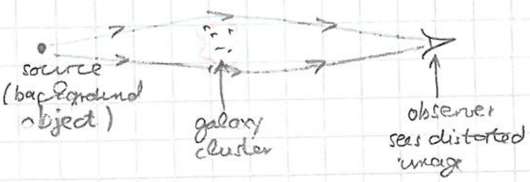


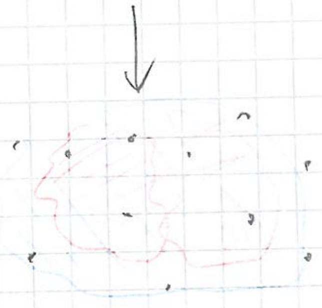
- Collisions of galaxy clusters

[show bullet cluster]

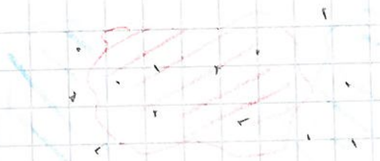
Expectation







Observation:



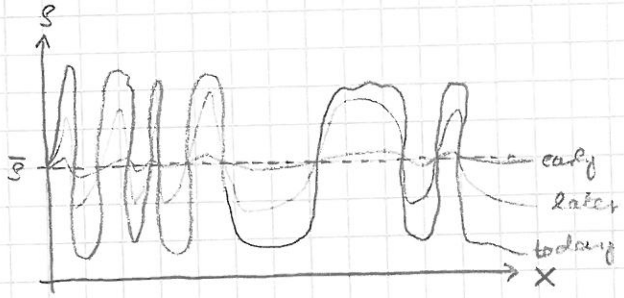
Large separation of mass distributions after collision

↓

a lot of non-interacting matter in galaxy clusters

- Structure formation

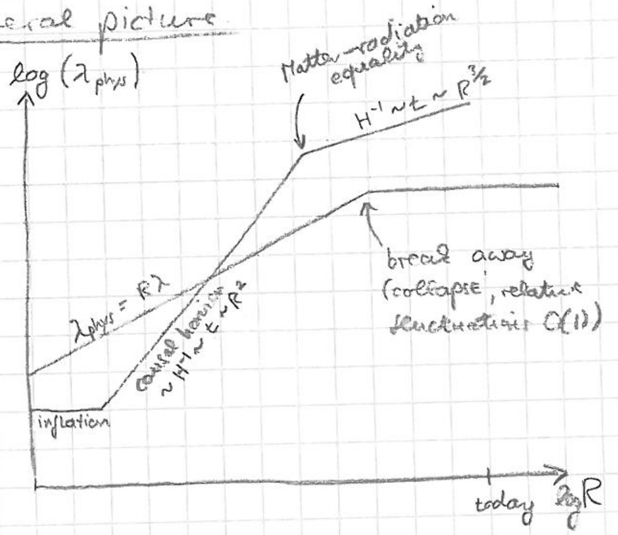
Jeans instability $\longrightarrow$ amplification of density fluctuations



- Governed by gravity alone
- Comparison of early and late observations (e.g. CMB and today)
  $\longrightarrow$ consistency check
  $\Rightarrow$ we need DM
- Work in multipole expansion, different terms = different length scales λ

end 16.05.2013

General picture



Kolb Turner; Knobel arXiv:1208.5931

Basic equations (fluid mechanics - Newtonian gravity)

1) Continuity equation: $\boxed{\frac{d\rho}{dt} + \vec{\nabla}(\rho\vec{v}) = 0}$ (1)

2) Euler equation (local balance of forces)

see Dentrader I, sec. 8.2

$$\boxed{\underbrace{\frac{\partial\vec{v}}{\partial t}}_{\text{acceleration}} + \underbrace{(\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{convective acceleration}} + \underbrace{\frac{1}{\rho}\vec{\nabla}p}_{\text{pressure gradients}} + \underbrace{\vec{\nabla}\phi}_{\text{gravity}} = 0} \qquad (2)$$

(e.g. liquid in pipes of different diameter)

3, Poisson equation

$$\boxed{\Delta\phi = 4\pi G \rho} \qquad (3)$$

Simplest solution (fully homogeneous Universe)

physical coordinate

$$\bar\rho(t) = \frac{\rho_0}{R^3(t)} \quad ; \quad \vec{\bar v}(t) = \frac{\dot R(t)}{R(t)}\vec r \quad ; \quad \vec\nabla\bar\phi = \frac{4\pi G \rho_0}{3}\vec r$$

Check: (1) $\frac{d\bar\rho}{dt} = -3\frac{\dot R}{R}\bar\rho = -\bar\rho\cdot(\vec\nabla\vec v)$ ✓ $\left[\vec\nabla\bar\rho = 0, \ \vec\nabla\vec r = 3\right.$

(2) $\frac{\partial\vec v}{\partial t} = \frac{\ddot R}{R}\vec r - \underbrace{\frac{\dot R^2}{R^2}\vec r}_{=(\vec r\cdot\vec\nabla)\vec r}$

Friedmann = Lemaître

$$= -\frac{4}{3}\pi G(\rho + 3p)\vec r - (\vec v\cdot\vec\nabla)\vec v \quad ✓ \qquad \left[\vec\nabla\bar p = w\vec\nabla\bar\rho = 0\right.$$

assume $\ll \rho$ (non-rel.)

(3) obvious

Consider small perturbations:

$$\rho(\vec x, t) = \bar\rho(t) + \rho_1(\vec x, t) \quad ; \quad |\rho_1| \ll |\bar\rho|$$

$$p(\vec x, t) = \bar p(t) + p_1(\vec x, t) \quad ; \quad |p_1| \ll |\bar p|$$

$$\vec v(\vec x, t) = \vec{\bar v}(t) + \vec v_1(\vec x, t) \quad ; \quad |\vec v_1| \ll |\vec{\bar v}|$$

$$(1) \Rightarrow \frac{d\bar{\rho}}{dt} + \frac{d\rho_1}{dt} + \vec{\nabla}\left[(\bar{\rho}+\rho_1)(\vec{\bar{v}}+\vec{v}_1)\right] = 0$$

$$\Leftrightarrow \frac{d\rho_1}{dt} + \bar{\rho}\,\vec{\nabla}\vec{v}_1 + (\vec{\nabla}\rho_1)\vec{\bar{v}} + (\vec{\nabla}\vec{\bar{v}})\,\rho_1 = 0$$

$$\Leftrightarrow \boxed{\frac{d\rho_1}{dt} + \bar{\rho}\,\vec{\nabla}\vec{v}_1 + (\vec{\nabla}\rho_1)\vec{\bar{v}} + 3\frac{\dot{R}}{R}\rho_1 = 0} \qquad (1')$$

$$(2) \Rightarrow \frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_1\cdot\vec{\nabla})\vec{\bar{v}} + (\vec{\bar{v}}\cdot\vec{\nabla})\vec{v}_1 + \frac{1}{\bar{\rho}}\vec{\nabla}p_1 + \vec{\nabla}\phi_1 = 0$$

$$\Leftrightarrow \boxed{\frac{\partial \vec{v}_1}{\partial t} + \frac{\dot{R}}{R}\vec{v}_1 + \frac{\dot{R}}{R}(\vec{r}\cdot\vec{\nabla})\vec{v}_1 + \frac{v_s^2}{\bar{\rho}}\vec{\nabla}\rho_1 + \vec{\nabla}\phi_1 = 0} \quad (2')$$

Speed of sound:

$$v_s^2 = \frac{\partial p}{\partial \rho}\bigg|_{adiabatic} \approx \frac{p_1}{\rho_1} = w$$

$$(3) \Rightarrow \boxed{\Delta\phi_1 = 4\pi G \rho_1} \qquad (3')$$

[due to superposition principle (linearity)

Solution: 1) Define $\delta = \frac{\rho_1}{\rho_0} \Rightarrow \dot{\rho}_1 = \dot{\delta}\,\bar{\rho} + \delta\,\dot{\bar{\rho}} = \dot{\delta}\,\bar{\rho} - 3H\delta\bar{\rho}$

$$(1') \Rightarrow \boxed{\dot{\delta} + \vec{\nabla}\vec{v}_1 + (\vec{\nabla}\delta)\vec{\bar{v}} = 0}$$

2) Use comoving coordinates: $\vec{r}\,' \equiv \frac{\vec{r}}{R(t)}$

$$\hookrightarrow \vec{\nabla}' = R\vec{\nabla} \quad ; \quad \frac{\partial}{\partial t}\bigg|_{\vec{r}\,'=const} = \frac{\partial}{\partial t}\bigg|_{\vec{r}=const} + \frac{1}{R}\vec{\bar{v}}\,\vec{\nabla}'$$

$$\Rightarrow \frac{\partial \delta}{\partial t}\bigg|_{\vec{r}\,'=const} - \cancel{\frac{1}{R}\vec{\bar{v}}\,\vec{\nabla}'\delta} + \frac{1}{R}\vec{\nabla}'\vec{v}_1 + \cancel{\frac{1}{R}\vec{\bar{v}}\,\vec{\nabla}'\delta} = 0 \qquad \bigg|\frac{\partial}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \delta}{\partial t^2}\bigg|_{\vec{r}\,'=const} + \frac{1}{R}\vec{\nabla}'\Big[\cancel{\frac{1}{R}\vec{\bar{v}}\,\vec{\nabla}'\vec{v}_1} - \frac{\dot{R}}{R}\vec{v}_1 - \cancel{\frac{\dot{R}}{R^2}(\vec{r}\cdot\vec{\nabla}')\vec{v}_1}$$

$$- \frac{1}{R}v_s^2\,\vec{\nabla}'\delta - \frac{1}{R}\vec{\nabla}'\phi_1\Big] - \frac{\dot{R}}{R^2}\vec{\nabla}'\vec{v}_1 = 0$$

$$\Leftrightarrow \frac{\partial^2 \delta}{\partial t^2}\bigg|_{\vec{r}\,'=const} \underbrace{-2H\cdot\frac{1}{R}\vec{\nabla}'\vec{v}_1}_{=2H\dot{\delta}} - \frac{1}{R^2}v_s^2\,\Delta'\delta - \frac{1}{R^2}\Delta'\phi_1 = 0$$

$$\Leftrightarrow \boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{v_s^2}{R^2}\Delta'\delta - 4\pi G\bar{\rho}\,\delta = 0}$$

3) Fourier transform:

$$\delta(\vec{x}', t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}'} \delta_k(t)$$

$$\hookrightarrow \boxed{\ddot{\delta}_k + 2H\dot{\delta}_k + \frac{v_s^2}{R^2}\Big(k^2 - R^2 \underbrace{\frac{4\pi G\bar{\rho}}{v_s^2}}_{=(\text{Jeans wave number } k_J)^2}\Big)\delta_k = 0} \qquad (*)$$

<u>Limiting cases</u>

i) <u>No expansion ($H = 0$, $R = 1$)</u>

$$\hookrightarrow \delta \sim \exp\left[\pm i\omega t \mp i\vec{k}\vec{x}\right]; \quad \omega = \sqrt{v_s^2(k^2 - k_J^2)}$$

For $k > k_J$: Propagating sound waves

For $k < k_J$: Exponentially growing and decaying modes

ii) <u>Matter-dominated Universe ($v_s = 0$), no curvature</u> (no k-term in Friedmann eq.)

$$\hookrightarrow R \sim t^{\frac{2}{3(1+w)}} \quad ; \quad \dot{R} = \frac{2}{3(1+w)t} R$$

$$(*) \Rightarrow \ddot{\delta}_k + \frac{4}{3(1+w)t}\dot{\delta}_k - \underbrace{4\pi G\bar{\rho}}_{= +\frac{3}{2}(\dot{R}/R)^2 = +\frac{2}{3}\frac{1}{(1+w)^2 t^2}} \cdot \delta_k$$

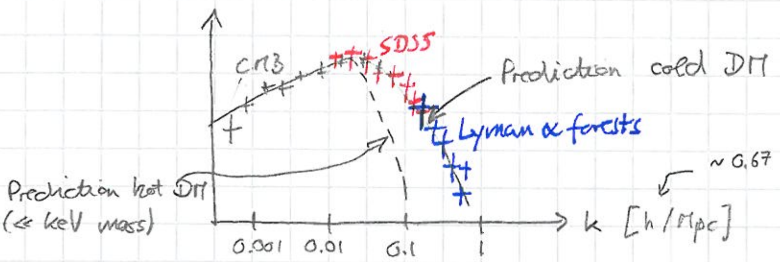
(Friedmann eq. $\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\bar{\rho}}{3}$)

$$\overset{w=0}{\Longleftrightarrow} \ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0$$

$$\Rightarrow \delta_k = c_1(\vec{k})\, t^{2/3} + c_2(\vec{k})\, t^{-1} \xrightarrow{\text{large } t} \sim t^{2/3}$$

<u>Comparing with data</u>

- Start from initial density perturbations (CMB anisotropies or theory — quantum fluctuations during inflation)
- Evolve in time
- Compare with observed distributions of galaxies (e.g. Sloan Digital Sky Survey). Observable: Matter power spectrum $P(\vec{k}) = |\delta_k|^2$

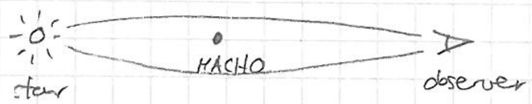


Conclusion: DM should be in the form of heavy ($\gg$ keV) particles.

- MACHO (Massive Compact Halo Object) searches

Question: Is DM in the form of Jupiter-like objects, white dwarfs etc.?

Method: Gravitational microlensing



$\rightarrow$ transient light amplification

Result: e.g. EROS (Experience de Recherche d'Objets Sombres) 2006:

$6 \cdot 10^{-8} M_\odot < m_{MACHO} < 15\, M_\odot$ ruled out as 100% DM

5.2 The DM abundance: Freeze-out

see Kolb-Turner

DM (assumed self-conjugate here) — Some SM particle + antiparticle (need not be a fermion)

Assume feeble DM-SM interaction $\chi\chi \leftrightarrow \bar{f}f$

Very early: DM in thermal equilibrium

Particle densities so high that even suppressed interactions are probable

Number density

$$n_{\chi,eq} = \frac{N_{eq,\chi}}{V} = \frac{N_{eq\,\chi}}{R^3(t)\cdot V_0}$$

($N_{eq,\chi}$: number, V: volume)

$$= g \cdot \int \frac{d^3p}{(2\pi)^3} \exp\left[-\left(E(p)-\mu\right)/T\right]$$

g: # of internal degrees of freedom (e.g. 2 for spin $\frac{1}{2}$)

μ: chemical potential (usually negligible)

$$= \frac{g}{2\pi^2} \int dp\, p^2 \exp\left[-\left(E(p)-\mu\right)/T\right]$$

$E = \sqrt{p^2+m^2}$

$\frac{dE}{dp} = \frac{p}{E}$

$$= \frac{g}{2\pi^2} \int dE\; E\sqrt{E^2-m^2}\; \exp\left[-(E-\mu)/T\right]$$

Later: T, ρ decrease $\rightarrow$ interactions freeze out

scattering / annihilation probability reduced due to lower density, as T drops, fewer and fewer particles have sufficient energy to initiate the backwards reaction

Governed by Boltzmann equation:

$$\boxed{\frac{dN_\chi}{dt} \overset{!}{=} \Gamma(\bar{f}f\rightarrow\chi\chi) - \Gamma(\chi\chi\rightarrow\bar{f}f) + \Gamma_{other}}$$

(Γ: rate; Γ_{other}: subdominant annihilation channels / elastic scattering)

Computation of Γ:

S-matrix: $S \equiv \exp\left[-i\int d^4x\, \mathcal{H}\right] \simeq 1 - i\int d^4x\,\mathcal{H} - \frac{1}{2}\left(\int d^4x\,\mathcal{H}\right)^2 + \ldots$

($\mathcal{H}$: Hamiltonian)

Matrix element:

$$(2\pi)^4\,\delta^{(4)}(p_1+p_2-k_1-k_2)\cdot \mathcal{M}(\chi\chi\rightarrow\bar{f}f) = \langle \bar{f}(p_1)\, f(p_2) \,|\, S \,|\, \chi(k_1)\,\chi(k_2)\rangle$$

Transition probability:

$$|\mathcal{M}|^2 \cdot \left[\delta^{(4)}(p_1+p_2-k_1-k_2)\right]^2 \cdot (2\pi)^8$$

$$\text{Use } \quad 2\pi\,\delta(\Delta E) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i\Delta E t}\, dt = \lim_{T\to\infty} \frac{2}{\Delta E} \sin\frac{\Delta E\, T}{2}$$

$$\left[2\pi\,\delta(\Delta E)\right]^2 = 2\pi\,\delta(\Delta E) \cdot \lim_{T\to\infty} \frac{2}{\Delta E} \sin\frac{\Delta E\, T}{2}$$

$$= \lim_{T\to\infty} 2\pi\,\delta(\Delta E) \cdot T$$

$\Rightarrow$ Transition rate [probability per time]

$$\Gamma = \frac{1}{\cancel{T}} (2\pi)^4 \cdot \cancel{T} \cdot V \, |\mathcal{M}|^2 \, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)$$

