What we observe as material bodies and forces
are nothing but shapes and variations
in the structure of space.
Erwiin Schrödinger

1. **Cosmic structure formation in perturbation theory**

In this problem, we develop the formalism of cosmic structure formation in the regime where density fluctuations are small. The relevant equations are (see lecture)

$$\frac{d\rho}{dt} + \vec{\nabla}(\rho\vec{v}) = 0, , \qquad \text{Continuity equation} \qquad (1)$$

$$\frac{d\vec{v}}{dt} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{1}{\rho}\nabla p + \vec{\nabla}\phi = 0\,, \qquad \text{Euler equation} \qquad (2)$$

$$\Delta\phi = 4\pi G\rho\,. \qquad \text{Poisson equation} \qquad (3)$$

Here, $\rho(\vec{r},t)$ is the matter density, $\vec{v}(\vec{r},t)$ is the velocity field, $p(\vec{r},t)$ is the pressure and ϕ is the gravitational potential.

(a) Show that in an expanding Universe, a solution to eqs. (1)–(3) is given by

$$\bar{\rho}(t) = \frac{\rho_0}{R^3(t)}\,, \qquad \bar{\vec{v}} = \frac{\dot{R}(t)}{R(t)}\vec{r}, \qquad \nabla\bar{\phi}_0 = \frac{4}{3}\pi G\rho_0\vec{r}\,. \qquad (4)$$

Here, ρ_0 is a constant, $R(t)$ is the usual scale factor of the Universe, and $\vec{r}$. Note that $\vec{r}$ is the *physical* coordinate (as opposed to the comoving coordinate $\vec{x} = \vec{r}/R(t)$ that is unaffected by cosmic expansion.)

(b) Consider small perturbations around the zeroth-order solution given by eqs. (1)–(3):

$$\rho(\vec{r},t) = \bar{\rho}(t) + \rho_1(\vec{r},t)\,, \qquad (5)$$

$$p(\vec{r},t) = \bar{p}(t) + p_1(\vec{r},t)\,, \qquad (6)$$

$$\vec{v}(\vec{r},t) = \bar{\vec{v}}(t) + \vec{v}_1(\vec{r},t)\,, \qquad (7)$$

$$\phi(\vec{r},t) = \bar{\phi}(t) + \phi_1(\vec{r},t)\,, \qquad (8)$$

where we assume $\rho_1 \ll \bar{\rho}$, $p_1 \ll \bar{p}$, $\vec{v}_1 \ll \bar{\vec{v}}$, $\phi_1 \ll \bar{\phi}$. Derive a system of differential equations for ρ_1, $\vec{v}_1$, ϕ_1 from (1)–(3).
Hint: To eliminate the p_1 from the equations, use the fact that the speed of sound v_s (which can be treated as a constant here) can be written as $v_s^2 = \partial p/\partial\rho|_{\text{adiabatic}} \simeq p_1/\rho_1$.

(c) To solve the equations derived in part (b), transform them to comoving coordinates $\vec{x} = \vec{r}/R(t)$ and use the definition

$$\delta(\vec{x},t) \equiv \rho_1(\vec{x},t)/\bar{\rho}(\vec{x},t)\,. \qquad (9)$$

Show that $\delta(\vec{x},r)$ satisfies the second order differential equation

$$\ddot{\delta} + 2H\dot{\delta} - \frac{v_s^2}{R^2}\Delta_{\vec{x}}\delta - 4\pi G\bar{\rho}\delta = 0\,. \qquad (10)$$

Here $\Delta_{\vec{x}}$ denotes the Laplace operator with respect to the comoving coordinate.

(d) Transform eq. (10) to Fourier space by writing

$$\delta(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\vec{x}} \delta_{\vec{k}}(t) \, . \tag{11}$$

Discuss the behavior of the Fourier modes $\delta_{\vec{k}}(t)$ in the following limiting cases

(a) A Universe without expansion ($H = 0$, $R = 1$)

(b) A matter-dominated Universe ($v_s = 0$) without curvature.