During the journey, Einstein explained his theory to me every day,
and when we arrived, I was convinced that he had understood it.
Chaim Weizmann, 1929

1. **Derivation of the Friedmann equation in flat spacetime**
Show that, for the energy-momentum tensor of a homogeneous, isotropic fluid, $T_{\mu\nu} = \text{diag}\big[\rho(t), -p(t), -p(t), -p(t)\big]$, the (00) component of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1}$$

yields, in a flat spacetime, the Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} \rho(t)\,. \tag{2}$$

Make the ansatz

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) \tag{3}$$

for the metric in flat space, and use the definitions of the Ricci tensor,

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha} \Gamma^\alpha{}_{\mu\nu} - \frac{\partial}{\partial x^\mu} \Gamma^\alpha{}_{\alpha\nu} + \Gamma^\alpha{}_{\mu\nu} \Gamma^\beta{}_{\alpha\beta} - \Gamma^\alpha{}_{\beta\nu} \Gamma^\beta{}_{\alpha\mu}\,, \tag{4}$$

and of the Christoffel symbols,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \left(\frac{\partial}{\partial x^a} g_{bd} + \frac{\partial}{\partial x^b} g_{ad} - \frac{\partial}{\partial x^d} g_{ab} \right). \tag{5}$$

2. **Friedmann-Robertson-Walker metric in curved spacetime**
   (a) If we allow for curved spacetime, the FRW metric eq. (3) can be generalized to

$$ds^2 = dt^2 - R^2(t) \begin{cases} d\psi^2 + \sin^2\psi (d\theta^2 + \sin^2\theta d\phi^2) & \text{closed Universe} \\ d\psi^2 + \psi^2 (d\theta^2 + \sin^2\theta d\phi^2) & \text{flat Universe} \\ d\psi^2 + \sinh^2\psi (d\theta^2 + \sin^2\theta d\phi^2) & \text{open Universe} \end{cases}. \tag{6}$$

   The three cases correspond to generalized spherical coordinates for a 3-sphere, a flat Universe, and a 3-hyperboloid, respectively. Show that this metric can be written in the more compact form

$$ds^2 = dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \big[d\theta^2 + \sin^2\theta\, d\phi^2\big] \right), \tag{7}$$

   with $k = 1$ for a closed Universe, $k = 0$ for a flat Universe, and $k = -1$ for an open Universe.

(b) In curved spacetime, the Friedmann equation takes the form

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho(t)\,. \tag{8}$$

Consider a "curvature dominated" Universe, i.e. a Universe with $\rho \simeq 0$, but $k \neq 0$. Is such a Universe, open, closed, or flat? How does R scale with t in a curvature dominated Universe? What is the equation of state parameter w for a curvature dominated Universe?

Two things are infinite: the Universe and human stupidity.
And I'm not sure about the Universe.
attributed to Albert Einstein

(c) A "particle horizon" is the boundary between the region of space that is in causal contact with an observer at a given point P and the region that is not. In other words, given the finite age of the Universe, light signals can only reach the observer if they were emitted within the horizon. Using the metric from eq. (7) and the solutions to the Friedmann equation from the lecture, show that particle horizons exist for all three spatial geometries ($k = 1$, $k = 0$, $k = -1$) in the case of matter or radiation domination.

(d) On large scales, the visible Universe today looks homogeneous and isotropic. If we assume that the very early Universe was not born in such a homogeneous state, what does this observation imply for particle horizons and the evolution of the very early Universe? In particular, can the homogeneity and isotropy be understood in a matter- or radiation-dominated cosmology?