Continuum of final states and of initial states

change normalization to Lorentz invariant one

$$\hookrightarrow \quad \Gamma = V \int \underbrace{\frac{d^3p_1}{(2\pi)^3 2E(p_1)}}_{\equiv d\Pi(p_1)} \frac{d^3p_2}{(2\pi)^3 2E(p_2)} \frac{d^3k_1}{(2\pi)^3 2E(k_1)} \frac{d^3k_2}{(2\pi)^3 2E(k_2)} \, \phi_\chi(\vec{k}_1)\, \phi_\chi(\vec{k}_2)$$

i.s. phase space densities

Identical i.s. particles ?

$$\cdot \, (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \, \overline{|\mathcal{M}|^2} \qquad (\square)$$

[Note: For final state bosons, include factor $(1 + \phi_f(\vec{p}_1))(1 + \phi_f(\vec{p}_2))$, for final state fermions $(1 - \phi_f(\vec{p}_1))(1 - \phi_f(\vec{p}_2))$. For low densities, $f \ll 1$

[Note: $\overline{|\mathcal{M}|^2}$ includes factor $\frac{1}{2}$ for identical i.s. particles, spin averaging factors, etc

$$\Rightarrow \quad \frac{dN_\chi}{dt} = \frac{d(n_\chi V)}{dt} = V \frac{dn_\chi}{dt} + n_\chi \frac{d(R^3 V_0)}{dt} = V \frac{dn_\chi}{dt} + 3 n_\chi H V$$

$$\Rightarrow \quad \frac{dn_\chi}{dt} + 3 n_\chi H = \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2) \qquad (*)$$

$$\cdot \left[\phi_f(p_1)\phi_f(p_2) - \phi_\chi(k_1)\phi_\chi(k_2)\right] (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \, \overline{|\mathcal{M}|^2} + \Gamma_{\text{other}}$$

assume $\mathcal{M}(\chi\chi \to \bar{f}f) = \mathcal{M}(\bar{f}f \to \chi\chi)$ (T invariance + CP invariance

- Assume $f, \bar{f}$ in thermal equilibrium

  $\hookrightarrow \phi_{\bar{f}}(p_{1,2}) = \exp\left[-E(p_{1,2})/T\right] \equiv \phi_{\bar{f},eq}(p_{1,2})$

  Since $E(p_1) + E(p_2) = E(k_1) + E(k_2)$

  $$\hookrightarrow \phi_{\bar{f},eq}(p_1) \cdot \phi_{f,eq}(p_2) = \exp\left[-(E(p_1) + E(p_2))/T\right]$$

  $$= \exp\left[-(E(k_1) + E(k_2))/T\right]$$

  $$= \phi_{\chi,eq}(k_1) \cdot \phi_{\chi,eq}(k_2)$$

- Use $\sigma_{\chi\chi\to\bar{f}f} = \frac{\text{Rate per target particle}}{\text{Flux of i.s. particles}} = \frac{\text{Rate per target particle}}{\text{Density of i.s. particles} \cdot \text{Velocity}}$

  Take (□), divide by $N = n_\chi \cdot V$ to obtain rate per target WIMP

  $$\hookrightarrow \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} = \frac{1}{n_\chi^2} \int d\Pi(p_1)\, d\Pi(p_2)\, (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|^2}$$

- Assume that after <u>chemical decoupling</u> (cessation of $\chi\chi \leftrightarrow \bar{f}f$), χ still stays in <u>kinetic equilibrium</u> ($\chi f \leftrightarrow \chi f$ still fast)

  $$\Rightarrow \phi_\chi = \frac{n_\chi}{n_{\chi,eq}} \phi_{\chi,eq}$$

- Thermally averaged x-sec

  $$\langle \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} \rangle \equiv \frac{1}{n_\chi^2} \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2)\, \phi_\chi(k_1)\, \phi_\chi(k_2) \cdot (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|^2} \quad (□□)$$

  $$= \frac{1}{n_{\chi,eq}^2} \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2)\, e^{-[E_\chi(k_1) + E_\chi(k_2)/T]} \cdot (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|^2}$$

  $$= \langle \sigma_{\bar{f}f\to\chi\chi} \cdot v_{rel} \rangle$$

$\Rightarrow$ Rewrite (*)

$$\boxed{\begin{aligned} \frac{dn_\chi}{dt} + 3 n_\chi H &= -\langle \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} \rangle \left[n_\chi^2 - n_{\chi,eq}^2 \right] + \Gamma_{other} \\ &= -\langle \underbrace{\sigma_{ann}}_{\text{total annihilation x-sec}} \cdot v_{rel} \rangle \left[n_\chi^2 - n_{\chi,eq}^2 \right] \end{aligned}} \quad (**)$$

- Detailed balance: In thermal equilibrium,

$$\Gamma^{eq}(\bar{f}f \to \chi\chi) = \Gamma^{eq}(\chi\chi \to \bar{f}f)$$

[Alternative: Write out Γ according to (II) above as (schematically)

$$\Gamma(\chi\chi\to\bar{f}f) = \int d\Phi_\chi \, d\underbrace{\bar{\Phi}_f}_{\text{phase space}} \underbrace{f(\chi)}_{\text{distribution function}} |\mathcal{M}|^2 \cdot \delta^{(4)}(\Sigma p_\chi - \Sigma p_f)$$

and use

$$f(\chi) = f^{eq}(f) \; ; \quad f(\chi) = \frac{n_\chi}{n_{\chi,eq}} f^{eq}(\chi) \; ;$$

$$\mathcal{M}(\chi\chi \to \bar{f}f) = \mathcal{M}(\bar{f}f \to \chi\chi)$$]

$\Gamma(\bar{f}f \to \chi\chi)$ determined by $n_f = n_{f,eq}$

$\hookrightarrow \quad \Gamma(\bar{f}f \to \chi\chi) = \Gamma^{eq}(\chi\chi \to \bar{f}f)$

- Use cross section $= \dfrac{\text{total rate}}{(\#\text{ of target particles})\cdot(\text{flux of beam particles})}$

$$\Gamma(\chi\chi \to \bar{f}f) = \underbrace{\langle}_{\text{averaging over thermal } \vec{p}\text{-distrib.}} \sigma(\chi\chi \to \bar{f}f) \cdot (n_\chi \cdot \underbrace{V}_{\text{volume}}) \cdot (n_\chi \cdot \underbrace{v_{rel}}_{\text{rel. velocity of two } \chi \text{ particles}}) \rangle$$

$$\Rightarrow \quad \frac{d(n_\chi \overbrace{V}^{\equiv V_0 R^3})}{dt} = \langle \sigma(\chi\chi \to \bar{f}f) \, v_{rel} \rangle (n_{\chi,eq}^2 - n_\chi^2)$$

$$\boxed{\frac{dn_\chi}{dt} + 3H n_\chi = -\langle \sigma(\chi\chi \to \bar{f}f) \, v_{rel} \rangle (n_\chi^2 - n_{\chi,eq}^2)}$$

Useful definition : $\boxed{Y = \frac{n}{s}}$

where s = entropy density.

Note : $s \cdot R^3 = \underbrace{const}_{\equiv s_0}$ (Entropy in comoving volume element conserved for adiabatic expansion)

$\hookrightarrow \dot{Y} = \frac{\dot{n}}{s} - Y\frac{\dot{s}}{s} = \frac{\dot{n}}{s} + 3\frac{\dot{R}}{R}Y$

$\Rightarrow$ (**) becomes $\boxed{\frac{dY}{dt} = -\langle\sigma_{ann} v_{rel}\rangle\, s\left[Y^2 - Y_{eq}^2\right]}$ (omitting index χ)

Solving the Boltzmann equation

More useful than t: $x = \frac{m_\chi}{T}$

To relate x and t:

see expression for n_{eq}

fermions

$$\rho \overset{\text{rad. dom.}}{=} \frac{\rho_0}{R^4} \overset{\text{rad. dom.}}{=} \frac{\rho_0}{(R_0\sqrt{t})^4} = \sum_{\substack{i=\text{rel.}\\ \text{species}}} \frac{g_i}{2\pi^2}\int_0^\infty \frac{E}{\exp[E/T]\pm 1} E^2 dE$$

bosons

$$= \sum_{\substack{i=\text{rel.}\\ \text{bosons}}} \frac{\pi^2}{30} g_i T^4 + \sum_{\substack{i=\text{rel.}\\ \text{fermions}}} \frac{7}{8}\cdot\frac{\pi^2}{30} g_i T^4$$

$$\equiv g^* \frac{\pi^2}{30} T^4 \qquad \left[g^* = \sum_{\text{bosons}} g_i + \frac{7}{8}\sum_{\text{fermions}} g_i\right.$$

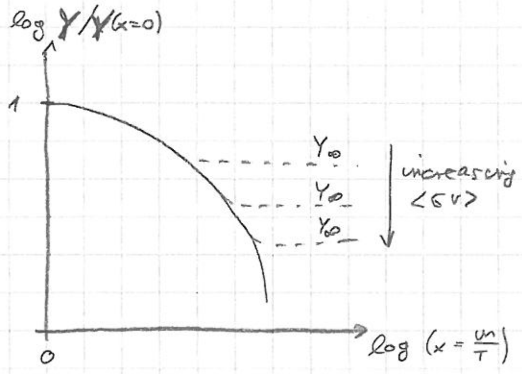
$\Rightarrow \bullet\ R \sim \frac{1}{T} \sim x \Rightarrow H = \frac{\dot{R}}{R} = \frac{\dot{x}}{x}$

$\bullet\ H \sim \frac{1}{t} \sim T^2 \sim \frac{m^2}{x^2}$

$\Rightarrow \frac{dY}{dt} = \frac{dY}{dx}\cdot x H(x) = \frac{dY}{dx}\frac{H(m)}{x}$ deduced from measured parameters (probably H_0)

$$\Rightarrow \boxed{\frac{dY}{dx} = -\langle\sigma_{ann} v_{rel}\rangle \frac{x s}{H(m)}\left(Y^2 - Y_{eq}^2\right)}$$

Numerical solution



Survival of the weakest

Freeze-out happens at $x = x_F \simeq 20$

↳ Yield at late times (neglect Y_{eq} compared to Y):

$$\frac{dY}{dx} = - \langle \sigma_{ann} v_{rel} \rangle \frac{x\, s}{H(m)} Y^2$$

$$\int_{Y(x_F)}^{Y(\infty)} \frac{1}{Y^2}\, dY = - \underbrace{\langle \sigma_{ann} v_{rel} \rangle}_{\text{assumed } v\text{- and } x\text{-independent}} \frac{1}{H(m)} \int dx \;\; x \cdot s$$

$$-\frac{1}{Y(\infty)} + \underbrace{\frac{1}{Y(x_F)}}_{\text{neglect}} = - \langle \sigma_{ann} v_{rel} \rangle \cdot \text{const}$$

Note: $s \sim \frac{1}{R^3} \sim T^3 \sim \frac{m^3}{x^3}$

↳ $\int dx\, x s \sim \int dx \frac{m^3}{x^2}$

$\sim \frac{m^3}{x_F}$

$\frac{1}{H(m)} \sim \frac{1}{m^2} \Rightarrow \text{const} \sim m$

$$\boxed{Y(\infty) = \frac{\text{const}}{\langle \sigma_{ann} v_{rel} \rangle}}$$

$$\Omega h^2 = \frac{\rho}{\rho_c} h^2 = \frac{m_\chi Y(\infty)\, s(t_0)}{\rho_c(t_0)} h^2 = \frac{\text{const}}{\langle \sigma_{ann} v_{rel} \rangle}$$

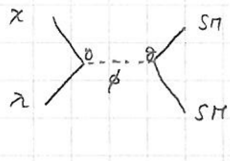
h: H_0 in units of 100 km s^{-1} Mpc^{-1}

Numerically, for $m_\chi \gtrsim 10$ GeV:

$$\boxed{\Omega h^2 \simeq \frac{2.5 \times 10^{-27} \frac{cm^3}{sec}}{\langle \sigma_{ann} v_{rel} \rangle}}$$

Planck CMB data: $\Omega h^2 = 0.1196 \pm 0.0031$

Assume annihilations mediated by particle ϕ, $m_\phi \sim m_\chi \sim 100$ GeV
coupling $\frac{g^2}{4\pi} \sim 0.01$



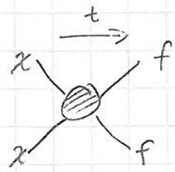
$\Rightarrow \langle \sigma_{ann} v_{rel} \rangle \sim$ few $\times 10^{-26} \frac{cm^3}{sec}$

$\boxed{\text{WIMP miracle}}$

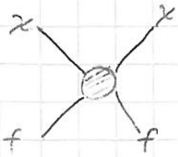
Strong motivation for ewk scale DM!

5.3 Direct DM detection

Freeze-out requires



Turn diagram around: DM - SM scattering



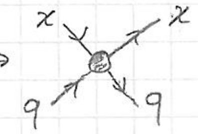
$\Rightarrow$ Galactic WIMPs detectable by scattering on SM particles (in particular nuclei for kinematic reasons)

Toy model: $\mathcal{L} \supset \sum_q \frac{m_q}{\Lambda^3} (\bar{\chi}\chi)(\bar{q}q)$

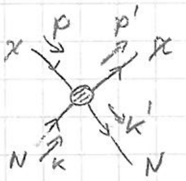
DM fermion

SM quark

cutoff scale

$\hookrightarrow$ 

$$= i\frac{m_q}{\Lambda^3}(\bar{u}_\chi u_\chi)(\bar{u}_q u_q)$$



$$= i\sum_q \frac{m_q}{\Lambda^3} \langle N|\bar{q}q|N\rangle [\bar{u}_\chi(p')u_\chi(p)][\bar{u}_N(k')u_N(k)] \equiv i\mathcal{M}$$

factorize spinor part (possible in non-rel. limit)

F.S. spin sum, I.S. spin avg.:

$$\frac{1}{4}\overline{|\mathcal{M}|^2} = \frac{1}{4\Lambda^6}\left[\sum_q m_q \langle N|\bar{q}q|N\rangle\right]^2 \operatorname{tr}\left[(\not{p}'+m_\chi)(\not{p}+m_\chi)\right]$$

$$\cdot \underbrace{\operatorname{tr}\left[(\not{k}'+m_N)(\not{k}+m_N)\right]}_{\substack{= \operatorname{tr}(\not{k}_1\not{k}_2) + 4m_N^2 \\ = 4(k_1 k_2 + m_N^2) \\ \simeq 8m_N^2 \text{ for non-relativistic process}}}$$

$$= \frac{1}{\Lambda^6}\left[\sum_q m_q \langle N|\bar{q}q|N\rangle\right]^2 \cdot 16\, m_N^2 m_\chi^2$$

$$\sigma_{\chi N} = \frac{\Gamma}{N_N j} \overset{\text{n.r.}}{=} \frac{\Gamma}{n_N \cdot V \cdot n_\chi \cdot v_\chi}$$

number of target nucleons

flux of incoming DM particles

see p. 24, (□)

note: ϕ_χ is normalized to $\int \frac{d^3p}{(2\pi)^3} \phi_\chi = n_\chi$

$$= \int \frac{d^3p'}{(2\pi)^3 \cdot 2E_{p'}} \frac{d^3k'}{(2\pi)^3 \cdot 2E_{k'}} (2\pi)^4 \delta^{(4)}(p+k-p'-k') \overline{|\mathcal{M}|^2}$$

$$\cdot \frac{1}{2E_p} \cdot \frac{1}{2E_k} \cdot \frac{1}{v_\chi}$$

$$\cong \frac{1}{4 m_\chi m_N \, v_\chi} \cdot \frac{1}{(2\pi)^2} \int \frac{d^3k'}{4E_{k'}E_{p'}} \, \delta(E_p + E_k - E_{p'} - E_{k'}) \, \overline{|\mathcal{M}|^2}$$

Note: $d^3k' = d\Omega \, k'^2 dk'$

Transform to c.m. system; use $\mathcal{M}$ independent of k', p':

$$\sigma_{CMS,\chi N} = \frac{4\pi}{16\pi^2 m_\chi m_N \underbrace{v_{\chi N}}_{\text{rel. velocity in c.m.s}}} \int \frac{dk' k'^2}{4E_{k'}E_{p'}} \, \delta\left(E_p + E_k - \sqrt{k'^2 + m_\chi^2} - \sqrt{k'^2 + m_N^2}\right) \overline{|\mathcal{M}|^2}$$

$$= \frac{1}{4\pi \, m_\chi m_N v_{\chi N}} \int \frac{dk' k'^2}{4E_{k'}E_{p'}} \left(\frac{k'}{E_{k'}} + \frac{k'}{E_{p'}}\right)^{-1} \delta(k' - \ldots) \overline{|\mathcal{M}|^2}$$

$\left[\delta(f(x)) = \frac{1}{|f'(x)|}\delta(x)\right]$

$$= \frac{1}{4\pi \, m_\chi m_N v_{\chi N}} \frac{k'}{4(E_{k'} + E_{p'})} \overline{|\mathcal{M}|^2}$$

