1. **East–west effect for charged cosmic rays**
   Argue that the flux of low-energy ($\lesssim$ few $\times$ 10 GeV) cosmic ray protons is larger from the west than from the east. Assume that cosmic rays are isotropic before they enter the Earth's magnetic field. What is the minimum energy a cosmic ray proton travelling in the (magnetic) equatorial plane must have in order to reach a detector on Earth?

2. **The GZK (Greisen-Zatsepin-Kuzmin) cutoff**
   Cosmic ray protons whose energy is sufficient to induce the reactions

$$p + \gamma_{\text{CMB}} \to p + \pi^0 \,, \qquad\qquad p + \gamma_{\text{CMB}} \to n + \pi^+ \tag{1}$$

   (inelastic scattering on CMB photon) loose energy very quickly and thus cannot reach the Earth. This is called the GZK (Greisen-Zatsepin-Kuzmin) cutoff.

   (a) What is the proton energy E_p at the cutoff? What velocity would a tennis ball ($m \sim 60$ g) of the same kinetic energy have?
   Hint: The average energy of CMB photons can be calculated from their temperature of 2.73 Kelvin. Assume for simplicity that all collisions are head-on.

   (b) What is the energy of the pions produced at rest in the reactions (1)?

   (c) Compute the maximum neutrino energy from the decay of these pions.

   Note: GZK neutrinos have not been detected yet, since their average flux on Earth is about 10 km^{-2} yr^{-1}, and their average interaction length is about 300 km.

3. **Dark matter capture in the Sun**
   When dark matter particles scatter on atomic nuclei in the Sun, they can loose enough energy to remain gravitationally bound to the Sun. Over astrophysical timescales, they can dissipate also their remaining kinetic energy through scattering, and eventually settle down in the core of the Sun. In the resulting dark matter cloud at the center of the Sun, dark matter annihilation is efficient, and if neutrinos are among the annihilation products, one expects a possibly observable flux of high-energy neutrinos from the Sun. (All other annihilation products will be absorbed in the Sun.)

   In the following, we derive an expression for the rate of dark matter capture in the Sun, following A. Gould, Astrophys. J. 321 (1987) 571.

   (a) Let $f(u)du$ be the (isotropic) DM velocity distribution far away from the Sun's gravitational field. (Here, $u \equiv |\vec{u}|$). Show that the number of DM particles entering a spherical region of radius R per unit time is given by

$$4\pi R^2 \cdot \frac{1}{4} n_\chi f(u) u\, du\, \frac{d(J^2)}{R^2 u^2 m_\chi^2}\,, \tag{2}$$

where J is the DM angular momentum, m_χ is the DM mass, and n_χ is the DM number density.

(b) Consider now a thin spherical mass shell of radius r and thickness dr, centered around the origin. Let $v_{\rm esc}(r)$ be the escape velocity of the shell, i.e. the velocity a particle located at radius r must have in order to escape the shell's gravitational field. We define $\Omega(v_{\rm esc}, w)$ as the probability per unit time for a dark matter particle of velocity w to scatter to a velocity $< v_{\rm esc}$ while travelling through the shell material. Show that the total DM capture rate of the shell is

$$dC = 4\pi r^2\, dr \int_0^\infty du\, n_\chi \frac{f(u)}{u} w\, \Omega\big(v_{\rm esc}, w\big)\,, \tag{3}$$

where $w = \sqrt{u^2 + v_{\rm esc}^2}$.

Hint: Show first that the time a DM particle spends inside the shell material is

$$\frac{2}{w} \frac{1}{\sqrt{1 - J^2/(m_\chi^2 r^2 w^2)}}\, dr\, \theta(r w m_\chi - J)\,. \tag{4}$$

Here $\theta(\dots)$ denotes the Heaviside step function.

(c) Next, we derive an expression for $\Omega(v_{\rm esc}, w)$. Compute the maximum fractional kinetic energy loss $\Delta E/E$ of a DM particle of mass m_χ scattering elastically on a nucleus of mass m_N, and the minimum fractional kinetic energy loss required for a DM particle of velocity w to scatter to a velocity $< v_{\rm esc}$. (Work in the nonrelativistic approximation).

(d) Using that kinetic energy losses $\Delta E/E$ in DM scattering are equally distributed between zero and the maximum value, show that

$$\Omega\big(v_{\rm esc}, w\big) = \frac{\sigma\, n_N}{w} \left(v_{\rm esc}^2 - \frac{(m_\chi - m_N)^2}{4 m_\chi m_N} u^2 \right) \theta\left(v_{\rm esc}^2 - \frac{(m_\chi - m_N)^2}{4 m_\chi m_N} u^2 \right). \tag{5}$$

Here σ is the DM–nucleus scattering cross section and n_N is the number density of nuclei.

(e) Assume the DM velocity follows a Maxwell-Boltzmann distribution

$$f(u) = \frac{4}{\bar{u}^3 \sqrt{\pi}} u^2 \exp(-u^2/\bar{u}^2)\,. \tag{6}$$

Derive an expression for the DM capture rate in the Sun per unit time, $C = \int dC$, neglecting the proper motion of the Sun relative to the galactic rest frame. (You do not need to evaluate the integrals analytically.)

(f) Estimate the total mass of the DM particles the Sun has captures during the 4.6×10^9 yrs of its existence. Compare the result to the total mass of the Sun. Assume $m_\chi = 100$ GeV, $n_\chi m_\chi = 0.3$ GeV/cm^3, $\bar{v} = 220$ km/sec, and $\sigma = 10^{-45}$ cm^2. You may neglect scattering on nuclei other than hydrogen here, but note that it can increase the capture rate by about two orders of magnitude! Assume that the solar mass was constant throughout its history, and that the Sun has a constant density.