Quintessence: The fifth and highest element in ancient and medieval philosophy that permeates all nature and is the substance composing the celestial bodies
Merriam Webster Online Dictionary

1. **Quintessence**
In this problem, we investigate the dynamics of so-called quintessence models, which attempt to explain dark energy in a dynamic way, without the need for a cosmological constant. The basic ingredient is a hypothetical real scalar field ϕ rolling down in a potential $V(\phi)$. The equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,. \tag{1}$$

Assuming spatial homogeneity, the components of the energy-momentum tensor are

$$\begin{aligned} \rho_\phi &= \tfrac{1}{2}\dot{\phi} + V(\phi)\,, \\ p_\phi &= \tfrac{1}{2}\dot{\phi} - V(\phi)\,. \end{aligned} \tag{2}$$

(a) Assume a flat Universe and an exponential potential of the form

$$V(\phi) = \bar{M}^4 \exp(-\lambda\phi/\bar{M})\,, \tag{3}$$

where λ is a free parameter and the reduced Planck mass $\bar{M}$ is defined by $\bar{M}^2 = 3/(8\pi G)$. Show that, for this potential, $\Omega_\phi \equiv 8\pi G\rho_\phi/3H^2$ and $w_\phi \equiv p_\phi/\rho_\phi$ satisfy the differential equations

$$\begin{aligned} \frac{d\Omega_\phi}{d\tau} &= -3\Omega_\phi(1-\Omega_\phi)(w_\phi - w_B)\,, \\ \frac{dw_\phi}{d\tau} &= -3(1-w_\phi^2) + \lambda(1-w_\phi)\sqrt{(1+w_\phi)\Omega_\phi}\,, \end{aligned} \tag{4}$$

where w_B is the equation of state parameter of the ordinary matter (or radiation) in the Universe and $\tau \equiv \ln(R/R_0)$. (Here, R is the usual scale factor and R_0 is the scale factor at some arbitrary but fixed reference time t_0.)
Hint: You will need the Friedmann equation, the Friedmann-Lemaître equation and the equation of motion (1).

(b) Find the nullclines of the system of differential equations (4), i.e. the trajectories in Ω_ϕ–w_ϕ space for which either $d\Omega_\phi/d\tau = 0$ or $dw_\phi/d\tau = 0$. Draw the nullclines for the parameter sets $\lambda = 5$, $w_B = 1/3$ and $\lambda = 2$, $w_B = 1/3$.

(c) Consider the fixed points at

$$(i) \qquad \Omega_\phi = (3/\lambda)^2(1+w_\phi) \qquad w_\phi = w_B \tag{5}$$

and at

$$(ii) \qquad \Omega_\phi = 1 \qquad w_\phi = (\lambda/3)^2 - 1 \tag{6}$$

Under what conditions do these fixed points exist within the physical region $0 \le \Omega_\phi \le 1$, $-1 \le w_\phi \le 1$? Convince yourself that in those cases where only one of the fixed points lies within the physical region, that fixed point is not repulsive. Demonstrate that it is in fact attractive by numerically solving eq. (4) in your favorite computer algebra system for a number of starting points and plotting the result.

(d) Discuss the physical implications of this quintessence model for the case where fixed point (i) is the global attractor and for the case where fixed point (ii) is the global attractor. Explain why each of these cases provides a phenomenologically viable model only during part of the evolution of the Universe. Take into account that observations require that the dark energy density at the time of Big Bang Nucleosynthesis (BBN), $\Omega_\phi^{\rm BBN}$, was much smaller ($\Omega_\phi^{\rm BBN} < 0.2$) than the dark energy density today ($\Omega_\phi^{\rm today} \sim 0.7$). Assume for simplicity that the matter content of the Universe was radiation-dominated throughout its history.

Discuss why an attractor model is nevertheless an interesting idea, solving the "cosmological smallness problem", i.e. the question why the cosmological "constant" we observe today is so small compared to other fundamental scales in physics, in particular $\bar{M}$, and the "coincidence problem" i.e. the question why the energy densities of matter and dark energy are similar today.

(e) Discuss qualitatively why a model with a double exponential potential of the form

$$V(\phi) = \bar{M}^4 \left[\exp\left(-\lambda_1 \frac{\phi}{\bar{M}} \right) + \exp\left(-\lambda_2 \frac{\phi + B}{\bar{M}} \right) \right], \tag{7}$$

with λ_1, λ_2 dimensionless parameters and B a parameter of mass dimension 1, can lead to a model that is phenomenologically viable throughout the whole history of the Universe.