$\left[k' \overset{c.m.s.}{=} \frac{m_N m_\chi v_{\chi N}}{m_N + m_\chi}\right]$

$$= \frac{1}{16\pi (m_N + m_\chi)^2} \overline{|\mathcal{M}|^2}$$

$$= \frac{m_\chi^2 m_N^2}{\pi \Lambda^6 (m_N + m_\chi)^2} \left[\sum_q m_q \langle N | \bar{q} q | N \rangle\right]^2$$

$$\left[\begin{aligned} k' &= m_\chi v_\chi = m_N v_N \\ &= m_\chi (v_{\chi N} - v_N) \\ &= m_\chi v_{\chi N} - \frac{m_\chi}{m_N} k' \\ \Rightarrow k' &= m_\chi v_{\chi N} \left(1 + \frac{m_\chi}{m_N}\right)^{-1} \\ &= \frac{m_\chi m_N v_{\chi N}}{m_\chi + m_N} \end{aligned}\right.$$

Or, use e.g. Peskin/Schröder eq. 4.84:

$$\left.\frac{d\sigma_{\chi N}}{d\Omega}\right|_{CMS} = \frac{1}{4 m_\chi m_N \underbrace{v_{\chi N}}_{\text{rel. velocity in CMS}}} \frac{k'}{(2\pi)^2 \cdot 4E_{cm}} \overline{|\mathcal{M}|^2}$$

$\left[k' \overset{c.m.s}{=} \frac{m_N m_\chi v_{\chi N}}{m_N + m_\chi}\right]$ $\sigma_{\chi N} = \frac{1}{16\pi (m_N + m_\chi)^2} \overline{|\mathcal{M}|^2}$

$$\boxed{\underbrace{\sigma_{\chi N}}_{= 4\pi \frac{d\sigma}{d\Omega}} = \frac{m_\chi^2 m_N^2}{\pi \Lambda^6 (m_N + m_\chi)^2} \left(\sum_q m_q \langle N | \bar{q} q | N \rangle\right)^2}$$

Also useful: $\frac{d\sigma}{dE_r}$

nuclear recoil energy

$E_r = E_{k'} - m_N = \frac{1}{2} m_N v_N'^2$

To calculate, note
$$E_r = \frac{1}{2} m_N v_N'^2$$
$$= \frac{1}{2} m_N \left(\frac{m_\chi \vec{v}_\chi}{m_\chi + m_N} + \vec{v}'_{N,\text{cms}} \right)^2$$
$$= \frac{1}{2} m_N \left[\frac{m_\chi^2 v_\chi^2}{m_\chi + m_N} + v'^2_{N,\text{cms}} + \frac{2 m_\chi v_\chi v'_{N,\text{cms}} \cos\theta}{m_\chi + m_N} \right]$$

$$dE_r = \underbrace{\frac{m_\chi m_N}{m_\chi + m_N}}_{\equiv \mu_N} v_\chi v'_{N,\text{cms}} \, d(\cos\theta)$$
$$= \mu_N v_\chi \frac{m_\chi v_\chi}{m_\chi + m_N} d(\cos\theta)$$

σ indep. of Ω

$$\Rightarrow \frac{d\sigma}{dE_r} = \int d\phi \underbrace{\frac{d\sigma}{d\Omega}}_{d\Omega = d\phi\, d(\cos\theta)} \frac{d(\cos\theta)}{dE_r} = 2\pi \frac{d\sigma}{d\Omega} \frac{m_N}{\mu_N^2 v_\chi^2} = \frac{\sigma}{2} \frac{m_N}{\mu_N^2 v_\chi^2}$$

- Count rate per time, energy, target mass:

DM number density

Target number density

for hypothetical hydrogen target
$$\boxed{\frac{dR_{\chi N}}{dE_r} = \int_{v_{min}(E_r)}^{\infty} dv_\chi \, v_\chi^2 \, d\Omega_{v_\chi} \, f_\oplus(\vec{v}_\chi) \, n_\chi v_\chi \frac{n_N}{\rho_N} \frac{d\sigma_{\chi N}}{dE_r}}$$

integral over DM velocity distribution

DM velocity distribution in Earth frame

DM density $\simeq 0.3$ GeV/cm^3
$$\boxed{\frac{dR_{\chi N}}{dE_r} = \frac{\rho_\chi}{m_\chi} \frac{\sigma_{\chi N}}{2\mu_N^2} \int_{v_{min}(E_r)}^{\infty} dv_\chi \, d\Omega_{v_\chi} \, v_\chi f_\oplus(\vec{v}_\chi)}$$

- Here: v_{min} = minimal velocity to achieve E_r. Derivation: head-on collision

$\chi \longrightarrow \overset{N}{\cdot} \dashrightarrow$

(i) $m_\chi v_\chi = m_\chi v_\chi' + m_N v_N'$
$$= m_\chi v_\chi' + m_N \sqrt{\frac{2E_r}{m_N}}$$

(ii) $\frac{1}{2} m_\chi v_\chi^2 = \frac{1}{2} m_\chi v_\chi'^2 + E_r$

$$\text{(ii)} \Rightarrow \frac{1}{2} m_\chi v_\chi^2 = \frac{1}{2} m_\chi \frac{1}{m_\chi^2} \left(m_\chi v_\chi - m_N \sqrt{\frac{2E_r}{m_N}} \right)^2 + E_r$$

$$\Leftrightarrow \quad 0 = \frac{1}{\cancel{2}m_\chi} \cdot \cancel{2}m_N E_r - \frac{\cancel{2}}{\cancel{2}m_\chi} m_\chi v_\chi \sqrt{2m_N E_r} + E_r$$

$$\Leftrightarrow \quad v_\chi = \left(1+\frac{m_N}{m_\chi}\right)\sqrt{E_r} \cdot \frac{1}{\sqrt{2m_N}}$$

$$\boxed{v_{min}(E_r) = \sqrt{\frac{m_N E_r}{2\mu_N^2}}}$$

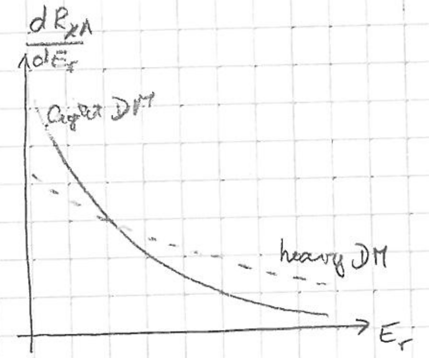
- Count rate for nuclear (non-hydrogen) target

nuclear form factor accounts for loss of coherence at large momentum transfer

$$\boxed{\frac{dR_{\chi A}}{dE_r} = \frac{\rho_\chi}{m_\chi}\left(\sigma_{\chi N} \cdot \frac{\mu_A^2}{\mu_N^2}\right) \cdot \frac{1}{2\mu_A^2} \cdot A^2 \; F^2\left(\sqrt{2m_N E_r}\right) \int_{v_{min}(E_r)}^{\infty} dv_\chi \, d\Omega_{v_\chi} \, v_\chi \, f_\oplus(\vec{v}_\chi)}$$

modify kinematic factors

matrix element larger by factor A (coherent scattering on all nucleons)



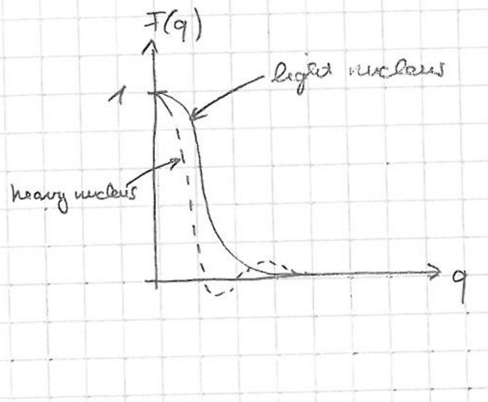
- The nuclear form factor:

Convenient parameterization:

$$F(q) = 3\, e^{-q^2 s^2/2} \; \frac{\sin qr - qr\cos qr}{(qr)^3}$$

$$s \equiv 1\,fm$$

$$r = \sqrt{(1.2 \cdot A^{1/3})^2 - 5s^2}$$



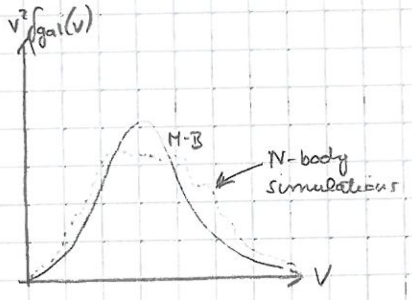
- The DM velocity distribution

Assumptions:
- isotropic (in the galactic frame)
- radially symmetric

Common assumption: Maxwell-Boltzmann distribution w/ cutoff

$$f_{gal}(\vec{v}) \sim \begin{cases} \exp\left[-|\vec{v}|^2/\bar{v}^2\right] - \exp\left[v_{esc}^2/\bar{v}^2\right] & \text{for } v < v_{esc} \\ 0 & \text{for } v \geq v_{esc} \end{cases}$$

$\bar{v}$ = velocity dispersion $\sim$ 220 km/s
v_{esc} = escape velocity $\sim$ 600 km/s



e.g. 1210.2328

In the Earth's rest frame:

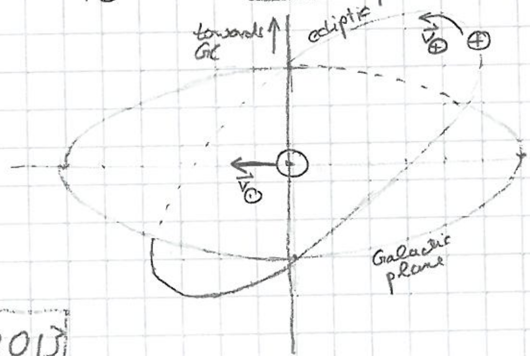
$$f_{\oplus}(\vec{v}) = f_{gal}(\vec{v} + \vec{v}_{\odot} + \vec{v}_{\oplus}(t))$$

$\vec{v}$: DM velocity in Earth frame

$\vec{v}_{\odot}$: motion of Sun rel. to GC ($\sim$ 220 km/s)

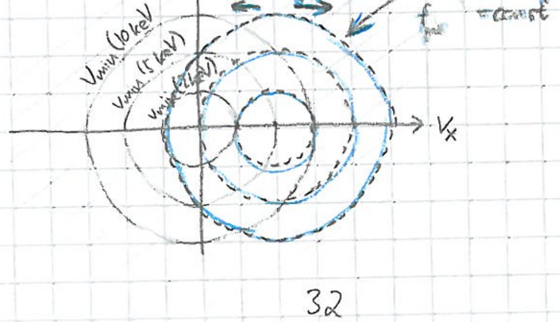
$\vec{v}_{\oplus}(t)$: velocity of Earth around Sun; $|\vec{v}_{\oplus}| \sim 30$ km/s.

$f_{\oplus}(\vec{v})$ is non-isotropic and varies throughout the year

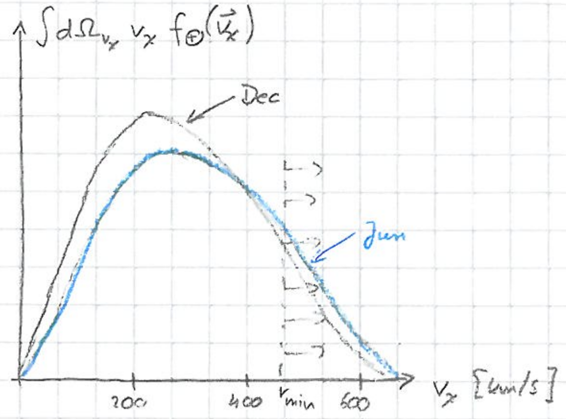


see e.g. Thomas' ISAPP 2011 lecture

end 06.06.2013



Remember: $\frac{dR_{\chi A}}{dE_r} \sim \int_{v_{min}}^{\infty} dv_\chi \, d\Omega_{v_\chi} \, v_\chi f_\oplus(\vec{v}_\chi)$



$\Rightarrow$ Expect annual modulation of count rate
Extremal around Jun 2 & Dec 2

- Direct detection experiments

a) Xenon (@ MPIK!) 1207.5988

[Show Xe-100 drawing and explain]

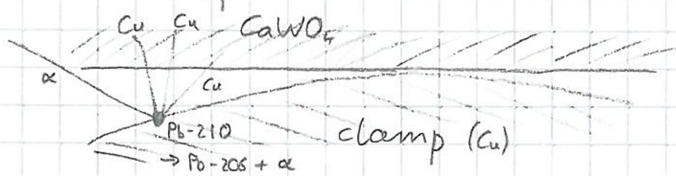
[Show Xe-100 S1-S2 plot, Xe-100 exclusion curve and explain]

- Two-phase xenon detector
- S1 signal: scintillation
  S2 signal: ionization
- S1/S2 different for nuclear recoils (= signal) and electron recoils (= background)

b) CRESST 1109.0702

[Show CRESST cryostat, CRESST module drawings, scintillation-vs-heat plot, and result (from Xe100 plot)]

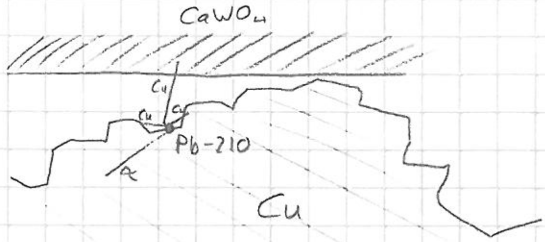
- $CaWO_4$ crystals
- Scintillation + phonon signals
- Superconducting phase transition thermometers $\rightarrow$ Cryogenic
  (use strong $R(T)$ gradient at T_c)
- Excess events seen
- Possible explanation (1203.1576)



radioactive impurity (unavoidable)
$^{210}Pb \rightarrow {}^{206}Pb + \alpha$

"sputtering" of secondary Cu atoms from clamp material

More realistic model: Surface roughness (not taken into account by CRESST)



Fewer sputtered Cu atoms reach detector
↳ lower event energy
→ larger BG for DM search

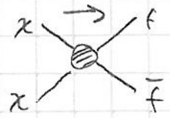
c) DAMA

[Show DAMA drawing, annual modulation plots, allowed regions (use Xe100 plot)]

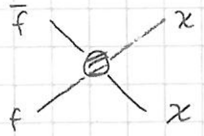
- very clean NaI(Tl) scintillator detectors
- no signal / BG discrimination
- use annual modulation as signature

5.4 DM production at colliders

Freeze-out requires



Flip diagram:



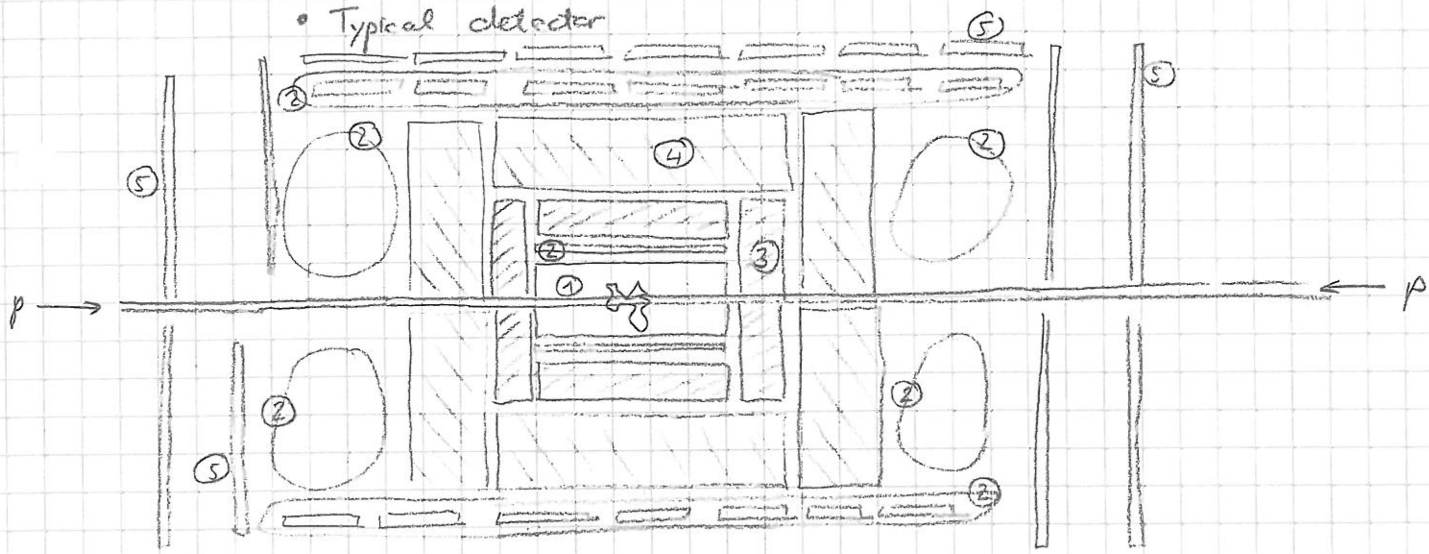
↳ DM production in particle collisions

5.4.1 Particle colliders and detectors

- Lepton colliders: $e^+ \rightarrow \leftarrow e^-$, e.g. LEP, KEKB
  - "Clean" collisions (no BG from spectator quarks, i.e. QCD, etc.)
  - Energy limited by synchrotron losses (in ring accelerators) ($P \sim E^4/m_e^4 r^2$, Jackson 14.31)
  - Beams polarisable

- Hadron colliders: $p \rightarrow \leftarrow p$ or $\bar{p}$, e.g. Tevatron, LHC
  - "Dirty" environment (lots of ISR, pile-up)
  - Initial parton momenta not known
  - Higher energies possible

- Typical detector



1. Inner detector — tracking
2. Magnets
3. El.-mag. calorimeter (small, typically high-Z material)
4. Hadronic calorimeter
5. Muon chambers

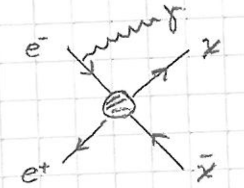
- Can only detect particles with em or strong interactions
- Can distinguish e, μ, γ, jet (to some degree: b-quarks, τ leptons)

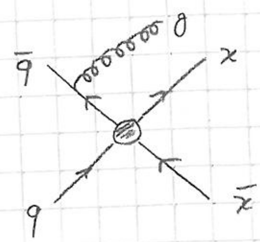
5.4.2 Missing energy signatures of DM

<u>Problem</u>: No visible f.s. particles in $f\bar{f} \to \chi\chi$

<u>One solution</u>: ISR

Only representative diagrams (one of many) shown!

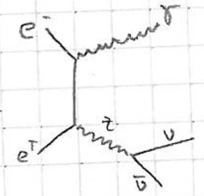




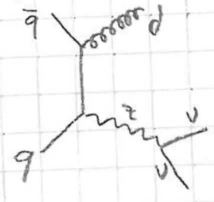
Signature: γ + Missing energy

jet + Missing transverse energy
(visible transverse $\vec{p}_T$ not balanced
$p_{\parallel}$ not known in hadron collider)

SM BG:

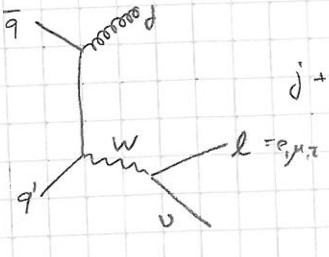


$\gamma + (Z \to \nu\bar{\nu})$

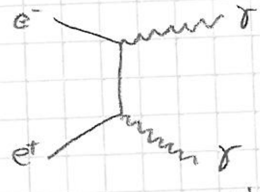


$j + (Z \to \nu\bar{\nu})$

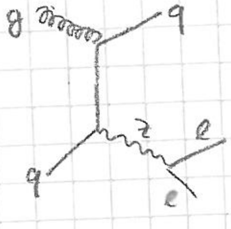
$e^+e^-\gamma$ (leptons missed)



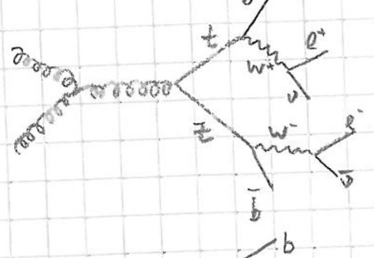
$j + (W \to \ell\nu)$
(ℓ missed)



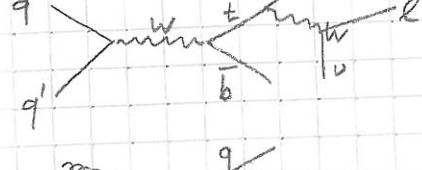
$\gamma\gamma$ (one γ missed)



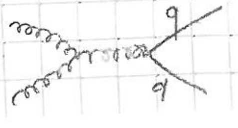
$j + (Z \to \ell\ell)$
(both leptons missed)



$t\bar{t}$ (leptons and extra jets missed)

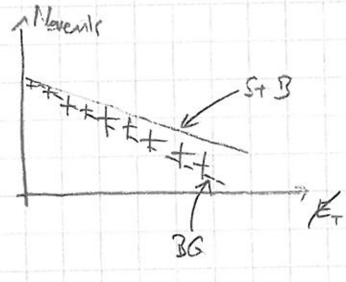


single top



QCD multijet
(jets mis-reconstructed)

Strategy: • Use generic DM model, e.g. $\mathcal{L} \supset \bar{\chi}\gamma_\mu \chi \; \bar{q}\gamma^\mu q$
• Simulate signal events
• Simulate all SM BGs (or determine from data)
• Develop cuts to enhance $S/\sqrt{B}$
• Determine systematic uncertainties in data and prediction
• Compare data with S+B prediction



E.g. CL_s upper limit on signal x-sec σ: 1204.3851
- Define test statistic log-likelihood ratio

$$LLR(\{N_i^{obs}\}|\sigma) = -2 \log \frac{\mathcal{L}(\{N_i^{obs}\}|\sigma)}{\mathcal{L}(\{N_i^{obs}\}|\sigma^{\text{best fit}})}$$

predicted # of events in i-th bin

observed # of events in i-th bin

$$\mathcal{L} = \text{Likelihood} = \prod_{i=\text{bins}} \underbrace{\frac{1}{N_i^{obs}!} e^{-N_i^{th}(\sigma)} \cdot \left[N_i^{th}(\sigma)\right]^{N_i^{obs}}}_{\text{Poisson distribution}}$$

probability of data $\{N_i^{obs}\}$ given theory parameter σ

- Determine PDF of LLR for each σ: $f(LLR|\sigma^{true}, \sigma^{test})$
e.g. by MC simulation; analytic approximations exist, see e.g. 1204.3851

- Define $CL_{S+B}(\{N_i^{obs}\}|\sigma^{test}) \equiv \int_{LLR(\{N_i^{obs}\}|\sigma^{test})}^{\infty} f(LLR'|\sigma^{test}, \sigma^{test})\, dLLR'$

If $\sigma^{true} = \sigma^{test}$ this is the probability that an experiment yields an LLR at least as "extreme" as the observed one (which is evaluated for x-sec σ^{test}) ...

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$$1 - CL_B(\{N_i^{obs}\}|\sigma) \equiv \int_{LLR(\{N_i^{obs}\}|\sigma^{test})}^{\infty} f(LLR'|0, \sigma^{test})\, dLLR'$$

If $\sigma^{true} = 0$, this is the probability that an experiment yields an LLR at least as "extreme" as the observed one (which is evaluated for x-sec σ^{test})

$$CL_s(\{N_i^{obs}\}|\sigma) \equiv \frac{CL_{S+B}(\{N_i^{obs}\}|\sigma)}{1 - CL_B(\{N_i^{obs}\}|\sigma)}$$

Requirement $CL_s(\{N_i^{obs}\}|\sigma) > \alpha$ defines confidence interval at CL $1-\alpha$ (conservative CL!)

Compared to simply using CL_{S+B} (or simply $\mathcal{L}(\{N_i^{obs}\}|\sigma)$) = Robust w.r.t. unknown systematics.

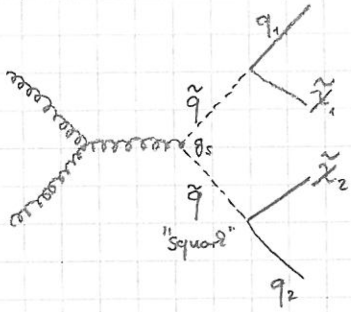
CL_s is large (σ allowed) if
- data well-described by signal <u>and</u> poorly by BG

[Show results from ATLAS or CMS monojet search]

5.4.3 Cascade decays

Idea: New particles with large production x-sec can decay to DM

Example: MSSM



- Large x-sec $(\sim \alpha_s^2)$
- Signature e.g. jets + MET

Mass reconstruction: The m_{T2} method hep-ph/0304226

Goal: Determine $m_{\tilde{q}}$ and $m_{\tilde{\chi}}$

Problem: Incomplete kinematical information

$$m_{\tilde{q}}^2 = (p_{q1} + p_{\chi 1})^2 = \underbrace{m_{q1}^2}_{\simeq 0} + m_\chi^2 + 2(E_{q1}E_{\chi 1} - \vec{p}_{q1}\vec{p}_{\chi 1})$$

$$= (p_{q2} + p_{\chi 2})^2$$

We measure: $E_{q1}, \vec{p}_{q1}, E_{q2}, \vec{p}_{q2}, \underbrace{\vec{p}_{\chi 1,T}}_{=\begin{pmatrix} p_{\chi 1,x} \\ p_{\chi 1,y}\end{pmatrix}} + \vec{p}_{\chi 2,T} = p_{T,\text{miss}}$

We can't measure: $E_{\chi 1}, E_{\chi 2}, \vec{p}_{\chi 1,T} - \vec{p}_{\chi 2,T}, p_{\chi 1,z}, p_{\chi 2,z}$

Transverse mass: $$m_T^2(p_q, p_\chi) \equiv \left[\begin{pmatrix} \sqrt{m_q^2 + \vec{p}_{q,T}^{\,2}} \\ \vec{p}_{q,T} \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{m_\chi^2 + \vec{p}_{\chi,T}^{\,2}} \\ \vec{p}_{\chi,T} \\ 0 \end{pmatrix}\right]^2$$

$$= m_q^2 + m_\chi^2 + 2\left(\underbrace{E_{q,T}}_{\equiv\sqrt{m_q^2 + \vec{p}_{qT}^{\,2}}} E_{\chi,T} - \vec{p}_{qT}\vec{p}_{\chi T}\right)$$

For $|\vec{p}_q| = |\vec{p}_{qT}|, |\vec{p}_\chi| = |\vec{p}_{\chi,T}|$: $m_T(p_q, p_\chi) = m_{\tilde{q}}$

Otherwise : $m_T(p_q, p_\chi) \leq m_{\tilde{q}}$

[Exercise : Proof !]

Proof: Suppose $\quad E_q E_\chi \;<\; E_{qT} E_{\chi T} + p_{qz} p_{\chi z}$

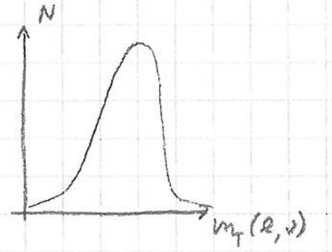
$$\Rightarrow (m_q^2 + p_q^2)(m_\chi^2 + p_\chi^2) \;<\; (m_q^2 + p_{qT}^2)(m_\chi^2 + p_{\chi T}^2) + p_{qz}^2 p_{\chi z}^2 + 2\sqrt{m_q^2 + p_{qT}^2}\sqrt{m_\chi^2 + p_{\chi T}^2}\; p_{qz} p_{\chi z}$$

$$\Leftrightarrow p_{qz}^2 (m_\chi^2 + p_{\chi T}^2) + p_{\chi z}^2 (m_q^2 + p_{qT}^2) \;<\; 2\sqrt{m_q^2 + p_{qT}^2}\sqrt{m_\chi^2 + p_{\chi T}^2}\; p_{qz} p_{\chi z}$$

$$\Leftrightarrow \left(p_{qz}\sqrt{m_\chi^2 + p_{\chi T}^2} - p_{\chi z}\sqrt{m_q^2 + p_{qT}^2} \right)^2 < 0 \qquad \lightning$$

$\square$

Useful e.g. for measuring W mass in
Plot m_T distribution, W mass is upper endpoint.



Here: Don't know $\vec{p}_{\chi 1,T}$, $\vec{p}_{\chi 2,T}$ separately

Solution:

$$m_{T2}^2 \equiv \min_{\vec{p}_{\chi 1,T},\, \vec{p}_{\chi 2,T} \,|\, \vec{p}_{\chi 1T} + \vec{p}_{\chi 2,T} = \vec{p}_{T,\text{miss}}^{\,\text{obs}}} \max\left[m_T^2(p_{q1}, p_{\chi 1}) \,;\, m_T^2(p_{q2}, p_{\chi 2}) \right]$$

- $m_{T2} \le m_{\tilde{q}}$ because $m_T \le m_{\tilde{q}}$
- $m_{T2} = m_{\tilde{q}}$ for specific configurations [Homework: find one!]

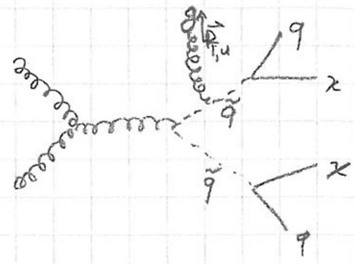
  [e.g. $\tilde{q}\tilde{q}$ production at rest, both decays in the transverse plane

- Need m_χ to compute m_{T2}.

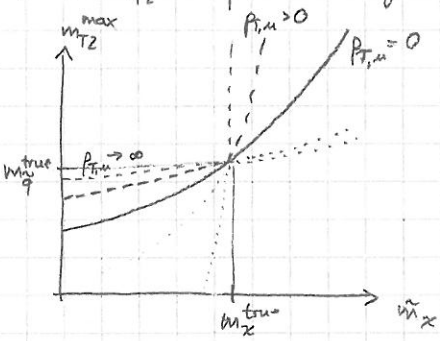
Determining m_χ:

0711.4008
0810.5576

- Define $m_{T2}(\tilde{m}_\chi)$ = value of m_{T2} replacing true m_χ by test value $\tilde{m}_\chi$
- Consider events with $\underbrace{\vec{p}_{\tilde{q}1T} + \vec{p}_{\tilde{q}2,T}}_{= -\vec{p}_{T,u} \text{ ("upstream momentum")}} \neq 0$



- Plot m_{T2} endpoint as function of $\tilde{m}_\chi$



Kink gives m_χ^{true} and $m_{\tilde{q}}^{true}$

Explanation: Two extremal configurations

- $\vec{p}_{\tilde{q}1} \upuparrows \vec{p}_{\tilde{q}2} \upuparrows \vec{p}_{T,u}$
- $\vec{p}_{\tilde{q}1} \upuparrows \vec{p}_{\tilde{q}2} \uparrow\downarrow \vec{p}_{T,u}$

$\Big\}$ and all $\vec{p}$ in transverse plane

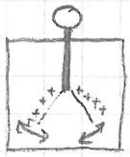
Both extremal configurations have to give same m_{T2} at $m_\chi = m_\chi^{true}$, but different functional dependence on $\tilde{m}_\chi$.

6. High energy cosmic rays

6.1 Experimental evidence

around 1800 : anomalous blackening of photographic plates (attributed to environmental radioactivity)

~1910 : Theodor Wulf, Domenico Pacini, Albert Gockel, ...
Electrometer experiments



measure discharge rate in various location (land, sea, underwater, in caves, on the Eiffel tower, ...)

→ radiation seemed to <u>increase</u> with altitude
→ extraterrestrial origin

Results not widely accepted

1912 : Victor Hess : balloon fleights with electrometers
→ evidence for cosmic rays

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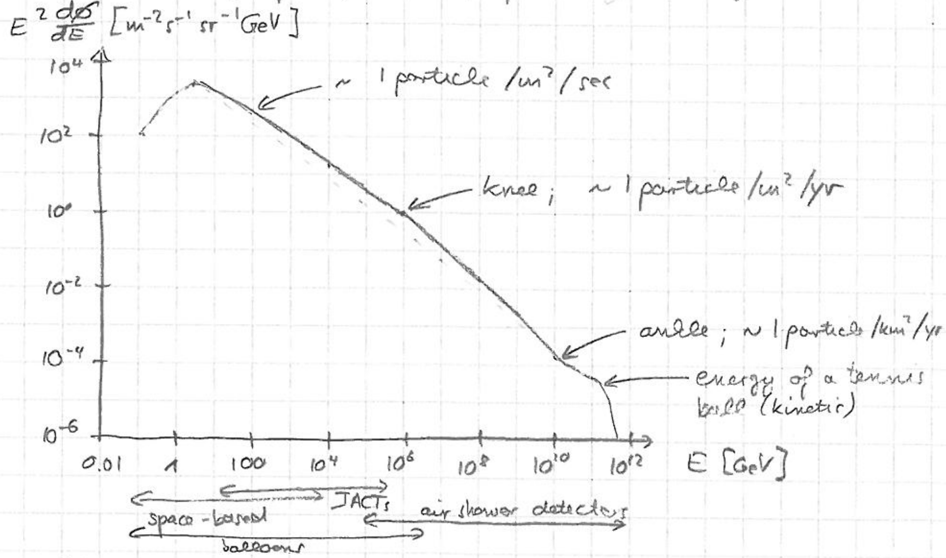
today : [show CR spectrum plot]

CR spectrum and composition well measured, but lots of open questions, especially at high E



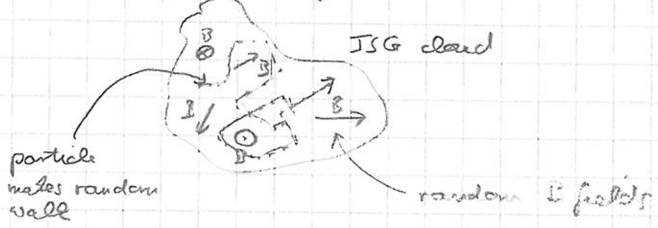
Many tools to study :
- space-based experiments (PAMELA, AMS-02, Fermi) : E ≲
- Imaging Air Čerenkov telescopes (HESS, MAGIC, VERITAS) (50 GeV – 50 TeV)
- Air shower detectors (Auger etc.)

[show pictures for each type of exp.]

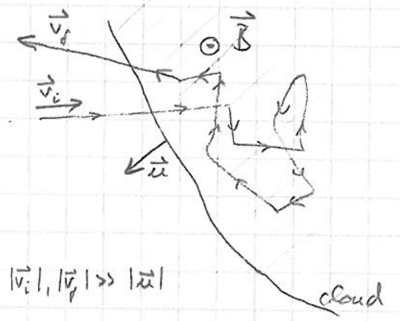
6.2 Cosmic ray acceleration: The Fermi mechanism (Fermi 1949)

Question: How can particles reach 10^{19} eV?

Consider dilute (~ 1 particle $/ cm^3$) interstellar gas with magnetic fields
$\rightarrow$ interactions only with em fields, no particle collisions



Moving magnetic cloud:



$|\vec{v}_i|, |\vec{v}_f| \gg |\vec{u}|$

- non-relativistic toy model

Observer frame S: $\vec{v}_i$, $\vec{v}_f$

Cloud frame S': $\vec{v}_i{}' = \vec{v}_i - \vec{u}$; $\vec{v}_f{}' = \vec{v}_f - \vec{u}$

$$\boxed{|\vec{v}_i{}'| = |\vec{v}_f{}'|}$$

$$\Rightarrow \Delta E = \frac{1}{2} m \left(v_f^2 - v_i^2 \right)$$

$$= \frac{1}{2} m \left(\vec{v}_f{}'^2 + 2 \vec{v}_f \vec{u} + \vec{u}^2 - \vec{v}_i{}'^2 - 2 \vec{v}_i \vec{u} - \vec{u}^2 \right)$$

$$= m \left(\vec{v}_f{}' - \vec{v}_i{}' \right) \vec{u}$$

For head-on collision: $\vec{v}_i \uparrow\downarrow \vec{u}$; $\vec{v}_i \uparrow\downarrow \vec{v}_f$; $\vec{v}_i{}' = -\vec{v}_f{}'$
w/ reflection

$$\Rightarrow \boxed{\Delta E = 2m \left(\vec{u}^2 - \underbrace{\vec{v}_i \vec{u}}_{<0} \right) > 0}$$

$\vec{v}_f = \vec{v}_f{}' + \vec{u}$
$= -\vec{v}_i{}' + \vec{u}$
$= -\vec{v}_i + 2\vec{u}$

For rear-end collision: $\vec{v}_i \uparrow\uparrow \vec{u}$, $\vec{v}_i \uparrow\downarrow \vec{v}_f$, $\vec{v}_i{}' = -\vec{v}_f{}'$
w/ reflection

$$\boxed{\Delta E = 2m \left(\vec{u}^2 - \underbrace{\vec{v}_i \vec{u}}_{>0} \right) < 0}$$

Energy gain in head-on collision is larger than energy loss in rear-end collision

↳ On average: Energy gain

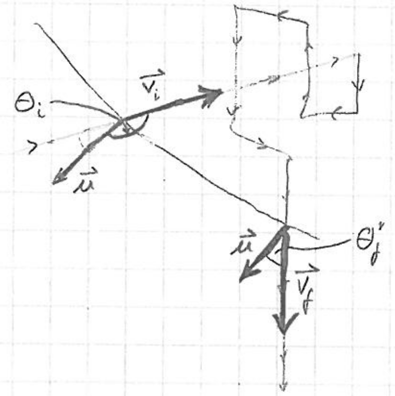
High E after many collisions

- Relativistic model

$$E_i' = E_i \gamma - \mu \gamma \underbrace{p_i}_{\simeq E_i \text{ for rel. particles}} \cos\Theta_i$$

$$\simeq E_i \gamma (1 - \mu \cos\Theta_i)$$

$$\gamma = \frac{1}{\sqrt{1-\mu^2}}$$



$$E_f \simeq E_f' \gamma (1 + \mu \cos\Theta_f')$$

Scattering in the cloud is elastic $\Rightarrow \boxed{E_i' = E_f'}$

$$\hookrightarrow E_f = \gamma^2 E_i (1 - \mu \cos\Theta_i)(1 + \mu \cos\Theta_f')$$

$$\left\langle \frac{\Delta E}{E} \right\rangle = \left\langle \gamma^2 \left[\underbrace{1 - \frac{1}{\gamma^2}}_{=\mu^2} + \mu (\cos\Theta_f' - \cos\Theta_i) - \mu^2 \cos\Theta_i \cos\Theta_f' \right] \right\rangle$$

Need average angles:

can reach cloud from any direction under any angle

- By assumption: $\langle \cos\Theta_f' \rangle = 0$ (isotropization in cloud, particles can leave cloud in any direction under any angle)
- Number of particles reaching the cloud per time δt: (assume Θ_i isotropically distributed)

$$dN \propto (1 - \mu \cos\Theta_i)\, \delta t$$

$$\Rightarrow \langle \cos\Theta_i \rangle = \frac{\int_{-1}^{1} d(\cos\Theta_i) \cos\Theta_i (1 - \mu\cos\Theta_i)}{\int_{-1}^{1} d\cos\Theta_i \, (1 - \mu\cos\Theta_i)}$$

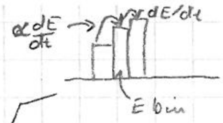
$$= -\frac{\frac{2}{3}\mu}{2} = -\frac{\mu}{3}$$

$$\Rightarrow \left\langle \frac{\Delta E}{E} \right\rangle = \frac{1}{1-\mu^2}\left[\mu^2 + \frac{\mu^2}{3} \right]$$

$$= \frac{4}{3} \frac{\mu^2}{1-\mu^2}$$

$$\boxed{\left\langle \frac{\Delta E}{E} \right\rangle \simeq \frac{4}{3}\mu^2 > 0}$$

- Energy gain on average
- 2^{nd} order in $\mu \longrightarrow$ 2^{nd} order Fermi acceleration
- Typically $\mu \sim 10 \frac{km}{s} \Rightarrow \frac{\Delta E}{E} \sim 10^{-8}$
- Distance between clouds $\sim ly$
  $\Rightarrow \tau_{acc} \sim 10^8$ yrs (Duration of acceleration process)
- At low E: energy loss import (Coulomb) $\rightarrow$ need different mechanism at low E



• Final spectrum:

$$\text{Diffusion eq}: \quad 0 = \frac{\partial N(E)}{\partial t} = -\frac{N(E)}{\tau_{esc}} - \frac{\partial}{\partial E}\left(\frac{dE}{dt} N(E)\right)$$

τ_{esc}: escape time from acceleration region

$$\simeq -\frac{N(E)}{\tau_{esc}} - \frac{\partial}{\partial E}\left(\frac{E}{\tau_{acc}} N(E)\right)$$

$$\Rightarrow -\frac{N}{\tau_{esc}} - \frac{N}{\tau_{acc}} = \frac{E}{\tau_{acc}}\frac{dN}{dE}$$

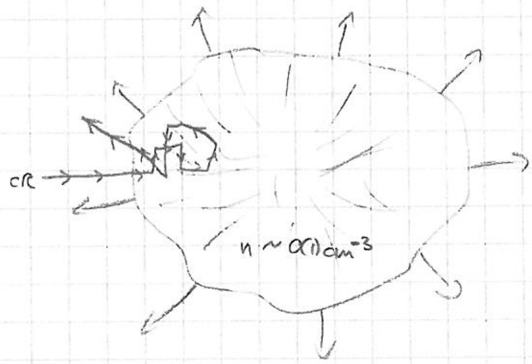
$$\frac{dN}{dE} = -\frac{N}{E}\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right)$$

$$\boxed{N \propto E^{-\alpha} \quad \text{with} \quad \alpha \simeq 1 + \frac{\tau_{acc}}{\tau_{esc}}}$$

Power law index α depends on
- velocity of clouds ($\tau_{acc} \sim 1/u^2$)
- density and size of acceleration region (τ_{esc})

6.3 Diffusive shock acceleration

Consider shock wave rather than random clouds, e.g. from supernova remnant



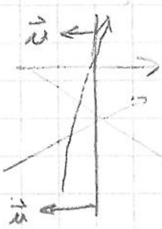
No rear-end collisions, only head-on!

$$\frac{\Delta E}{E} = \underbrace{\gamma^2}_{\simeq 1} \underbrace{u}_{= v_{upstream} - v_{downstream}\ (\neq v_{shock})} \left(\cos\theta_f' - \cos\theta_i\right) + \mathcal{O}(u^2) \qquad \text{(see above)}$$

Particles reaching cloud per time per solid angle

$$dN \propto \cos\theta_i \, d\Omega \, dt$$

$$\Rightarrow \langle \cos\theta_i \rangle \overset{u \simeq 0}{=} \frac{\int_{-1}^{0} d\cos\theta_i \, \cos^2\theta_i}{\int_{-1}^{0} d\cos\theta_i \, \cos\theta_i} = -\frac{2}{3}$$



particles can reach shock from only one direction (relative to $\vec{u}$), but under any angle

By similar arguments:

$$\langle \cos\theta_f' \rangle = +\frac{2}{3}$$

$$\Rightarrow \boxed{\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3} u > 0}$$

- much larger gain than for 2^{nd} order Fermi acceleration
- $\propto u$ → "1st order Fermi acceleration"

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After n acceleration cycles:

$$E_n = (1 + \underbrace{\tfrac{4}{3} u}_{\equiv k})^n E_0$$

To produce particle with energy E, need $\frac{\log E/E_0}{\log(1+k)}$ cycles

Include escape probability P_{esc}

$$\hookrightarrow N(\geq E) = N_0 (1 - P_{esc})^{\frac{\log E/E_0}{\log(1+k)}}$$

$$= N_0 \left(\frac{E}{E_0}\right)^{\frac{\log(1-P_{esc})}{\log(1+k)}}$$

Expand $\log(1-P_{esc}) \simeq -P_{esc}$; $\log(1+k) \simeq k$

$$\hookrightarrow N(\geq E) \simeq N_0 \left(\frac{E}{E_0}\right)^{-P_{esc}/k}$$

$$\boxed{N(E) = \frac{dN(\geq E)}{dE} \propto \left(\frac{E}{E_0}\right)^{-\alpha} \quad \text{with} \quad \alpha = \frac{P_{esc}}{k} + 1}$$

see slides by Régis Terrier

Using hydrodynamic conservation laws, one can show that, for strong shocks ($v_{upstream} \gg v_{sound}$): $\underline{\underline{\alpha = 2}}$.

6.4 The GZK cutoff (Greisen - Zatsepin - Kuzmin)

Consider $\gamma + p \rightarrow p/n + \pi^0/\pi^+$ (large x-sec !)

in collisions of UHECR with CMB photons ($E_\gamma \sim 2 \cdot 10^{-4}\,eV$)

Energetics: Threshold (in cms) : $m_\pi \sim 135$ MeV

$$\hookrightarrow \quad E_{cms} = \sqrt{2\, E_{CR}\, E_\gamma (1 - \cos\Theta) + m_p^2} \overset{!}{=} m_\pi + m_p$$

$$\overset{\Theta = 0}{\Longleftrightarrow} \quad \boxed{E_{GZK} \sim 10^{20}\,eV \simeq 15\,J}$$

CR with $E > E_{GZK}$ loose energy very quickly.

$\hookrightarrow$ Do not expect events above E_{GZK}

6.5 The Hillas plot

Question: Max. energy an accelerator can magnetically contain

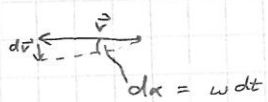
$$E^{max} \simeq \underbrace{Ze}_{\text{CR charge}} \cdot \underbrace{R}_{\text{accelerator size}} \cdot \underbrace{B}_{\text{avg. magnetic field}} \cdot \underbrace{c^{-1}}_{\text{CR velocity}}$$

[based on expression for relativistic gyroradius:

$$\frac{dp_\mu}{d\tau} = Z e F_{\mu\nu} u^\mu \qquad [\text{Jackson 12.1 and 12.2}]$$

$p = \gamma m v$, $\tau = \frac{t}{\gamma}$, $u = \frac{dx^\mu}{d\tau}$

$$\Longrightarrow \frac{d\vec{v}}{dt} = \vec{v} \times \underbrace{\vec{\omega}_B}_{\equiv \frac{Ze\vec{B}}{\gamma m}} = \text{angular frequency}$$



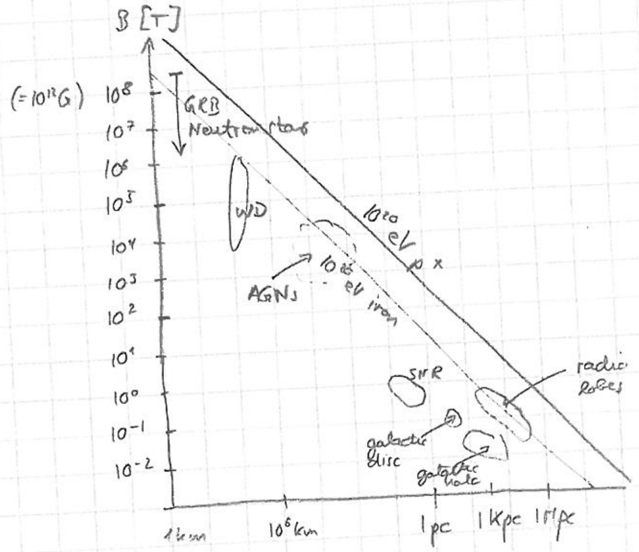
$$\omega_B = \frac{2\pi}{T} = \frac{2\pi |\vec{v}|}{2\pi R}$$

T: revolution time

$$\Rightarrow \quad Ze \cdot B \cdot R = E \cdot v$$

UHE CR can only be produced in accelerators with large B field and large R

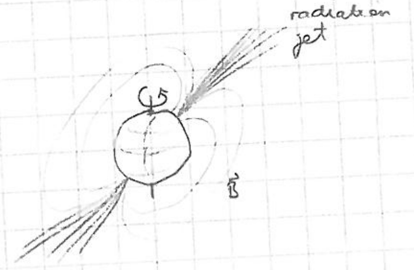
↳ Hillas plot [show!]



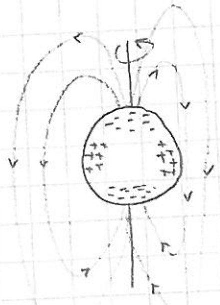
6.6 Pulsars as cosmic e^+e^- accelerators and γ ray sources

Pulsar = fast rotating neutron star with large B-field

f can be ~ kHz



Field configuration:



- charge separation $\Rightarrow$ strong $\vec{E}$-fields
- plasma outside the star cancels E-field components parallel to B-field if dense enough (particles move only along B field lines)
- acceleration in regions of low plasma density $\rightarrow$ requires open field lines

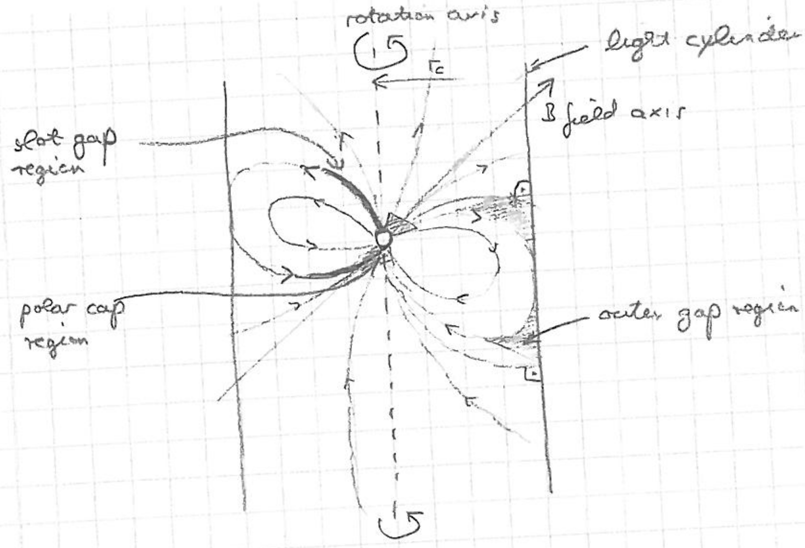
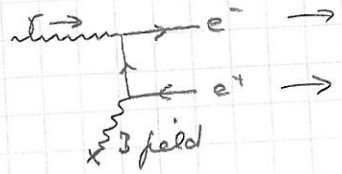
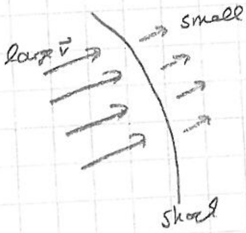


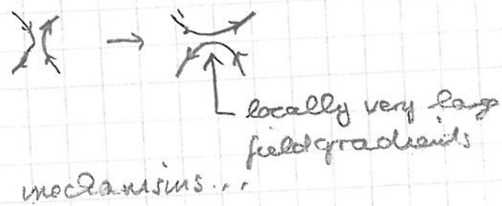
Image from
Science 322 (2008) 1221

- e^- from pulsar surface or surrounding plasma can easily reach $E \sim 100$ GeV
- γ emission by curvature radiation (= synchrotron emission due to propagation along curved B field lines)
- e^+e^- pair production via



- Pulsar wind nebulae:
  Relativistic particles from pulsar stream outward, form standing shock wave ⟹
  - 1st order Fermi acceleration
  - magnetic flux reconnection



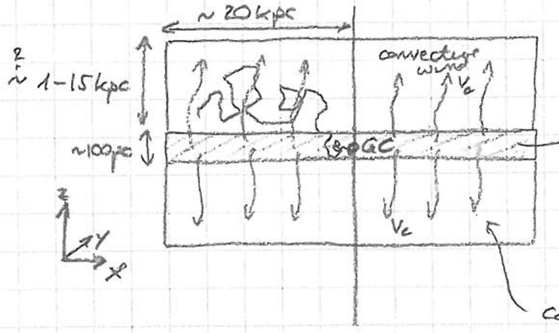


  - other mechanisms...

- Many details <u>not understood</u>

6.7 Cosmic ray transport

How do high-E charged particles travel from their sources to the Earth?



Cirelli et al.:
1012.4515

Salati, Cargèse 2009
lectures

Master equation (diffusion - loss eq.)

density of particles per unit energy

$$\underbrace{\frac{\partial f(t,\vec{x},E)}{\partial t}}_{=0 \text{ in steady state}} - K \cdot \Delta f + \frac{\partial}{\partial E}\left[b(E) \cdot f \right]$$

diffusion coefficient (assumed x-indep.)

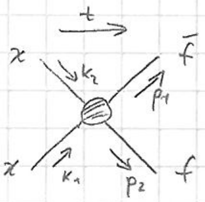
loss coefficient function

$$+ \underbrace{\frac{\partial}{\partial z}(V_c f) \cdot \mathrm{sgn}\, z}_{\text{convective wind}} = \underbrace{Q(t,\vec{x},E)}_{\text{source term}}$$

$$\Big[\text{ Flux } \quad J = K \cdot \vec{\nabla} f \quad \Rightarrow \quad \frac{\partial f}{\partial t} = \vec{\nabla} J + \dots = K \Delta f + \dots$$

6.7 Indirect DM detection

Freeze-out requires



This can still happen today.

Remember (□), p.24: rate per volume element

$$\frac{\Gamma}{V} = \left(\frac{1}{2}\right)\int \underbrace{d\Pi(p_1)}_{=\frac{d^3p_1}{(2\pi)^3\cdot 2E(p_1)}}\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2)\, \phi_\chi(k_1)\, \phi_\chi(k_2)$$

$$\cdot (2\pi)^4\, \delta^{(4)}(p_1+p_2-k_1-k_2)\, \overline{|\mathcal{M}|^2}$$

only for self-conjugate DM to avoid double counting.

$$\overset{(□□)\ p.25}{=} n_\chi^2 \langle \sigma_{\chi\chi\to f\bar f}\, v_{rel}\rangle \qquad (\text{or} \to \text{p. 25a})$$

Strong dependence on DM number density

6.7.1 DM distribution in the Milky Way

Nano, N-body simulations
www.usm.uni-muenchen.de/..

from N-body simulations (linearized method from sec. 5.1 not applicable)

Idea: Trace evolution of many ($10^8 - 10^9$) "particles" with mass $m \sim 10^3 M_\odot - 10^4 M_\odot$ — 0809.0898

Force:
$$\vec F_i = \sum_{j\neq i} \frac{G m_i (\vec x_i - \vec x_j)}{(|\vec x_i - \vec x_j|^2 + \varepsilon^2)^{2/3}} \qquad (*)$$

force on i-th particle

regulator to avoid divergence

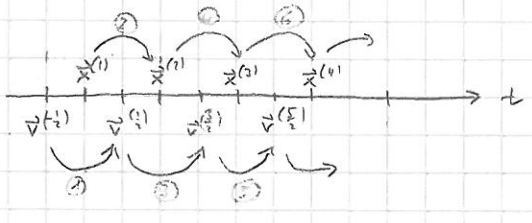
Time step:
$$\boxed{\begin{aligned} \vec x_i^{(n+1)} &\equiv \vec x_i^{(n)} + \vec v_i^{(n+\frac12)}\,\Delta t \\ \vec v_i^{(n+\frac12)} &\equiv \vec v_i^{(n-\frac12)} + \frac{1}{m}\,\vec F_i\big(\vec x_1^{(n)}, \dots \vec x_N^{(n)}\big)\,\Delta t \end{aligned}}$$

at $t = t^{(n)}$

at $t = t^{(n+\frac12)}$

particle mass

[Note: For $\Delta t \to 0$: $\dot{\vec x} = \vec v$; $\dot{\vec v} = \vec F/m$]



"Leapfrog scheme" makes algorithm T-invariant (to order $(\Delta t)^2$)

[In both time directions, first compute next $\vec v$, then use it to compute next $\vec x$.

Accuracy: Assume exact solution is $\vec{X}(t)$, $\vec{V}(t)$

Discretisation error in n-th step

$$\delta^{(n)} \equiv \frac{1}{m}\vec{F}(\vec{X}(t^{(n)})) - \frac{1}{\Delta t}\left[\vec{V}(t^{(n+1)}) - \vec{V}(t^{(n)})\right]$$

$$= \frac{1}{m}\vec{F}(X(t^{(n)})) - \frac{1}{(\Delta t)^2}\left[X(t^{(n+1)}) - 2X(t^{(n)}) + X(t^{(n-1)})\right]$$

Taylor-expand:

$$X(t^{(n\pm1)}) = X(t^{(n)}) \pm \Delta t\,\dot{X}(t^{(n)}) + \frac{1}{2}(\Delta t)^2 \underbrace{\ddot{X}(t^{(n)})}_{=\frac{1}{m}\vec{F}(X(t^{(n)}))} + \ldots$$

$$\pm \frac{1}{6}(\Delta t)^3 \dddot{X}(t^{(n)}) + \frac{1}{24}(\Delta t)^4 \frac{d^4X(t^{(n)})}{dt^4}$$

$$\Rightarrow \boxed{\delta^{(n)} = \frac{(\Delta t)^2}{12}\frac{d^4X(t^{(n)})}{dt^4}}$$

$\Rightarrow$ Method is 2^{nd} order accurate

Numerical tricks: Naab and Scholarpedia

Main bottleneck: Sum in (*) $\sim O(N^2)$

- Tree methods:
  - Divide cubic volume into 8 sub-cubes
  - If sub-cube contains <2 particles $\to$ done, continue with next sub-cube
  - If sub-cube contains ≥2 particles $\to$ sub-divide again
  - and so on

  $\Rightarrow$ Each tree node = large pseudo-particle

  When computing $\vec{F}$:
  - For small $|\vec{x}_i - \vec{x}_j|$: exact evaluation (use tree "leaves")
  - For large $|\vec{x}_i - \vec{x}_j|$: use larger and larger pseudo-particles (higher tree nodes)

  $\hookrightarrow$ $O(N \log N)$ operations

- Mesh methods:
  - Compute $\vec{F}$ on discrete grid (solve Poisson eq. with FFT) then use this force field instead of (*)
  - Adaptive methods: Use smaller grid spacing in "interesting" regions; adapt dynamically

- Hybrid methods: (*) or tree method on small scales (more accurate) mesh method on large scale (faster)

Results

[Show Aquarius video www.mpa-garching.mpg.de/aquarius/Aq-A-2-evolv.mp4]

[Show Aquarius halo profiles]

[r^{-2}: radius where $\rho(r)$ has "isothermal" value -2]

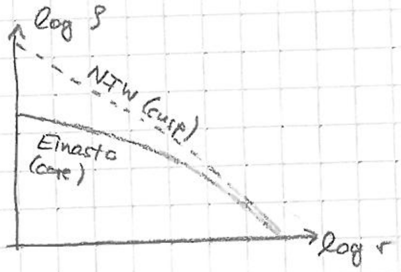
Phenomenological fits:

Navarro-Frenk-White (NFW):
(divergent @ $r=0$ $\Rightarrow$ "cusp")

$$\boxed{\rho^{NFW}(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}}$$

Einasto:
(finite @ $r=0$ $\Rightarrow$ "core")

$$\boxed{\rho^{Einasto}(r) = \rho_0 \exp\left[-\frac{2}{\alpha}\left[\left(\frac{r}{r_0}\right)^\alpha - 1\right]\right]}$$

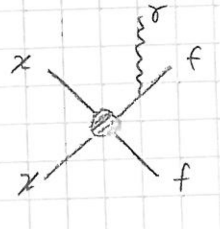


Complications

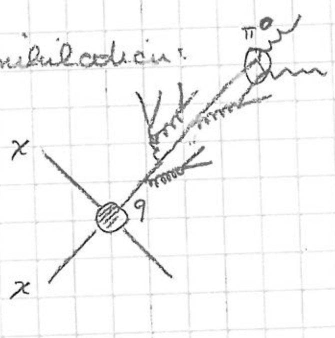
- Subhalos (dwarf galaxies)
- Effect of baryons

6.7.2 Gamma rays

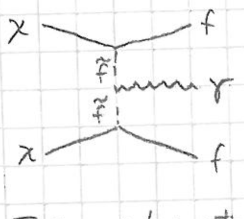
Prompt γ rays from DM annihilation:



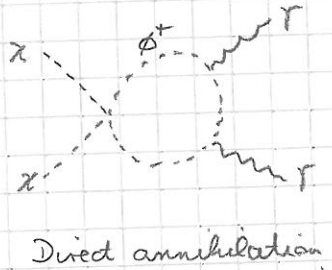
Final state radiation (FSR)



π^0 decay (for hadronic final state)



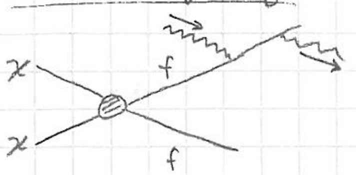
Internal bremsstrahlung ($\tilde{f}$ e.g. sfermion in SUSY)



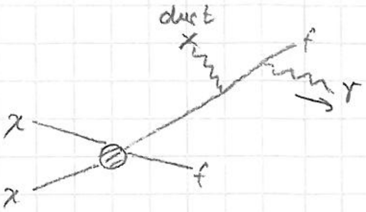
Direct annihilation (only through loops)

$\Rightarrow$ $\boxed{\text{highly model-dependent}}$

Secondary γ rays:



Inverse Compton scattering (ICS) on CMB + starlight



Bremsstrahlung (less important)

Exercise 10 from S Profumo's TASI 2012 lecture ?
$\frac{1-\lambda}{\sqrt{1-\lambda^2}} = \left(\frac{1+\lambda}{\sqrt{1-\lambda^2}}\right)^{-1}$

Flux: $\left[m^{2} s^{-1} sr^{-1} GeV^{-1} \right]$

line of sight in direction ψ from within $\Delta\Omega$

$$\boxed{\phi_\gamma = \frac{\Delta\Omega}{4\pi} \underbrace{\left[\frac{1}{\Delta\Omega}\int_{\Delta\Omega} d\Omega \int dl(\psi)\; \rho^2_{DM}(l,\psi)\right]}_{\text{"J-factor" (contains all astrophysics dependence)}} \frac{\langle\sigma v_{rel}\rangle}{(2)m_\chi^2}\frac{dN_\gamma}{dE_\gamma}}$$

avoid double counting for self-conjugate DM

$\Gamma = \langle\sigma v_{rel}\rangle \cdot \frac{\rho^2}{m_\chi^2}$

injection spectrum

end 04.07.2013

Places to look for γ rays from DM annihilation:

- Galactic Center (high DM density, lots of astrophysical BG sources)
- Dwarf galaxies (overdense regions in the DM halo with few baryonic objects)
- Diffuse γ rays from all over the Sky (except center + disc of MW)
- galaxy clusters
- .....

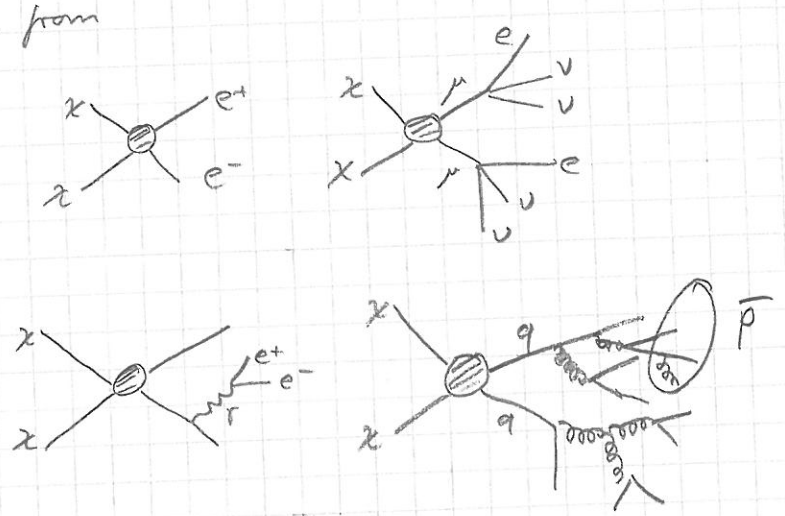
Results:

[show GC spectrum fit + morphology (fig. 2 in 1207.6047)

show dSph limits]

6.7.3 Charged cosmic rays

e.g. from



- e^+ and e^-
  - cannot travel too far $\rightarrow$ locally produced
  - BG from pulsars

  [show AMS-02 spectrum]

- $\bar{p}$, $\bar{D}$, ...
  - produced in hadronic DM annihilation

  [show PAMELA $\bar{p}$ spectrum]

7. Axions

7.1 The strong CP problem

Thesis of Wilco J. den Dunnen (Amsterdam, 2008)

Consider QCD Lagrangian:

$$\mathcal{L}_{QCD} = i\bar{\Psi}\not{D}\Psi - \overline{\Psi_L} M \Psi_R - \overline{\Psi_R} M^\dagger \Psi_L$$
$$- \frac{1}{4}\mathrm{tr}\, F^a_{\mu\nu} F^{\mu\nu,a} + \boxed{\frac{\theta g^2}{16\pi^2}\,\mathrm{tr}\, F^a_{\mu\nu}\tilde{F}^{\mu\nu,a}}$$

with

$$D_\mu = \partial_\mu - i g A^a_\mu t^a$$

gauge group generators

gauge group structure constants

$$F_{\mu\nu} = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\right) t^a$$

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma,a}\, t^a$$

Infinitesimal gauge transformation

$$\Psi(x) \to (1 + i\alpha^a(x)\, t^a)\, \Psi(x)$$

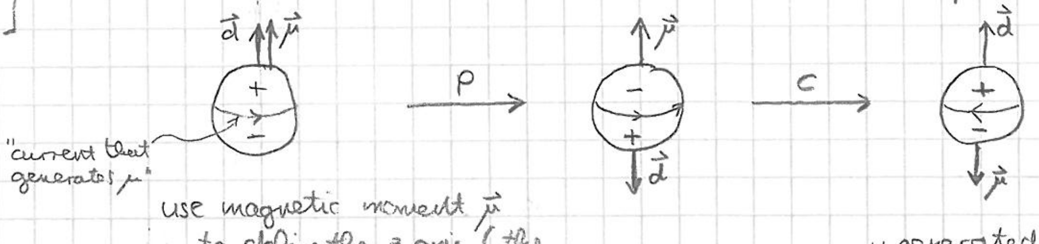
$$A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g}\partial_\mu \alpha^a(x) + f^{abc} A^b_\mu(x)\, \alpha^c(x)$$

Consequences of the extra term:

- P and CP odd: $F^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \neq 0$ only if μ, ν, ρ, σ are different $\to$ 3 spatial components, which flip sign under P. (C does not affect A^a_μ since it is a real field.)

[see Wikipedia article on neutron EDM]

- Observable CP violation: Neutron electric dipole moment



use magnetic moment $\vec{\mu}$ to define the z axis (the only preferred direction in the neutron)

μ generated by circular currents, which are reversed under C

$\Rightarrow$ CP maps $\vec{\mu}\cdot\vec{d} \longrightarrow -\vec{\mu}\cdot\vec{d}$

we know $\vec{\mu} \neq 0$ $\Rightarrow$ CP conservation only if $\vec{d} = 0$

Experimental searches constrain $|\vec{d}| < 0.29 \cdot 10^{-25}\, e\, cm$

One can show (using heavy theoretical machinery):

$$d_n \cong \Theta \cdot 10^{-16}\, e \cdot cm$$

$$\Rightarrow \boxed{\Theta \ll 10^{-10}}$$

$\boxed{\text{Strong CP problem: Why is } \Theta \text{ so small?}}$

Relation to axial $U(1)_A$ transformations

following Peskin/Schröder sec. 19.2

Consider transformation $\psi \to \psi' \equiv (1 + i\underbrace{\alpha}_{\text{infinitesimal}}\gamma^5)\psi$

$$\overline{\psi} \to \overline{\psi'} = \overline{\psi}(1 + i\alpha\gamma^5)$$

$$\Rightarrow \mathcal{L}_{QCD} \longrightarrow \mathcal{L}'_{QCD} = i\overline{\psi}(1+i\alpha\gamma^5)\gamma^\mu D_\mu(1+i\alpha\gamma^5)\psi$$

$$-\left[M\overline{\psi}(1+i\alpha\gamma^5)\left(\tfrac{1+\gamma^5}{2}\right)(1+i\alpha\gamma^5)\psi + h.c.\right]$$

$$- \text{gauge kinetic term} - \Theta\text{-term}$$

$$= \mathcal{L}_{QCD} - \left[2i\alpha M \overline{\psi}_L \psi_R + h.c.\right] \qquad (*)$$

$\Rightarrow$ Massless QCD ($M = 0$) should be invariant .....

However: Quantum effects break this classical symmetry ($\to$ <u>axial anomaly</u>)

Deeper reason: $S = \int d^4x\, \mathcal{L}_{QCD, M=0}$ invariant, but functional measure $\mathcal{D}\psi$ in path integrals is not:

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, \mathcal{D}A \exp\left[i\int d^4x \left(\overline{\psi}\, i\not{D}\,\psi + \frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{\Theta g^2}{16\pi^2}\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\right]$$

Only fermions affected by $U(1)$ $\Rightarrow$ Consider only

$$Z_\psi \equiv \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \exp\left[i \int d^4x \; \overline{\psi} \, i\not{D} \, \psi \right]$$

Write $\psi, \overline{\psi}$ in terms of eigenfunctions of $i\not{D}$

$$i\not{D} \, \phi_m = \lambda_m \phi_m$$

$$\hat{\phi}_m (i\not{D}) \equiv -i (D_\mu \hat{\phi}_m) \gamma^\mu = \lambda_m \hat{\phi}_m$$

[Minus sign: $\phi^\dagger A = \phi^\dagger A^\dagger = (A\phi)^\dagger$; $A = A^\dagger$]

$$\Rightarrow \psi(x) = \sum_m a_m \phi_m(x) \quad ; \quad \overline{\psi}(x) = \sum_m \hat{a}_m \hat{\phi}_m(x)$$

anticommuting (Grassmann) coefficients

$$\Rightarrow \mathcal{D}\psi \, \mathcal{D}\overline{\psi} = \prod_m da_m \, d\hat{a}_m$$

Apply axial transformation:

$$a'_m = \sum_n \int d^4x \; \phi^\dagger_m(x) \, (1 + i\alpha\gamma^5) \, \phi_n(x) \, a_n$$

$$\equiv a_m + \sum_n \underbrace{C_{mn}}_{= \int d^4x \, \phi^\dagger_m \, i\alpha\gamma^5 \phi_n} a_n$$

$$\Rightarrow \mathcal{D}\psi' = \left[\det (1 + C) \right]^{-1} \mathcal{D}\psi \qquad (\text{analogously for } \mathcal{D}\overline{\psi})$$

[Consider $\int da_m \, f(a_m) = \int da_m \, (A + B a_m) = B$

Substitute $a_m \to (1 + c) a_m \equiv a'_m$

$\hookrightarrow \int da_m \left(A + \frac{B}{1+c} a'_m \right) = \int da'_m \, J \left(A + \frac{B}{1+c} a'_m \right) \stackrel{!}{=} B$

$\Rightarrow J = 1+c \quad ; \quad da'_m = \frac{1}{J} da_m$]

Jacobian: $\det (1 + C) = \exp\left[\operatorname{tr} \log (1 + C) \right] \stackrel{C \text{ infinitesimal}}{=} \exp(\operatorname{tr} C)$

$$= \exp\left[i\alpha \int d^4x \sum_n \phi^\dagger_n \gamma^5 \phi_n \right]$$

[$\neq \exp\left[i\alpha \int d^4x \; \operatorname{tr} \gamma^5 \right] = 1$

because need to regularize infinite sum.]

$$\log \det(1+G) = \lim_{M\to\infty} i\alpha \int d^4x \sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) e^{\lambda_n^2/M^2}$$

end 11.07.2013

Note: For free particles ($A_\mu \to 0$) or large momentum:

$$\lambda_m^2 = k^2 = E_{\vec k}^2 - \vec k^2$$

After Wick rotation to Euclidean space: $E_{\vec k}^2 \to -E_{\vec k}^2$

$$\Rightarrow \lambda_m^2 < 0$$

$$= \lim_{M\to\infty} i\alpha \int d^4x \langle x | \operatorname{tr} \gamma^5 e^{(i\not{D})^2/M^2} | x\rangle$$

trace over Dirac indicies and gauge group indicies

Rewrite $(i\not{D})^2$:

$$(i\not{D})^2 = -\left(\partial_\mu - igA_\mu^a t^a\right) \underbrace{\gamma^\mu\gamma^\nu}_{=\frac{1}{2}\left(\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{=2g^{\mu\nu}} + \underbrace{[\gamma^\mu,\gamma^\nu]}_{-2i\sigma^{\mu\nu}}\right)} \left(\partial_\nu - igA_\nu^b t^b\right)$$

$$= -D^2 + i\sigma^{\mu\nu}\left(-ig(\partial_\mu A_\nu^b)t^b - g^2 A_\mu^a A_\nu^b t^a t^b\right)$$

$$= -D^2 + \tfrac{1}{2} i\sigma^{\mu\nu}\left[-ig\left(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b\right)t^b - g^2 A_\mu^a A_\nu^b [t^a, t^b]\right]$$

$$= -D^2 + \frac{g}{2}\sigma^{\mu\nu}\left[\partial_\mu A_\nu^b - \partial_\nu A_\mu^b + g f^{acb} A_\mu^a A_\nu^c\right] t^b$$

$$= -D^2 + \frac{g}{2}\sigma^{\mu\nu}F_{\mu\nu}$$

In the limit $M \to \infty$:

- expand $(i\not{D})^2$ in A_μ^a (assume A_μ^a and its derivative bounded from above, while M is not)
- Derivative = momentum k, only large k relevant $\Rightarrow$ keep only term containing two derivatives in the numerator, expand the rest

$$\log \det(1+G) = \lim_{M\to\infty} i\alpha \int d^4x \operatorname{tr}\left[\gamma^5 \frac{1}{2!}\left(\frac{g}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right] \langle x| e^{-\partial^2/M^2}|x\rangle$$

traces with γ^5 need at least 4 Dirac matrices to be non-zero

- $$\langle x | e^{-\partial^2/M^2}|x\rangle = \lim_{x\to y} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{k^2/M^2}$$

$$\overset{\text{Wick rotation}}{\underset{k^0\to ik_E^0}{=}} i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2}$$

$$= \frac{iM^4}{16\pi^2}$$

- surface of unit 3-sphere: $2\pi^2$
- $\int_0^\infty x^3 e^{-x^2/M^2} dx = \frac{M^4}{2}$

$$\bullet \quad \mathrm{tr}\, \gamma^5 \sigma^{\mu\nu} \sigma^{\rho\tau} = -\frac{1}{4} \cdot (-4i)\, \varepsilon^{\mu\nu\rho\tau} - (\mu \leftrightarrow \nu) - (\rho \leftrightarrow \tau) + (\mu \leftrightarrow \nu,\ \rho \leftrightarrow \tau)$$

$$= 4i \varepsilon^{\mu\nu\rho\tau}$$

$$\Rightarrow \log \det (1 + G) = \lim_{M \to \infty} i\alpha \int d^4x\, \frac{g^2}{8M^4}\, 4i\varepsilon^{\mu\nu\rho\tau}\, \mathrm{tr}\, F_{\mu\nu} F_{\rho\tau} \cdot \frac{iM^4}{16\pi^2}$$

$$= -i\alpha \int d^4x\, \frac{g^2}{16\pi^2}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Result:

$$\boxed{Z_\psi \to Z'_\psi = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \exp\left[i \int d^4x\, \bar{\psi}\, i\not{D}\, \psi + i\alpha \frac{g^2}{8\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu,a} \right]}$$

In massless QCD, the Θ-term can be removed by a $U(1)_A$ rotation ($\to$ absorbed into field redefinitions)

In massive QCD, Θ can be traded for complex phases in the mass matrix. See (*) and its generalization to non-infinitesimal transformations. What remains unchanged is

$$\Theta_{\text{eff}} \equiv \underbrace{N_f}_{\text{factors } \mathcal{D}\psi \mathcal{D}\bar{\psi} \text{ for every quark flavor}} \Theta + \arg \det M$$

Peccei Quinn
PRL 38 (1977) 1440
PRD 16 (1977) 1791

7.2 The Peccei – Quinn mechanism

Thesis of Wiloodan Dammar

Peccei, hep-ph/0607268

Idea: Add a dynamical field that contributes to arg det M, and show that its vev is at $\langle a \rangle + \Theta_{\text{eff}} = 0$

Toy model:

fermion — complex scalar

$$\mathcal{L} = -\frac{1}{4} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi + y\, \phi\, \bar{\psi}_L \psi_R + y^* \phi^+ \bar{\psi}_R \psi_L$$

Yukawa coupling

$$- (\partial_\mu \phi)^+ (\partial^\mu \phi) - V(\phi^+, \phi) + \frac{\Theta g^2}{16\pi^2} \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Axial $U(1)_{PQ}$ transformations:

$$\psi_L \to e^{i\alpha} \psi_L \; ; \quad \psi_R \to e^{-i\alpha} \psi_R \; ; \quad \phi \to e^{2i\alpha} \phi$$

- as above: absorb θ into complex phase of y
- assume ϕ develops vev
- show that $\boxed{\arg\langle\phi\rangle + \arg y = 0 \mod 2\pi}$

Outline of proof: Goal: Compute effective potential (= scalar potential including quantum corrections)

Partition function (with "external source") $Z[J] = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}\phi\, \mathcal{D}A\; e^{i\int d^4x\,(\mathcal{L} + J\phi)}$

$$\Rightarrow \langle\phi\rangle = -i\,\frac{\delta \log Z[J]}{\delta J}$$

Effective action: $\Gamma[\langle\phi\rangle] \equiv -i\log Z[J] - \int d^4y\, J(y)\,\langle\phi\rangle(y)$

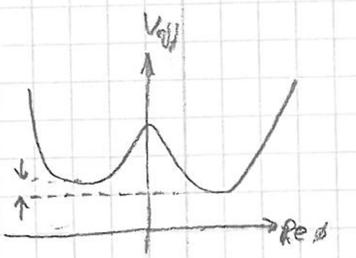
$$\Rightarrow \frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle(x)} = -i\int d^4y\,\frac{\delta\log Z[J]}{\delta J(y)}\,\frac{\delta J(y)}{\delta\langle\phi\rangle(x)}$$

$$- J(x) + \int d^4y\,\frac{\delta J(y)}{\delta\langle\phi\rangle(x)}\,\langle\phi\rangle(y)$$

$$= -J(x)$$

$\Rightarrow$ for zero external source, $\langle\phi\rangle$ is the minimum of $\Gamma[\langle\phi\rangle]$.

Effective potential: $V_{eff}[\langle\phi\rangle] \equiv -\frac{1}{V\cdot T}\,\Gamma[\langle\phi\rangle]$



Here: - Terms depending on $\arg\langle\phi\rangle$ have to come from Yukawa couplings
- Assume $y \ll 1$ and expand

$\hookrightarrow V_{eff}[\langle\phi\rangle] \sim V_{class}(|\langle\phi\rangle|^2) + y\langle\phi\rangle K + y^*\langle\phi\rangle^* K$

real because the rest of $\mathcal{L}$ is symmetric under $\psi_L \leftrightarrow \psi_R$

$\sim -\cos(\arg\langle\phi\rangle + \arg y)\cdot |y|\cdot|\langle\phi\rangle|\cdot A$

sign not obvious

$\Rightarrow$ Extremal at $\boxed{\arg\langle\phi\rangle + \arg y = 0}$

7.3 The axion

- Write $\phi = f_a e^{i(\alpha + \frac{a}{f_a})}$, neglecting radial excitations (heavy!)

f_a: $|\langle\phi\rangle| \gg M_W$

α: $\arg\langle\phi\rangle$

a: dynamic field "axion"

Excitations of $a \Leftrightarrow$ chiral rotations. Using the axial anomaly, we can make the transformation

$$y f_a e^{i(\alpha + \frac{a}{f_a})} \bar{\Psi}_L \Psi_R + \text{h.c.} \longrightarrow y f_a e^{i\alpha} \bar{\Psi}_L \Psi_R + \text{h.c.}$$

$$+ \frac{a g^2}{16\pi^2 f_a} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Low-E axion phenomenology depends on $a - A_\mu$ couplings since ϕ, Ψ are heavy.

If Ψ is charged under several gauge group factors, there is an $a F_{\mu\nu} \tilde{F}^{\mu\nu}$ coupling for each of them, in particular

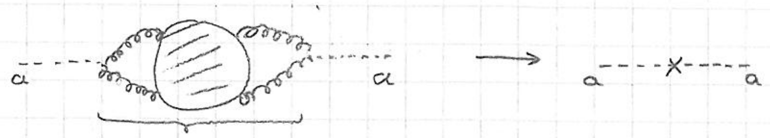
$$\boxed{\frac{a e^2 q_\Psi^2}{16\pi^2 f_a} F_{\mu\nu} \tilde{F}^{\mu\nu}}$$

q_Ψ: em charge of Ψ

$F_{\mu\nu}$, $\tilde{F}^{\mu\nu}$: em field strength tensor

- Axion mass: [Note: $m_a = 0$ at the classical (Lagrangian) level

Bardeen Peccei Yanagida
Nucl Phys B 279 (1987) 401



has nonzero vev below Λ_{QCD}

$$\hookrightarrow \mathcal{L} \supset \frac{a^2}{f_a^2} \cdot O(\Lambda_{QCD}^4) \Rightarrow m_a \sim \frac{O(\Lambda_{QCD}^2)}{f_a} \ll \Lambda_{QCD}$$

Calculation in chiral perturbation theory yields

Wantz, Shellard
0910.1066

$$\boxed{m_a = m_\pi \frac{f_\pi}{f_a} \frac{\sqrt{m_u m_d}}{(m_u + m_d)} \times \text{model-dependent } O(1) \text{ correction factor}}$$

- Adding axions to the SM :

e.g. KSVZ (Kim Shifman Vainshtein Zakharov) model :

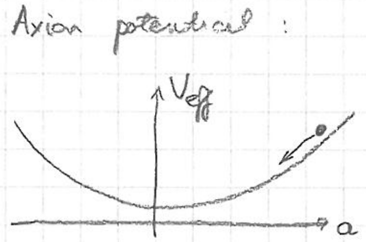
add scalar field ϕ with $\langle\phi\rangle \gg M_W$, and heavy quark Q with $m_Q \sim \langle\phi\rangle$; both charged under $U(1)_{PQ}$, all other fields uncharged

Q charged under $SU(3)$ and $U(1)_{em}$

↳ axion couples to gluons and photons

7.4 Production of axions in the early Universe: The misalignment mechanism

Axion potential:



assume random initial value

e.o.m: $\boxed{\ddot{a} + 3H\dot{a} + m_a^2 a = 0}$ (see chapter on inflation)

Early times: H very large $\rightarrow$ $\dot{a} \sim 0$

Later: Field "rolls down" the potential and oscillates
small m_a $\rightarrow$ shallow potential $\rightarrow$ slow oscillations,
but large amplitude if $\langle a \rangle_{initial}$ large

amplitude $\sim$ # of particles

osc. frequency $\sim$ energy of particles

$\Rightarrow$ production of many <u>non-relativistic</u> axions
(in spite of their low mass!)

$\Rightarrow$ $\boxed{\text{Cold DM candidate}}$

7.5 Detecting axions

Starting point : $\mathcal{L} \supset \frac{a}{f_a} \frac{e^2 q_\psi^2}{16\pi^2} F^{\mu\nu} \tilde{F}_{\mu\nu}$

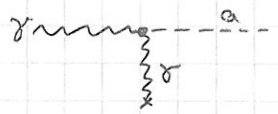
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\varepsilon^{0123} = -\varepsilon_{0123} = 1$]

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L} \supset \frac{a}{f_a} \frac{e^2 q_\psi^2}{32\pi^2} \left(-4 \vec{E}\cdot\vec{B}\right)$$

$$= \boxed{-\frac{a}{f_a} \frac{\alpha q_\psi^2}{\pi} \vec{E}\cdot\vec{B}}$$



Photon can convert into axion in external $\vec{B}$-field ($\vec{E}$-field)
if polarization along (orthogonal to) $\vec{B}$-field ($\vec{E}$-field)
and propagation direction orthogonal to $\vec{E}$- or $\vec{B}$-field.

[show experiments and constraints]

8. The baryon asymmetry of the Universe

8.1 Introduction

Murayama
CERN lectures
25.05.2010

Observation: Only matter in the Universe, (almost) no antimatter

- Absence of annihilation lines (511 keV, 938 MeV, etc.)
- Low antimatter fraction in cosmic rays
  (→ mostly secondary production

Koichi Funakubo:
"Baryon Asymmetry of the Universe"

Early Universe: Matter and antimatter thermally produced
Later: Annihilation via B (baryon number) and L (lepton number) conserving SM processes

⇒ There must have been more matter than antimatter when the annihilations happened

Asymmetry is tiny: CMB & BBN: $\boxed{\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} \sim 6 \times 10^{-10}}$

baryon density

photon density

"Big Bang Nucleosynthesis"
production of elements heavier than H at $T \sim 100\,\text{keV}$

Where did the asymmetry come from?

↳ Initial conditions?
→ any initial asymmetry erased by inflation

↳ Particle physics processes!

Denis V. Perepelitsa:
"Sakharov conditions for baryogenesis"

Requirements (Sakharov conditions):

- $\boxed{\text{B violation}}$
- $\boxed{\text{C and CP violation}}$

↳ without C violation $\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$

with C violation, but without CP violation:

$$\Gamma(X \to q_L q_L) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R)$$

$$\Rightarrow \Gamma(X \to q_L q_L) - \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L) =$$

$$\Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) - \Gamma(X \to q_R q_R)$$

⇒ asymmetries in LH and RH sectors cancel

• $\boxed{\text{Departure from thermal equilibrium}}$

↳ in thermal equilibrium, all d.o.f. equally populated (chemical potentials ensure this)

Actually, the SM satisfies these:

- $\not{C}$, $\not{CP}$: weak interactions, CKM phase
- out of equilibrium: bubble formation during ewk phase transition → interactions at bubble boundaries out of equilibrium
- B violation → see next section

.... but the numbers don't work out.
(e.g. phase transition not 1st order at $m_H \hat{=} 125$ GeV, CKM phase too small, ...)

8.2 Sphalerons: B+L violation in the SM

Consider pure $SU(2)_L$ theory:

$$\mathcal{L} = -\frac{1}{4} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi}_L \not{D} \Psi_L$$

Axial anomaly

$$\partial_\mu \bar{\Psi} \gamma^\mu \gamma^5 \Psi = -\frac{g^2}{16\pi^2} \underbrace{N_f}_{\#\text{ of fermion families}} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

[Proof: As in sec. 7.1 with $\alpha \to \alpha(x)$; vary w.r.t. $\alpha(x)$ in the end; see Peskin/Schröder sec. 19.2; $\bar{\Psi} i \not{D} \Psi \to \bar{\Psi}\gamma^\mu \Psi (-\partial_\mu \alpha) \to + \alpha(x) \partial_\mu j^\mu$]

B current: $j^\mu_B \equiv \bar{\Psi}_L \gamma^\mu \Psi_L = \frac{1}{2} \bar{\Psi} \gamma^\mu \Psi - \frac{1}{2} \bar{\Psi} \gamma^\mu \gamma^5 \Psi$

$$\Rightarrow \partial_\mu j^\mu_B = \frac{g^2}{2 \cdot 8\pi^2} N_f \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Similarly
$$\partial_\mu \underbrace{j^\mu_L}_{\text{Lepton number current}} = \frac{g^2}{2 \cdot 8\pi^2} N_f \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

⇒ B+L not conserved, B−L is conserved

Connection to vacuum structure of non-Abelian gauge theories:

B+L violating processes ("sphaleron processes") change from one vacuum state to another

Consider non-infinitesimal gauge transformation

$$\Psi_L \to U(x)\,\Psi_L \quad ; \quad U = e^{i\alpha^a(x)\frac{\sigma^a}{2}} \in SU(2)$$

$$A_\mu \to U A_\mu U^{-1} - (\partial_\mu U)U^{-1}$$

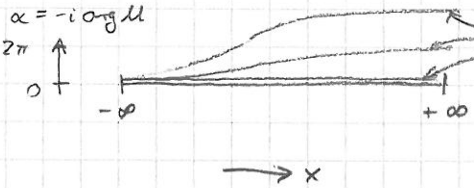
Vacuum state = "pure gauge":

$$A_\mu = -(\partial_\mu U)U^{-1}$$

Choose boundary conditions

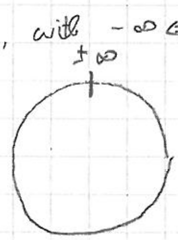
$$U \xrightarrow{|x|\to\infty} 1 \quad ; \quad A_\mu \xrightarrow{|x|\to\infty} 0 \quad \text{mod } 2\pi$$

Simple analogy: $U(1)$ in 1+1 dim



different configurations that cannot be continuously transformed into each other

Since $\lim\limits_{|x|\to-\infty} A_\mu(x) = \lim\limits_{|x|\to+\infty} A_\mu(x)$, can "compactify" $\mathbb{R}$ to S^1 (a circle), with $-\infty \Leftrightarrow 0$; $+\infty \Leftrightarrow 2\pi \hat{=} 0$



Consider map $S^1 \to U(1)$; $x \mapsto e^{i\alpha(x)} \equiv U$

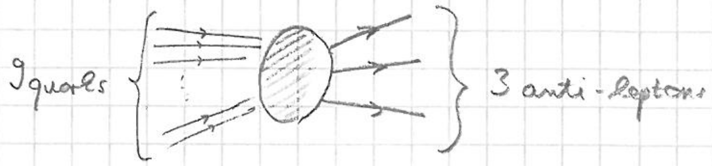
Classify vacuum gauge field configurations by "winding number" = number of times U travels around the circle corresponding to $U(1)$ as x travels once around the circle corresponding to compactified space.

We don't see B+L violation in perturbative calculations

↳ must be non-perturbative processes

Detailed investigation reveals:

Sphaleron processes:



$$\Rightarrow \Delta B = -3 \quad ; \quad \Delta L = -3$$

- energy barrier of order M_W
- accessible in early Universe at $T \gtrsim M_W$

8.3 Leptogenesis

Idea: - Generate lepton asymmetry in L and CP violating decays of heavy ($\gg M_W$) particles out of thermal equilibrium
- Sphaleron processes will convert ΔL to ΔB until $\Delta B \sim \Delta L$ ($\sim$ thermal equilibrium)

The heavy particles are RH neutrinos N_R, which could simultaneously explain why LH neutrinos are so light via the seesaw mechanism

$$SM + N_R: \quad \mathcal{L} \supset - \underbrace{m_D}_{\text{"Dirac mass"} \sim M_W} \bar{\nu}_L N_R - y_D h \bar{\nu}_L N_R$$

$$+ \frac{1}{2} \underbrace{m_M}_{\text{"Majorana mass"} \gg M_W} \overline{(N_R)^c} N_R + h.c.$$

where $(N_R)^c = \underbrace{\hat{C}}_{\text{particle-antiparticle conjugation}} N_R \equiv -i\gamma^2 N_R^*$.

Physical meaning: $\hat{C}$ transforms RH particle to LH antiparticle ($\gamma^5 \hat{C} \psi = +i\gamma^2\gamma^5\psi^* = -\hat{C}(\gamma^5\psi)^*$)

No reason why LH antiparticle $\neq$ LH particle if particle is uncharged (like N_R)

Mass terms couple LH fields to RH fields
↳ $\overline{(N_R)^c} N_R$ is a valid mass term

Integrating out N_R: Write $n \equiv \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix}$

$$\hookrightarrow \mathcal{L} \supset -\frac{1}{2} \overline{\left(\nu_L \;\; (N_R)^c \right)^c} \begin{pmatrix} 0 & m_D \\ m_D^{(T)} & m_M \end{pmatrix} \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix} + h.c.$$

$$= -\frac{1}{2} \overline{(\nu_L)^c} m_D (N_R)^c - \frac{1}{2} \overline{((N_R)^c)^c} m_M (N_R)^c$$

$$-\frac{1}{2} \overline{((N_R)^c)^c} m_D^{(T)} \nu_L + h.c.$$

$$\overline{(\nu_L)^c} m_D (N_R)^c = \overline{(-i\gamma^2 \nu_L^*)} m_D (-i\gamma^2 N_R^*)$$

$$= \nu_L^T \gamma^{2\dagger} i \gamma^0 m_D - i\gamma^2 N_R^*$$

$$= \nu_L^T \quad \gamma^0\gamma^2\gamma^0\gamma^0\gamma^2 m_D N_R^*$$

$$\gamma^0 \cdot \gamma^0 = -1$$

$$= -\nu_{L\alpha}\, \gamma^0_{\alpha\beta}\, N^*_{R\beta} \quad \text{spinor indices}$$

fermion anticomm.

$$= +N^*_{R\beta}\, \gamma^0_{\beta\alpha}\, \nu_{L\alpha}\, m_D$$

use $\gamma^0_{\alpha\beta} = \gamma^0_{\beta\alpha}$

$$= \overline{N_R}\, \nu_L$$

$$\Rightarrow \mathcal{L} \supset -\tfrac{1}{2} m_D \overline{N_R} \nu_L - \tfrac{1}{2} \overline{(N_R)^c}\, m_M N_R - \tfrac{1}{2} m_D \overline{N_R} \nu_L + \text{h.c.}$$

$\square$

Diagonalize mass matrix $\rightarrow$ eigenvalues $\sim m_M, -\frac{m_D^2}{m_M}$

(fix sign of second eigenvalue by rephasing fields)

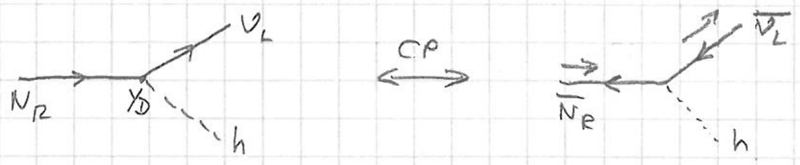
$$\hookrightarrow \quad \mathcal{L} \supset -\frac{1}{2} \frac{m_D^2}{m_M} \overline{(\nu_L')^c}\, \nu_L'$$

with $\nu_L' \simeq \nu_L + \mathcal{O}\left(\frac{m_D}{m_M}\right)(N_R)^c$

$\hookrightarrow$ with $m_D \sim 100$ GeV, $m_M \sim 10^{14}$ GeV, we obtain

$$m_\nu \equiv \frac{m_D^2}{m_M} \sim 0.1\,\text{eV}$$

Heavy neutrino decay:



$$i\mathcal{M} = i y_D \, \bar{u}_\nu \underbrace{P_R}_{=\frac{1+\gamma^5}{2}} u_N \qquad\qquad i\overline{\mathcal{M}} = i y_D^* \, \bar{v}_\nu P_L v_N$$

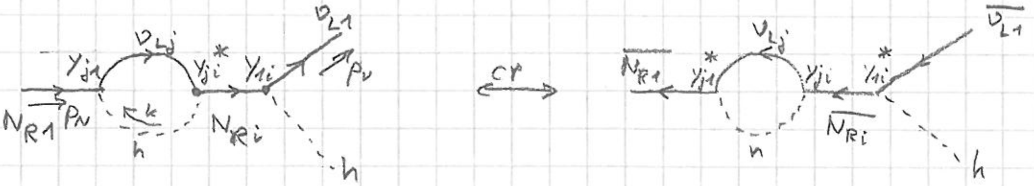
$$|\mathcal{M}|^2 = |y_D|^2 \, \mathrm{tr}\left[(\not{p}_N + m_R) P_L \not{p}_\nu P_R\right]$$

$$= \frac{1}{2}|y_D|^2 \, \mathrm{tr}\, \not{p}_N \not{p}_\nu$$

$$= 2|y_D|^2 \, p_N \cdot p_\nu \qquad = |\overline{\mathcal{M}}|^2$$

$$\Rightarrow \quad A_{CP} \equiv \frac{|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2}{|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2} = 0$$

↳ no ~~CP~~ at tree level, but there are loop corrections, and there is more than one neutrino flavor.



$$i\mathcal{M}_{loop} \cong \sum_{ij} \int \frac{d^4k}{(2\pi)^4} \bar{u}_{\nu 1} \, i y_{1i} P_R \frac{i(\not{p}_N + m_{Ni})}{p_N^2 - m_{Ni}^2} \, i y_{ji}^* P_L \frac{i(\not{p}_N + \not{k})}{(p_N + k)^2} \, i y_{j1} P_R u_{N1} \cdot \frac{i}{k^2}$$

$$= \underset{\substack{\uparrow \\ i^6}}{-} \sum_{i,j} y_{1i} y_{ji}^* y_{j1} \, \bar{u}_{\nu 1} P_R \frac{\not{p}_N}{p_N^2 - m_{Ni}^2} \left[\int \frac{d^4k}{(2\pi)^4} \frac{\not{p}_N + \not{k}}{(p_N + k)^2} \cdot \frac{1}{k^2}\right] u_{N1}$$

Idea: $|\mathcal{M}_{tree} + \mathcal{M}_{loop}|^2 \neq |\overline{\mathcal{M}}_{tree} + \overline{\mathcal{M}}_{loop}|^2$

$\left(\text{use } |\mathcal{M}_{loop}|^2 = |\overline{\mathcal{M}}_{loop}|^2 \text{ and } |\mathcal{M}_{tree}|^2 = |\overline{\mathcal{M}}_{tree}|^2\right)$

$$\Leftrightarrow \quad 2\,\mathrm{Re}\, \mathcal{M}_{tree} \mathcal{M}_{loop}^* \neq 2\,\mathrm{Re}\, \overline{\mathcal{M}}_{tree} \, \overline{\mathcal{M}}_{loop}^*$$

$$\Leftrightarrow \quad \cancel{\arg \mathcal{M}_{tree}} - \arg \mathcal{M}_{loop} \neq \cancel{\arg \overline{\mathcal{M}}_{tree}} - \arg \overline{\mathcal{M}}_{loop}$$

$\Leftrightarrow$ [Phase from couplings] · [phase from loop integral]

$\neq$ [Phase from couplings]* · [Phase from loop int.]

Loop integrals yield complex phases if particles in the loop can go on-shell simultaneously

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not p_N + \not k}{(p_N+k)^2} \frac{1}{k^2} = \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\not p_N + \not k}{\left[x(p_N^2 + 2kp_N + k^2) + (1-x)k^2\right]^2}$$

$$\underset{\ell \equiv k + p_N x}{=} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\not \ell + (1-x)\not p_N}{\left[\ell^2 + x(1-x)p_N^2\right]^2}$$

$$= \int_0^1 dx\, (1-x)\not p_N \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 + x(1-x)p_N^2\right]^2}$$

$$= \int_0^1 dx\, (1-x)\not p_N \frac{i}{(4\pi)^2} \Big(\underbrace{\frac{2}{\varepsilon} - \gamma + \log 4\pi}_{\text{removed by renormalizati}} - \log \big[\underbrace{-x(1-x)p_N^2}_{<0}\big]$$

$\Rightarrow \log\left[-x(1-x)p_N^2\right]$ imaginary!

[Note: If particles in loop were heavy, we would have obtained $\log\left[-x(1-x)p_N^2 + x m_\nu^2 + (1-x) m_h^2\right] \in \mathbb{R}$]

$$\Rightarrow \mathcal{M}_{tree} \mathcal{M}_{loop}^* \propto y_{11} \cdot \sum_{i,j} y_{1i}^* y_{ji} y_{j1}^* \frac{1}{m_{N1}^2 - m_{Ni}^2} \cdot \underbrace{i}_{\text{from loop}}$$

$$\overline{\mathcal{M}_{tree}}\, \overline{\mathcal{M}_{loop}}^* \propto y_{11}^* \sum_{i,j} y_{1i} y_{ji}^* y_{j1} \frac{1}{m_{N1}^2 - m_{Ni}^2} \cdot i$$

$$\Rightarrow A_{CP} = \frac{2\,\mathrm{Re}\, \mathcal{M}_{tree} \mathcal{M}_{loop}^* - 2\,\mathrm{Re}\, \overline{\mathcal{M}_{tree}}\, \overline{\mathcal{M}_{loop}}^*}{|\mathcal{M}_{tree} + \mathcal{M}_{loop}|^2 + |\overline{\mathcal{M}_{tree}} + \overline{\mathcal{M}_{loop}}|^2} \neq 0$$

[<u>Note</u>: For just one ν flavor, we would have obtained

$$\mathcal{M}_{tree} \mathcal{M}_{loop}^* \propto |y_{11}|^4 \cdot i = \overline{\mathcal{M}_{tree}}\, \overline{\mathcal{M}_{loop}}^*$$

$\Rightarrow$ multiple flavors essential]

Note: Full calculation: add'l diagrams: