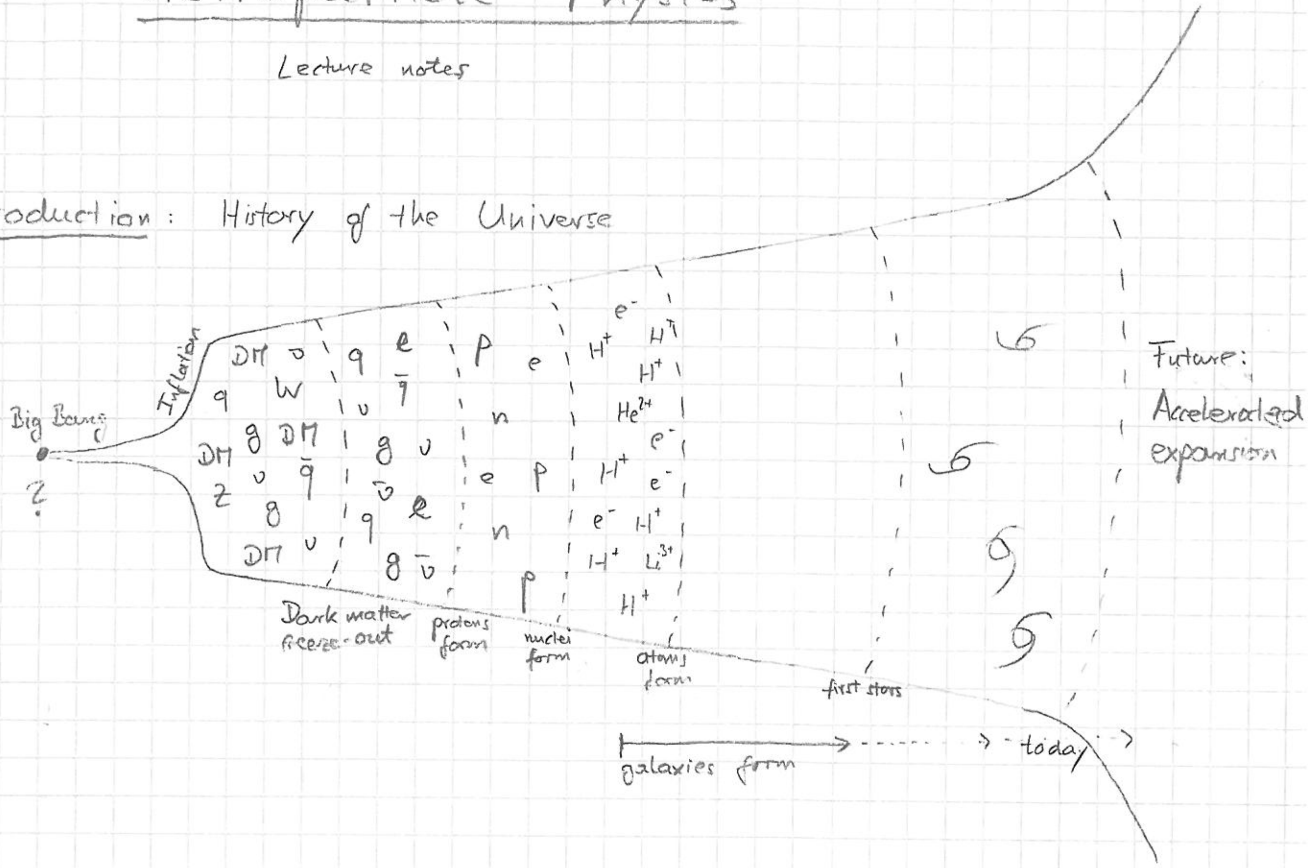
Astroparticle Physics

Lecture notes

1. Introduction: History of the Universe



E [GeV]	10^{2}	10^{-1}	10^{-4}	$3 \cdot 10^{-10}$	10^{-12}	2.3×10^{-13}
T [K]	10^{15}	10^{12}	10^{9}	3000	15	2.73
t [s, yrs]	10^{-10} s	10^{-5} s	10^{2} s	3×10^{5} yrs	10^{9} yrs	14×10^{9} yrs

Natural units

$$\hbar = c = 1$$

$$\Rightarrow \hbar \cdot c = 6.58 \cdot 10^{-16}\,\text{eV} \cdot \text{s} \times 3 \cdot 10^{8}\,\frac{\text{m}}{\text{s}} = 197 \cdot 10^{-9}\,\text{eV} \cdot \text{m}$$

$$\Rightarrow \boxed{1\,\text{m} = 5.066 \cdot 10^{6}\,\text{eV}^{-1}}$$

$$6.58 \cdot 10^{-16}\,\text{eVs} = 1 \quad \Rightarrow \boxed{1\,\text{s} = 1.52 \cdot 10^{15}\,\text{eV}^{-1}}.$$

Literature

E. Kolb, M. Turner: The Early Universe
Westview Press, 1994, ISBN 0-201-62674-8
(Textbook on cosmology, relatively old, but still ~~the~~ standard book)

M. Peskin, D. Schroeder: An Introduction to Quantum Field Theory
Westview Press, 1995, ISBN 0-201-50397-2
(Practitioner's introduction to QFT, Feynman calculus, and the Standard Model)

R. Wald: General Relativity
The University of Chicago Press, 1984, ISBN 0-226-87033-2
(Standard textbook on GR)

[Conventions: Lindner, Kolb-Turner
(Wald: different metric)]

2. Big bang theory

Einstein's equations:

$$\underbrace{\overset{\text{Ricci tensor}}{R_{\mu\nu}} - \frac{1}{2} g_{\mu\nu} \overset{\text{Ricci scalar}}{R} - \overset{\text{cosmological constant} \simeq 0}{\Lambda} \overset{\text{metric}}{g_{\mu\nu}}}_{\text{geometry}} = \underbrace{8\pi G \overset{\text{energy-momentum tensor}}{T_{\mu\nu}}}_{\text{matter}}$$

Metric:

$$(g_{\mu\nu}) = \underbrace{\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}}_{=\eta_{\mu\nu}\text{ flat metric (Minkowski metric) "normal coordinates"}} + \underbrace{\delta g_{\mu\nu}}_{\text{curvature}}$$

Riemann curvature tensor

$$\frac{\partial g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right)$$

$$\Leftrightarrow \quad g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \dots \quad \leftarrow$$

for small x

[$x_k = g_{kj}\, x^j$

$\delta_{ki} = g_{kj,i}\, x^j + g_{ki}$

$0 = g_{kj,il}\, x^j + g_{kl,i} + g_{ki,l}$

$0 \overset{\text{normal coord.}}{=} g_{km,il} + g_{kl,im} + g_{ki,lm}$

$= g_{km,il} + g_{ki,lm} + g_{kl,mi}$ (*

Use $\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left(g_{bd,a} + g_{ad,b} - g_{ab,d}\right)$ to express $R_{\mu\nu\rho}{}^{\sigma}$ in terms of the metric. This requires the identity

$R_{\mu\nu\rho}{}^{\sigma} = \Gamma^\sigma{}_{\mu\rho,\nu} - \Gamma^\sigma{}_{\nu\rho,\mu}$

Then use (*), together with $g_{\alpha\beta,\mu\nu} = g_{\mu\nu,\alpha\beta}$ (which can be shown using (*))]

Ricci tensor:

$$R_{\mu\nu} \equiv R_{\mu\alpha\nu}{}^{\alpha} \qquad \left[\text{"} \hat{=}\ \overset{\text{Laplace}}{\Delta} g_{\mu\nu} \text{"}\right]$$

Ricci scalar:

$$R = R^\mu{}_\mu$$

In cosmology, we observe:
- Universe looks homogeneous at large scales
- No preferred direction (isotropy)
- at large scales, only gravity matters

$$\Rightarrow (T^\mu{}_\nu) = \mathrm{diag}\big(\underset{\uparrow\ \text{energy density}}{\rho(t)},\ p(t),\ p(t),\ \overset{\text{pressure}}{p(t)}\big)$$

Space-like components:
force = $\partial\vec{p}/\partial t$
pressure = force per area = $\partial\vec{p}/\partial^3 x$
Note: KG field: $\partial_\mu \phi \Rightarrow p_\mu \phi$
consider Fourier-expansion of ϕ, note $\frac{1}{\sqrt{E_k}}$ in denominator

Motivate ansatz (in flat spacetime)

$$ds^2 = dt^2 - R^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

$\Rightarrow$ (00) component of Einstein equation ($\Lambda = 0$):

$$\boxed{\left(\underbrace{\frac{\dot{R}}{R}}_{\text{Hubble function}}\right)^2 + \overset{=0 \text{ in flat spacetime}}{\frac{k}{R^2}} = \frac{8\pi G}{3}\, \rho(t) \quad \text{Friedmann equation}} \qquad (1)$$

[Homework: Derive!]
[Curved spacetime: Homework]

$$\boxed{\text{value today}: \quad \frac{\dot{R}(t_0)}{R(t_0)} \equiv H_0 \quad (\text{Hubble constant}) \\ \equiv \underset{\sim 74.3 \pm 1.5}{h_0} \cdot 1\ \mathrm{km\ s^{-1}\ Mpc^{-1}}}$$

(ii) components: $$\boxed{2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p} \qquad (2)$$

Classical "derivation" of (1):

Expansion of sphere:

$$r = R(t) \cdot r_0$$

$$v = \dot{r} = \dot{R} \cdot \frac{r}{R}$$

Energy conservation:

$$\frac{1}{2} \cancel{m} v^2 - G \frac{\cancel{m} M}{r} = \text{const} \equiv -\frac{k}{2} m r_0^2$$

$$\left(\frac{\dot{R}}{R}\right)^2 \frac{r^2}{2} - G \cdot \frac{\frac{4}{3}\pi r^3 \rho}{r} = -\frac{k}{2} r_0^2$$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8}{3}\pi G \rho$$

Caution: Correct result, but not the real situation

Behavior of ρ:

$$\frac{d}{dt}\left[R^2 (1)\right] \Rightarrow 2\dot{R}\ddot{R} = \frac{8\pi G}{3}\left(\dot{\rho} R^2 + 2\rho R \dot{R}\right) \qquad (3)$$

$$\frac{(2)-(1)}{2} \Rightarrow \boxed{\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G \left(\rho + 3p\right)} \quad \text{Friedmann-Lemaître equation} \qquad (4)$$

$$(4) \text{ in } (3) \Rightarrow -\cancel{2}\dot{R} \cdot \frac{4}{\cancel{3}\pi} \cancel{G} \cancel{R} (\rho + 3p) = \frac{8\pi \cancel{G}}{\cancel{3}}\left(\dot{\rho} R^{\cancel{2}} + 2\rho \cancel{R} \dot{R}\right)$$

$$\Leftrightarrow 3\dot{R}\rho + 3\dot{R}p + R\dot{\rho} = 0$$

We write $\boxed{p(t) = \underset{\text{Equation of state parameter}}{w}\ \rho(t)}$

$$\hookrightarrow \quad 3\dot{R}\rho(1+w) = -R\dot{\rho}$$

$$\boxed{\frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R}(1+w)} \qquad (5)$$

$$\Leftrightarrow \frac{d}{dt}\log\rho = -3(1+w)\frac{d}{dt}\log R \qquad \Big| \int dt;\ \exp$$

$$\boxed{\rho = c\, R^{-3(1+w)}} \qquad (6)$$

end 18.04.2013

Insert into Friedmann eq. (1) for flat Universe:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8}{3}\pi G c \, R^{-3(1+\omega)}$$

$$R^{\frac{3\omega+1}{2}} \, dR = \sqrt{\tfrac{8}{3}\pi G c} \; dt$$

$$\frac{2}{3(1+\omega)} R^{\frac{3(1+\omega)}{2}} = \sqrt{\tfrac{8}{3}\pi G c} \; t$$

$$\boxed{R \sim t^{\frac{2}{3(1+\omega)}}} \qquad (7)$$

Special cases:

$\omega = \frac{1}{3}$: Relativistic gas (radiation domination)
$p = \frac{1}{3}\rho$
$\rho \sim R^{-4}$ $\qquad R \sim t^{1/2}$

$\omega = 0$: Nonrelativistic matter (matter domination)
$p = 0$
$\rho \sim R^{-3}$ $\qquad R \sim t^{2/3}$

$\omega = -1$: Vaccuum domination
$p = -\rho$
$\rho = \text{const.}$ $\qquad R \sim e^{Ht}$ (see sec. 3.3)

3. Inflation

3.1 The flatness problem

Wrong on blackboard on 25.04.2013

$(1) \Rightarrow$ For $k = 0$: $\rho = \frac{3H^2}{8\pi G} \equiv \rho_c$ (critical density)

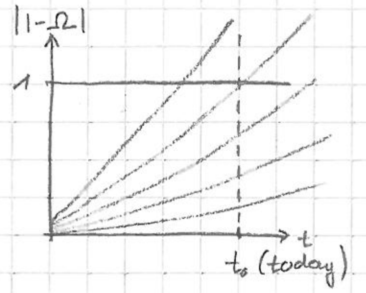
Today: $\rho_{c,0} \equiv \rho_c(t_0) \simeq 10^{-26}\,\text{kg m}^{-3} \simeq 5$ H atoms $/\text{m}^3$

$$\boxed{\begin{array}{c}\text{Definition: Dimensionless density} \\ \Omega \equiv \frac{\rho}{\rho_c}\end{array}}$$

Note: (6), (7) $\quad \frac{\rho}{\rho_c} \sim \frac{R^{-3(1+\omega)}}{\dot{R}^2/R^2} \sim \frac{t^{-3(1+\omega)\cdot\frac{2}{3(1+\omega)} + 2\frac{2}{3(1+\omega)}}}{t^{(\frac{2}{3(1+\omega)}-1)\cdot 2}} \sim t^0$ (→ self-consistency of assumption $k=0$!)

But: Allow for $k \neq 0$

$$\hookrightarrow \quad 1 - \Omega = -\frac{k}{\dot{R}^2} \overset{(7)}{\sim} \frac{1}{\left[t^{\frac{2}{3(1+w)}-1}\right]^2} \sim t^{\frac{2}{3}\frac{1+3w}{1+w}}$$



Extreme fine-tuning of initial value to get $1-\Omega \sim \mathcal{O}(1)$ today

Estimate: $T = 0.1$ MeV $(t \simeq 100$ s$)$; $w = \frac{1}{3}$ for most of history (on a log scale)

$$\Rightarrow 1 - \Omega \lesssim 10^{-15}$$

$T = 100$ GeV $(t \simeq 10^{-10}$ s$)$; $w = \frac{1}{3}$

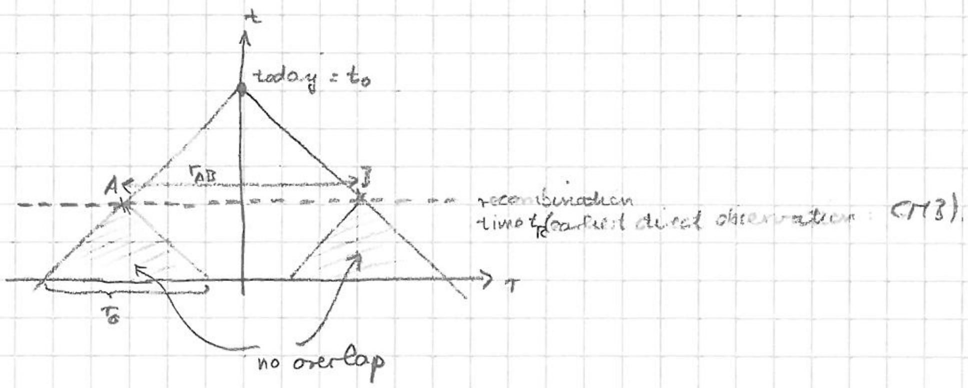
$$\Rightarrow 1 - \Omega \lesssim 10^{-27}$$

There is no reason why $1-\Omega$ should be so tiny

$\hookrightarrow$ $\boxed{\text{Flatness problem}}$

3.2 The horizon problem

Consider backward light cone



$$ds^2 = dt^2 - R^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \doteq 0 \quad \text{on the light cone}$$

$$\hookrightarrow \frac{1}{R(t)}dt = \frac{dr}{\sqrt{1-kr^2}}$$

In a flat Universe: $$\frac{r_{AB}}{r_0} = \frac{2\int_{t_R}^{t_0} \frac{dt}{R(t)}}{2\int_0^{t_R}\frac{dt}{R(t)}} \quad \begin{matrix} \leftarrow \text{matter dominated} \\ \leftarrow \text{radiation dominated} \end{matrix}$$

for correct dimension; comes from matching RD ↔ MD

$$\frac{r_{AB}}{r_0} \sim \frac{\int_{t_R}^{t_0} t^{-\frac{2}{3}}\, dt}{\int_0^{t_R} t^{-\frac{1}{2}}\, dt} \sim \frac{\left(t_0^{1/3} - t_R^{1/3}\right)\cdot t_R^{1/6}}{t_R^{1/2}} \gg 1$$

⇒ A and B were never in causal contact, still they "look the same" (large-scale homogeneity)

↳ $\boxed{\text{Horizon problem}}$

3.3 The cosmological constant Λ

Einstein equation with $\Lambda \neq 0$: $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$

Define $T_{\mu\nu}' = T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu}$

$$= \mathrm{diag}\left(\rho + \frac{\Lambda}{8\pi G} ; p - \frac{\Lambda}{8\pi G} ; p - \frac{\Lambda}{8\pi G} ; p - \frac{\Lambda}{8\pi G}\right)$$

Note: For vacuum-dominated Universe: $\omega_\Lambda = p/\rho = -1$

Modified Friedmann equation (cf. (1))

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

Modified Friedmann-Lemaître equation (cf. (4))

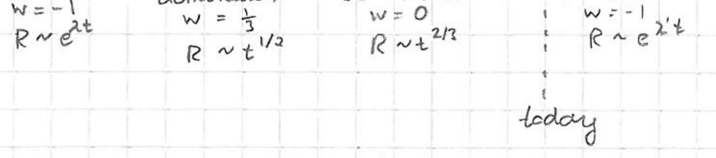
$$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G \underbrace{(\rho + 3p)}_{\substack{= \rho(1+3\omega_M) \\ > 0}} + \frac{\Lambda}{3}$$

- $\Lambda \ll 4\pi G \rho (1 + 3\omega_M)$ ⇒ as before, Λ negligible
- $\Lambda = 4\pi G \rho (1 + 3\omega_M)$ ⇒ no acceleration / deceleration
- $\Lambda \gg 4\pi G \rho (1 + 3\omega_M)$ ⇒ $\frac{\ddot{R}}{R} = \frac{\Lambda}{3}$; $\Lambda < 0$: oscillatory solution

$\Lambda > 0$: $\boxed{R = R_0 e^{\sqrt{\Lambda/3}\, t}}$

Inflation

Assume an epoch of inflation shortly after the Big Bang



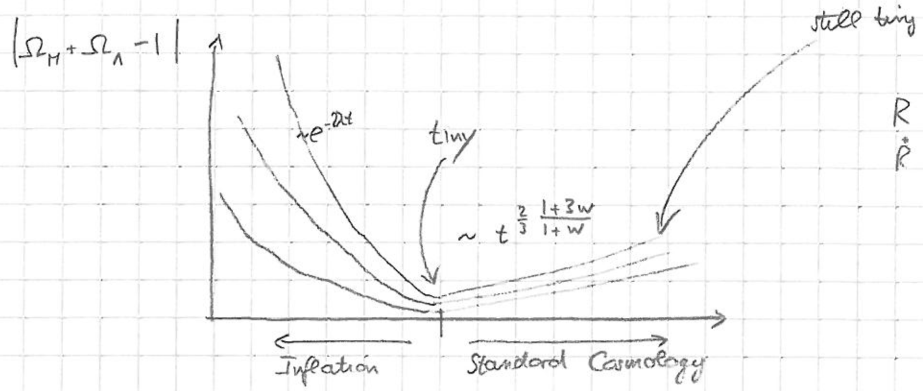
- Flatness problem solved:

Friedmann equation:

$$1 = -\underbrace{\frac{k}{R^2 \underbrace{H^2}_{=\frac{\dot{R}^2}{R^2}}}}_{\equiv \Omega_k} + \underbrace{\frac{8\pi G}{3H^2}\rho}_{\substack{\equiv \Omega_M \\ = \rho/\rho_c}} + \underbrace{\frac{\Lambda}{3H^2}}_{\equiv \Omega_\Lambda}$$

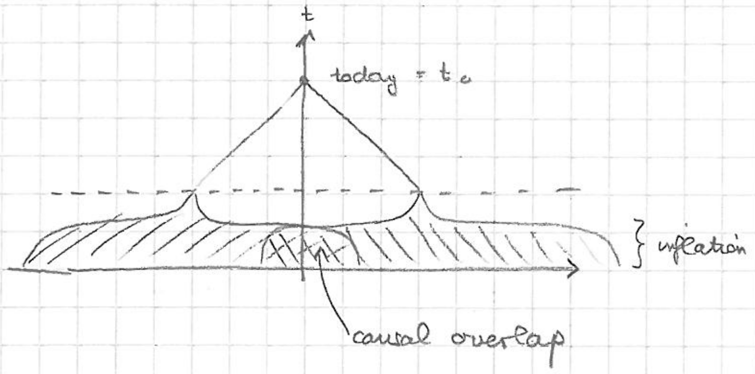
Inflation — $R \sim e^{\lambda t} \Rightarrow \Omega_k \sim e^{-2\lambda t} \to 0$

$\hookrightarrow \Omega_M + \Omega_\Lambda \to 1$



$$R \sim t^{\frac{2}{3(1+w)}}$$
$$\dot{R} \sim t^{\frac{2}{3(1+w)}-1}$$
$$\sim t^{\frac{-1-3w}{3(1+w)}}$$

- Horizon problem solved



3.4 Λ in QFT

Particles correspond to excitations of fields $A_\mu(x)$, $\psi(x)$, $\phi(x)$

e.g. scalar field: $\mathcal{L}(\phi(x), \partial_\mu\phi(x)) = (\partial_\mu\phi)^+ (\partial_\mu\phi) - V(\phi^+\phi)$

scalar potential $V(\phi^+\phi) = m^2\phi^+\phi + \frac{\lambda}{2}(\phi^+\phi)^2$

Action $S = \int d^4x\, \mathcal{L}$

Require $\delta S = 0 \Rightarrow \int d^4x \left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \underbrace{\delta(\partial_\mu\phi)}_{=\partial_\mu\delta\phi} + \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi\right] = 0$ | Partial integration fields 0 at boundaries

$$\Leftrightarrow \int d^4x \left[-\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} + \frac{\delta\mathcal{L}}{\delta\phi}\right]\delta\phi = 0$$

Requires $\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = \frac{\delta\mathcal{L}}{\delta\phi}$ (Euler Lagrange eq.) (ELE)

Scalar field: $\left[\partial_\mu\partial^\mu + m^2 + \lambda\phi^+\phi\right]\phi^+ = 0$

Question. How to combine with gravity?
Is there a Lagrangian which leads to Einstein's equation via the ELE?

Yes! $\mathcal{L} \equiv \mathcal{L}_{GR}(g_{\mu\nu}) + \mathcal{L}_{Matter}(g_{\mu\nu}, A_\mu, \psi, \phi, \dots)$

end 25.04.2013

$$\mathcal{L}_{GR} = -\frac{1}{16\pi G}\underbrace{\sqrt{g}}_{\equiv -\det g_{\mu\nu}}(R + 2\Lambda)$$

$$\left[\frac{\partial(\det A)}{\partial\alpha} = \det A\, \mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial\alpha}\right)\right.$$

$$\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}\cdot(-g)\, g^{\mu\nu}\delta g_{\mu\nu}$$

Reference: Wald & Wikipedia

$$\delta\mathcal{L}_{GR} = -\frac{1}{16\pi G}\left[\delta\sqrt{-g}\,(R + 2\Lambda) + \sqrt{-g}\,\delta(R_{\mu\nu}g^{\mu\nu})\right]$$

$$\overset{\text{see next page}}{=} -\frac{1}{16\pi G}\Big[(R + 2\Lambda)\tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} - \sqrt{-g}\,R^{\mu\nu}\delta g_{\mu\nu}$$

$$+ \sqrt{-g}\,\underbrace{g^{\mu\nu}\delta R_{\mu\nu}}\Big]$$

can be shown to yield tot
covariant derivative

To compute $\delta R^{\mu\nu}$ use $R_{\mu\nu\rho}{}^{\sigma} \overset{\text{Wald 3.4.4}}{=} \Gamma^{\sigma}_{\mu\rho,\nu} - \Gamma^{\sigma}_{\nu\rho,\mu} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\sigma}$

and definition of covariant derivative:

$$\nabla_\mu t^\nu \overset{\text{Wald 3.1.15}}{=} \partial_\mu t^\nu + \Gamma^{\nu}_{\mu\rho}t^\rho$$

$$\nabla_\mu t_\nu = \partial_\mu t_\nu - \Gamma^{\rho}_{\mu\nu}t_\rho$$

$$\begin{aligned}
\nabla_\mu t_\nu &= g_{\nu\rho}\nabla_\mu t^\rho \\
&= g_{\nu\rho}\partial_\mu t^\rho + \tfrac{1}{2}g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha}t^\alpha \\
&= \partial_\mu t_\nu - g_{\nu\rho,\mu}t^\rho + \tfrac{1}{2}g_\nu{}^\sigma(g_{\alpha\sigma,\mu} + g_{\mu\sigma,\alpha} - g_{\mu\alpha,\sigma})t^\alpha \\
&= \partial_\mu t_\nu + \tfrac{1}{2}\big(-2g_{\nu\alpha,\mu} + g_{\nu\alpha,\mu} - g_{\alpha\sigma,\mu}\underbrace{g_\nu{}^\sigma{}_{,\mu}}_{=0} \\
&\qquad + g_{\nu\mu,\alpha} - g_{\mu\alpha,\nu}\big)g^{\alpha\beta}t_\beta \\
&= \partial_\mu t_\nu - \tfrac{1}{2}g^{\alpha\beta}(g_{\nu\alpha,\mu} + g_{\mu\alpha,\nu} - g_{\mu\nu,\alpha})t_\beta \\
&= \partial_\mu t_\nu - \Gamma^{\beta}_{\mu\nu}t_\beta
\end{aligned}$$

$$\hookrightarrow \quad \delta R_{\mu\nu\rho}{}^{\sigma} = \delta\left(\partial_\nu \Gamma^\sigma{}_{\mu\rho}\right) - \delta\left(\partial_\mu \Gamma^\sigma{}_{\nu\rho}\right) + \delta\left(\Gamma^\alpha_{\mu\rho}\right)\Gamma^\sigma_{\alpha\nu}$$

$$+ \Gamma^\alpha{}_{\mu\rho}\,\delta\Gamma^\sigma{}_{\alpha\nu} - \delta\Gamma^\alpha{}_{\nu\rho}\cdot\Gamma^\sigma{}_{\alpha\mu} - \Gamma^\alpha{}_{\nu\rho}\,\delta\Gamma^\sigma{}_{\alpha\mu}$$

Note: $\Gamma^\sigma{}_{\mu\rho}$ is not a tensor, but $\delta\Gamma^\sigma{}_{\mu\rho}$ is because it can be written as (for some vector field t^ρ)

Note: $\Gamma^\alpha_{\mu\rho} = \Gamma^\alpha$

$$\delta\left(\Gamma^\sigma{}_{\mu\rho}\, t^\rho\right) = \delta\left(\nabla_\mu t^\sigma\right) - \partial_\mu \delta t^\sigma$$

$$\left(\delta\Gamma^\sigma_{\mu\rho}\right) t^\rho + \Gamma^\sigma_{\mu\rho}\delta t^\rho = \delta\left(\nabla_\mu t^\sigma\right) - \partial_\mu \delta t^\sigma$$

$$\left(\delta\Gamma^\sigma_{\mu\rho}\right) t^\rho = \underbrace{\delta\left(\nabla_\mu t^\sigma\right)}_{\text{covariant}} - \underbrace{\nabla_\mu \delta t^\sigma}_{\text{covariant}}$$

$\Rightarrow$ Can form covariant derivative

$$\nabla_\nu \delta\Gamma^\sigma{}_{\mu\rho} = \partial_\nu \delta\Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\nu\alpha}\delta\Gamma^\alpha{}_{\mu\rho}$$

$$- \Gamma^\alpha{}_{\nu\mu}\delta\Gamma^\sigma{}_{\alpha\rho} - \Gamma^\alpha{}_{\nu\rho}\delta\Gamma^\sigma{}_{\mu\alpha}$$

$$\Rightarrow \delta R_{\mu\nu\rho}{}^{\sigma} = \nabla_\nu \delta\Gamma^\sigma{}_{\mu\rho} + \cancel{\Gamma^\alpha{}_{\nu\mu}\,\delta\Gamma^\sigma{}_{\alpha\rho}} - \nabla_\mu\delta\Gamma^\sigma{}_{\nu\rho} - \cancel{\Gamma^\alpha{}_{\mu\nu}\,\delta\Gamma^\sigma{}_{\alpha\rho}}$$

$$\Rightarrow \delta R_{\mu\rho} = \delta R_{\mu\alpha\rho}{}^{\alpha} = \nabla_\alpha \delta\Gamma^\alpha{}_{\mu\rho} - \nabla_\mu \delta\Gamma^\alpha{}_{\alpha\rho}$$

$$\Rightarrow \delta R = \delta\left(R_{\mu\rho}\, g^{\mu\rho}\right) = \underbrace{\left(\nabla_\alpha \delta\Gamma^\alpha{}_{\mu\rho}\right) g^{\mu\rho} - \left(\nabla_\mu \delta\Gamma^\alpha{}_{\alpha\rho}\right) g^{\mu\rho}}_{g^{\mu\rho} \text{ can be pulled into } \nabla \text{ since } \nabla_\alpha g_{\mu\nu} = 0}$$

$$+ \underbrace{R_{\mu\rho}\,\delta g^{\mu\rho}}_{= -R^{\mu\rho}\,\delta g_{\mu\rho}}$$

In the action: $\int d^4x \overset{\text{can be pulled into } \nabla}{\widetilde{\sqrt{g}}}\nabla_\alpha t^\alpha = 0$ (boundary term)

← need derivation with lower index here, i.e. works only for $\delta R_{\mu\nu}$, not $\delta R^{\mu\nu}$

$$\Rightarrow \delta\mathcal{L}_{GR} = -\frac{1}{16\pi G}\left[(R + 2\Lambda)\frac{g^{\mu\nu}}{2} - R^{\mu\nu}\right]\sqrt{-g}\,\delta g_{\mu\nu}$$

$$\rightarrow \quad \mathcal{L}_{\text{Matter}} = \sqrt{-g}\,\tilde{\mathcal{L}} \overset{\text{scalar}}{\underset{\text{field}}{=}} \sqrt{-g}\left[(\nabla_\mu\phi)^\dagger(\nabla^\mu\phi) - V(\phi^\dagger\phi)\right]$$

$$\rightarrow \quad \delta\mathcal{L}_{\text{Matter}} = \tilde{\mathcal{L}} \cdot \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} + \sqrt{-g}\,\frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}$$ + field variation (neglected here)

Note: Taking variations in $\nabla_{\mu\nu}$ will double-count compared to the case where we vary $g_{\mu\nu}$

$$\delta(\mathcal{L}_{GR} + \mathcal{L}_{Matter}) = -\frac{1}{16\pi G}\left[(R + 2\Lambda)\frac{g^{\mu\nu}}{2} - R^{\mu\nu}\right] + \frac{1}{2}\tilde{\mathcal{L}} g^{\mu\nu} + \frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}$$

$$\Rightarrow \quad R^{\mu\nu} - \frac{1}{2}R g^{\mu\nu} = 16\pi G\left[\underbrace{-\frac{1}{2}\tilde{\mathcal{L}} g^{\mu\nu} - \frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}}_{\equiv \frac{T^{\mu\nu}}{2}} + \frac{\Lambda}{16\pi G} g^{\mu\nu}\right]$$

$$= 8\pi G T^{\mu\nu} + \Lambda g^{\mu\nu}$$ ... Einstein's equation

Assume scalar field and note $\nabla_\mu \phi = \partial_\mu \phi$

$$\tilde{\mathcal{L}} = (\partial_\mu \phi)^+ (\partial_\nu \phi) g^{\mu\nu} - V(\phi^+ \phi)$$

Note: The fundamental field is $\partial_\mu \phi$, not $\partial^\mu \phi$
→ have to include the $g^{\mu\nu}$ to pull the index up

$$\hookrightarrow T^{\mu\nu} = -\tilde{\mathcal{L}} g^{\mu\nu} - 2\frac{\delta\tilde{\mathcal{L}}}{\delta g_{\mu\nu}}$$

Approx const g?

$$= -(\partial_\rho \phi)^+ (\partial^\rho \phi) g^{\mu\nu} + V g^{\mu\nu} + 2\frac{\delta\tilde{\mathcal{L}}}{\delta g^{\alpha\beta}} g^{\mu\alpha} g^{\nu\beta}$$

$$\left[\delta A^{-1} = -A^{-1}\delta A A^{-1}\right.$$

$$= -(\partial_\rho \phi)^+ (\partial^\rho \phi) g^{\mu\nu} + 2(\partial^\mu \phi)^+ (\partial^\nu \phi) + V(\phi^+ \phi) g^{\mu\nu}$$

$$\rho = T^0{}_0 = T^{00} = (\partial_0 \phi)^+ (\partial_0 \phi) + \frac{1}{R^2}(\underset{=\partial_i}{\vec{\nabla}} \phi)^+ (\vec{\nabla}\phi) + V(\phi^+ \phi)$$

$$p = T^i{}_i = R^2 T^{ii} = R^2\left[\frac{1}{R^2}(\partial_0 \phi)^+ (\partial_0 \phi) - \frac{1}{R^4}(\vec{\nabla}\phi)^+ (\vec{\nabla}\phi) + 2\frac{1}{R^4}(\partial_i \phi)^+ (\partial_i \phi) - \frac{1}{R^2}V(\phi^+ \phi)\right]$$

(no summation)

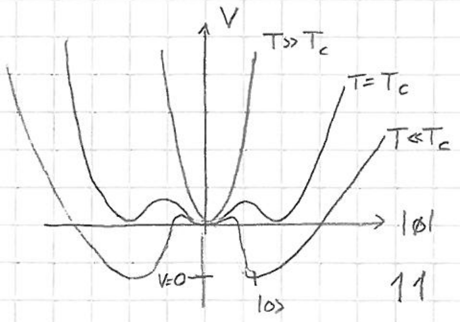
For nearly constant ϕ $(\partial^0 \phi \simeq 0;\ \partial^i \phi \simeq 0)$

$$\hookrightarrow \rho = V(\phi^+ \phi) = -p \quad \Rightarrow \quad \boxed{w = \frac{p}{\rho} = -1}$$

Behaves like cosmological constant!

Typical scalar potential:

$$V = -\mu^2 \phi^+\phi + \lambda(\phi^+\phi)^2 + \underbrace{\lambda(\phi^+\phi)^2 \ln \phi^+\phi}_{\text{small quantum correction}} + \underbrace{\overset{\text{temperature}}{T^2} \phi^+\phi}_{T \neq 0 \text{ correction}} + \ldots$$



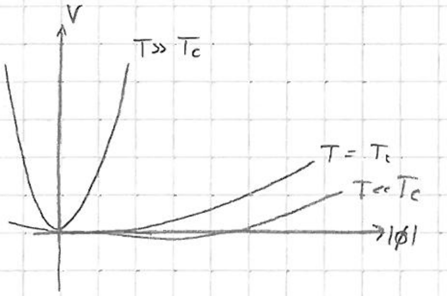
Phase transition at $T = T_c$

("Old inflation": Alan Guth, 1981)

- $V \gg 0$ → Universe inflates → at $T < T_c$, tunnelling into true vacuum begins → when tunnelling ends, inflation is over

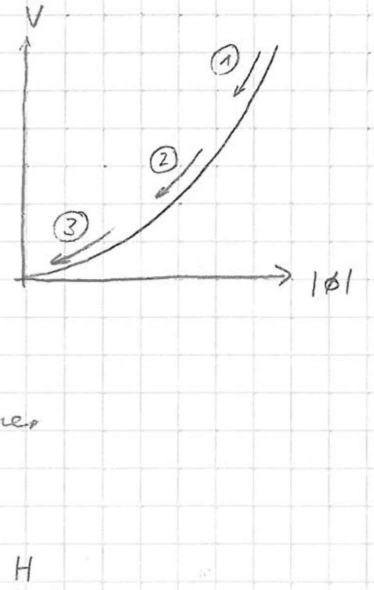
Problems: - Outside "true vacuum bubbles," inflation continues → may never end in parts of the Universe
- Universe almost empty after inflation
  → need mechanism to reheat
- Properties of different bubbles different (e.g. curvature)
  → expect a lot of inhomogeneities on large scales

- "New inflation" (Andrei Linde, 1982)



- "Slow roll" of field into true vacuum
- Eventually, $\partial^\mu \phi$ becomes non-negligible
  ↳ oscillations around true vacuum = inflaton particles
- Inflaton decay → produces lots of ordinary particles → "reheating"

- "Chaotic inflation"



- $|\phi|$ starts at random value (large)
- EoM: (assume $\vec{\nabla}\phi \simeq 0$):

$$\mathcal{D}_t \frac{\delta \mathcal{L}}{\delta(\mathcal{D}_t \phi^\dagger)} - \frac{\delta \mathcal{L}}{\delta \phi^\dagger} = 0$$

$$\mathcal{L} = \sqrt{-g}\left((\mathcal{D}^\mu \phi)^\dagger (\mathcal{D}_\mu \phi) - V \right)$$

see lecture notes Lindner for derivation

Covariant derivative: $\mathcal{D}_\mu \phi \equiv \partial_\mu \phi$

$$\mathcal{D}_\mu A^\mu \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu$$

① Very large H
↳ $\ddot{\phi}$ negligible
$\dot{\phi}$ small ($|3H\dot{\phi}| \simeq \frac{dV}{d\phi^\dagger}$)
→ slow roll

② V decreases, H decreases
Friction term $3H\dot{\phi}$ becomes less important

③ Friction negligible
$\ddot{\phi} + \frac{dV}{d\phi^\dagger} = 0$
→ oscillations around minimum
→ reheating

$$\Rightarrow \mathcal{D}_t \left(\sqrt{-g}\, \partial_t \phi \right) + \sqrt{-g} \frac{\partial V}{d\phi^\dagger} = 0$$

$$\frac{1}{\sqrt{-g}} \partial_t (-g \partial_t \phi) + \sqrt{-g} \frac{dV}{d\phi^\dagger} = 0$$

$$\sqrt{-g}\, \ddot{\phi} + \frac{1}{\sqrt{-g}} \dot{\phi} \cdot 3 R^2 \dot{R} + \sqrt{-g} \frac{dV}{d\phi^\dagger} = 0$$

$$(g_{\mu\nu}) = \left(\begin{matrix} 1 & & & \\ & -R(t) & & \\ & & -R(t) & \\ & & & -R(t \end{matrix} \right.$$

$$\Rightarrow g = -R^3$$

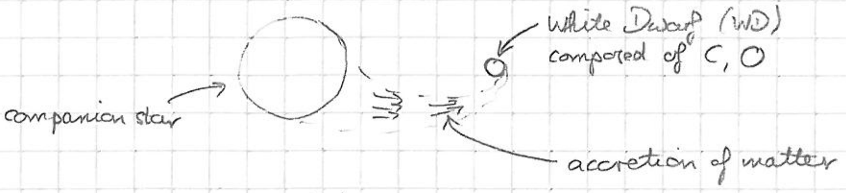
$$\boxed{\ddot{\phi} + 3 \underset{\overset{\uparrow}{= \frac{\dot{R}}{R}}}{H} \dot{\phi} + \frac{dV}{d\phi^\dagger} = 0}$$

4. Dark energy

1998 : High-z supernova search team (Adam Riess, Brian Schmidt et al.)
Supernova Cosmology Project (Saul Perlmutter et al.)

Observations of Type Ia SN at high redshift

- Type Ia SN:



Binary system

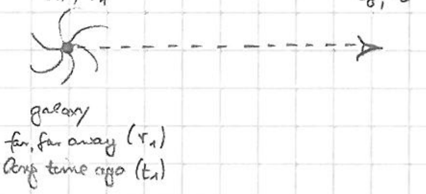
When WD mass reaches Chandrasekhar Limit ~1.4 $M_\odot$
T high enough to re-ignite fusion
=> runaway fusion, thermonuclear explosion, WD destroyed

Common progenitor mass → similar E-output in all Type Ia SN

→ $\boxed{\text{"Standard Candles"}}$

- Redshift

Distant source emits light



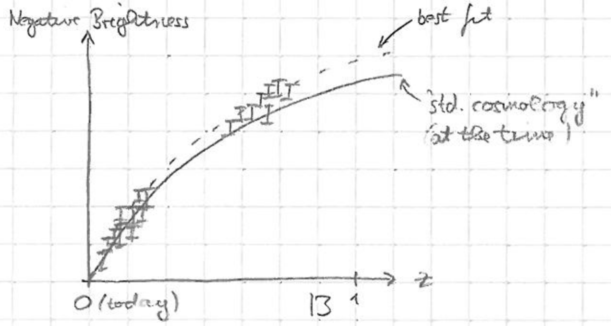
Distance between wave crests: $\lambda_1 = a(t_1)\,dr$

Today: $\lambda_0 = ~~R~~ a(t_0)\,dr$

Redshift: $\frac{\lambda_0 - \lambda_1}{\lambda_1} = \boxed{\frac{a(t_0)}{a(t_1)} - 1 \equiv z}$

Here was a
typo on 02.05.201

- Observation



Two measurements of distance disagree!

Hypothesis: $a(t_1)$ was smaller than assumed

$\boxed{\Rightarrow \text{Expansion accelerating}}$

end 02.05.2013

Remember: Matter domination : $R \sim t^{2/3}$; $\dot{R} \sim t^{-\frac{1}{3}}$; $\dot{R} \sim -t^{-\frac{4}{3}} < 0$:-(

Radiation domination : $R \sim t^{1/2}$; $\dot{R} \sim t^{-\frac{1}{2}}$; $\ddot{R} \sim -t^{-\frac{3}{2}} < 0$:-(

Vacuum domination : $R \sim e^{\lambda t}$; $\dot{R} \sim e^{\lambda t}$; $R \sim e^{\lambda t} > 0$:-)
(Λ or inflation)

A new form of vacuum energy : c.c. Λ or scalar potential of new field ϕ

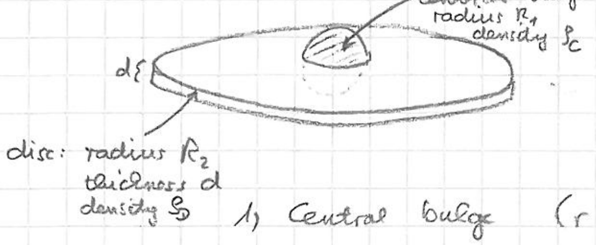
$\rightarrow$ $\boxed{\text{Dark Energy}}$

Alternative : Einstein equations wrong, GR modified

5. Dark Matter

5.1 The path to the dark side: Evidence for DM

- Jan Oorth (1932), Vera Rubin (1970): <u>Galaxy rotation curves</u> (v vs. r diagram



Visible stars → estimate mass distribution
→ compute force on test body at distance r,
mass m → compute <u>orbital velocity</u> v
(for stable orbit)

1) Central bulge ($r \leq R_1$):

bulge density

$$F = \frac{G m M(r)}{r^2} \overset{!}{=} m \frac{v^2}{r} \quad ; \quad M(r) = \frac{4}{3}\pi r^3 \rho_C$$

$$\Rightarrow \boxed{v = \sqrt{\frac{4}{3}\pi G \rho_C}\, r}$$

~~2) Disc ($R_1 \leq r \leq R_2$):~~

Disc cannot be treated as point mass !

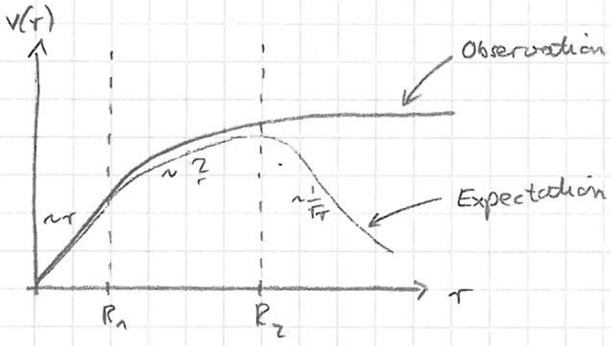
~~$M(r) = \frac{4}{3}\pi R_1^3 \rho_C + \pi r^2 d \rho_D$~~

~~$\Rightarrow v = \sqrt{\frac{G}{r}\cdot\left(\frac{4}{3}\pi R_1^3 \rho_C + \pi r^2 d \rho_D\right)} \xrightarrow{r \text{ large}} \boxed{\sqrt{\pi G d \rho_D}\cdot\sqrt{r}}$~~

3) Outside the disc ($r \gg R_2$)

$$M(r) = \frac{4}{3}\pi R_1^3 \rho_C + \pi R_2^2 d \rho_D \equiv M_0$$

$$v = \sqrt{G \cdot M_0} \cdot \frac{1}{\sqrt{r}}$$



Observation:
Flat rotation curves
→ Invisible (dark?) matter

$$v(r) \sim \text{const} \sim \sqrt{\frac{G M(r)}{r}}$$

$$\hookrightarrow M(r) \sim r$$

Spherical DM halo: $M(r) = 4\pi \int_0^r \rho(r) r'^2$

$$\hookrightarrow \boxed{\rho(r) \sim \frac{1}{r^2}}$$

~~DM disc: $M(r) = 2\pi d \int_0^r \rho(r)\, r\, d$~~

~~$\hookrightarrow \boxed{\rho(r) \sim \frac{1}{r}}$~~

- Fritz Zwicky, 1933 : Dynamics of galaxy clusters

Virial theorem : $\boxed{\underbrace{\sum}_{\text{sum over all galaxies in the cluster}} \underbrace{E_{kin}}_{\text{from Doppler shifts}} = -\frac{1}{2}\sum \underbrace{E_{pot}}_{\text{from brightness } + V = \frac{GMm}{r}}}$

Observation : $\boxed{\sum E_{kin} \cong -\frac{1}{2} \cdot 170 \sum E_{pot}}$

↳ More gravitational pull than explained by luminous matter

- The cosmic microwave background (CMB)

[show WMAP or Planck CMB skymap]

At $t_{rec} \simeq 300\,000$ yrs : e^- and p^+ recombine to H $\Rightarrow$ Universe becomes transparent

↳ Photons emitted at t_{rec} observable today

$$T(t_{rec}) \cong 3000K \xrightarrow{\text{redshift } z \simeq 1100} \boxed{T(t_0) = 2.73 \text{ K}}$$

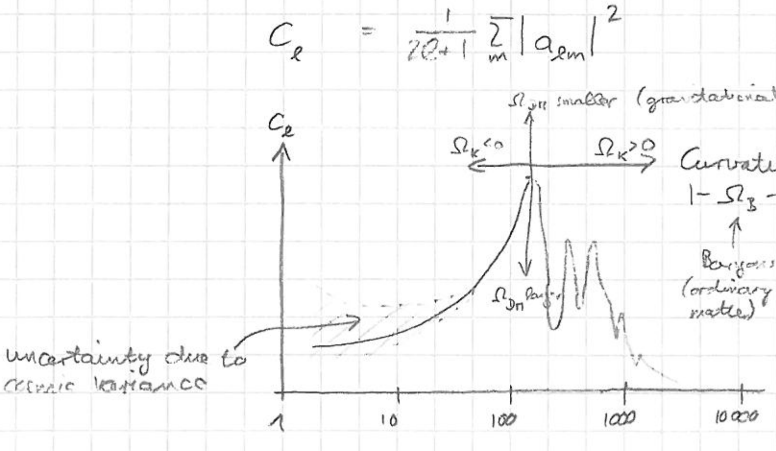
Tiny fluctuations of T from primordial quantum fluctuations; strongest at eigenfrequencies of the plasma

↳ CMB Power spectrum

$$T(\underbrace{\theta, \phi}_{\text{galactic coordinates}}) = \sum_{\ell m} a_{\ell m} \overset{\text{spherical harmonic } \left[\sim e^{im\phi} \overset{\text{associated legendre function}}{P_\ell^m}(\cos\theta)\right]}{Y_{\ell m}}(\theta, \phi)$$

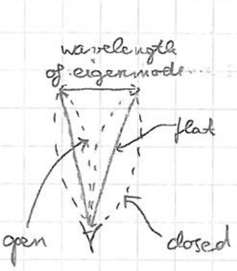
$$a_{\ell m} = \int d(\cos\theta)\, d\phi \;\; Y^*_{\ell m}(\theta, \phi)\; T(\theta, \phi)$$

$$C_\ell = \frac{1}{2\ell+1} \sum_m |a_{\ell m}|^2$$



Curvature

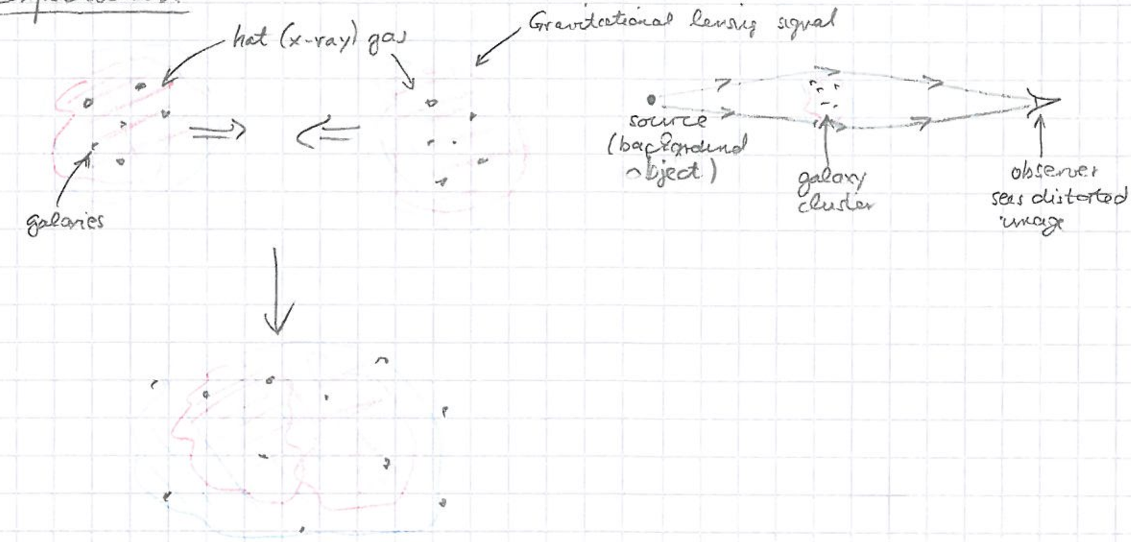
$$1 - \underset{\substack{\uparrow \\ \text{Baryons} \\ \text{(ordinary matter)}}}{\Omega_B} - \underset{\substack{\uparrow \\ \text{Pressureless} \\ \text{matter}}}{\Omega_{DM}} - \underset{\substack{\uparrow \\ \text{cosmological} \\ \text{constant}}}{\Omega_\Lambda} \equiv \Omega_k = -\frac{k}{R^2}$$



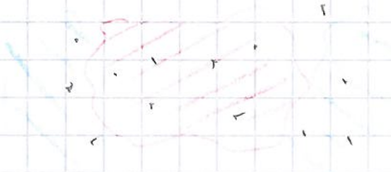
- **Collisions of galaxy clusters**

  [show bullet cluster]

  Expectation



  Observation:



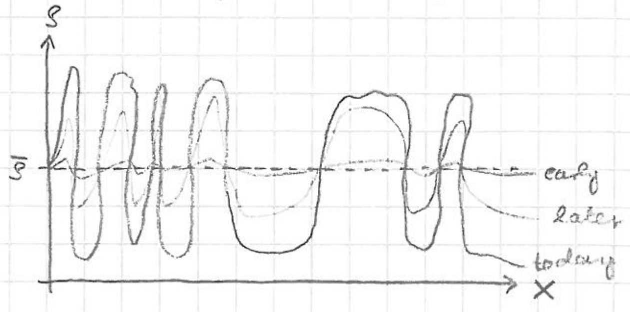
  Large separation of mass distributions after collision

  ↓

  a lot of non-interacting matter in galaxy clusters

- Structure formation

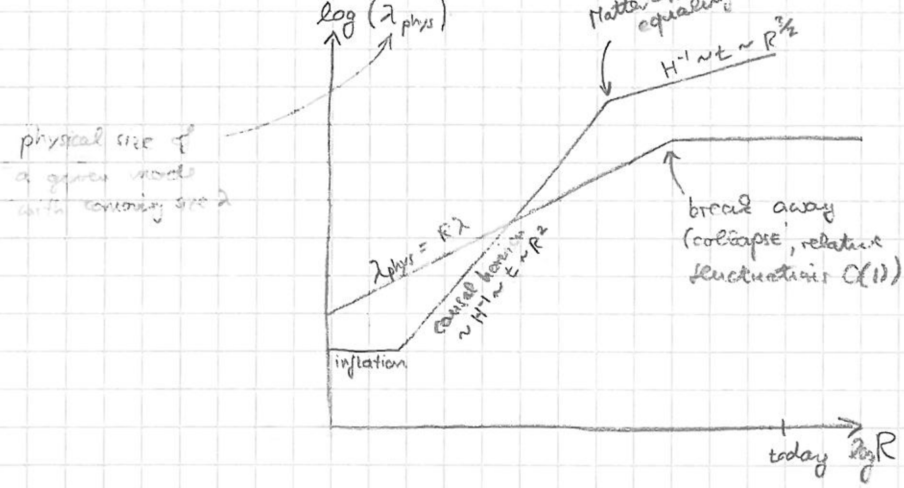
Jeans instability → amplification of density fluctuations



- Governed by gravity alone
- Comparison of early and late observations (e.g. CMB and today)
  → consistency check
  ⇒ we need DM
- Work in multipole expansion, different terms = different length scales λ

end 16.05.2013

General picture



Kolb Turner;
Knobel arXiv:1208.5931

Basic equations (fluid mechanics – Newtonian gravity)

1) Continuity equation:

$$\boxed{\frac{d\rho}{dt} + \vec{\nabla}(\rho\vec{v}) = 0} \qquad (1)$$

2) Euler equation (local balance of forces)

see Dentrioder I, sec. 8.2

$$\boxed{\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p + \vec{\nabla}\phi = 0} \qquad (2)$$

↑ acceleration
↑ convective acceleration (e.g. liquid in pipes of different diameter)
↑ pressure gradients
↑ gravity

3) Poisson equation

$$\boxed{\Delta\phi = 4\pi G \rho} \qquad (3)$$

Simplest solution (fully homogeneous Universe)

physical coordinate

$$\bar\rho(t) = \frac{\rho_0}{R^3(t)} \quad ; \quad \vec{\bar v}(t) = \frac{\dot R(t)}{R(t)}\vec r \quad ; \quad \vec\nabla\bar\phi = \frac{4\pi G\rho_0}{3}\vec r$$

Check: (1) $\frac{d\bar\rho}{dt} = -3\frac{\dot R}{R}\bar\rho = -\bar\rho\cdot(\vec\nabla\vec v)$ ✓ $\left[\vec\nabla\rho = 0,\ \vec\nabla\vec r = 3\right.$

(2) $\frac{\partial\vec{\bar v}}{\partial t} = \frac{\ddot R}{R}\vec r - \frac{\dot R^2}{R^2}\vec r$ $= (\vec r\cdot\vec\nabla)\vec r$

Friedmann-
=
Lemaître

$$-\frac{4}{3}\pi G(\rho + 3p)\vec r - (\vec v\cdot\vec\nabla)\vec v \quad ✓ \qquad \left[\vec\nabla\bar p = w\vec\nabla\rho =\right.$$

↑ assume ≪ ρ (non-rel.)

(3) obvious

Consider small perturbations:

$$\rho(\vec x, t) = \bar\rho(t) + \rho_1(\vec x, t) \quad ; \quad |\rho_1| \ll |\bar\rho|$$

$$p(\vec x, t) = \bar p(t) + p_1(\vec x, t) \quad ; \quad |p_1| \ll |\bar p|$$

$$\vec v(\vec x, t) = \vec{\bar v}(t) + \vec v_1(\vec x, t) \quad ; \quad |\vec v_1| \ll |\vec{\bar v}|$$

$$\phi(\vec x, t) = \bar\phi(t) + \phi_1(\vec x, t) \quad ; \quad |\phi_1| \ll |\bar\phi|$$

Strategy:
- Define $\delta \equiv \frac{\rho_1}{\bar\rho}$
- Insert in (1)
- Take time derivative → eq. for $\ddot\delta$
- Eliminate p_1 using $p = w\rho$
- Eliminate $\vec v_1$ using (2)
- Eliminate ϕ_1 using (3)
- Eliminate $\vec v_1$ using (1) again

Helpful: work in comoving coordinates $\vec r\,' \equiv \frac{\vec r}{R(t)}$

↳ Equation for $\ddot\delta$ in terms of known quantities

- Fourier expand δ → Equations for individual Fourier coefficients δ_k
- Solve

$$(1) \Rightarrow \frac{d\bar{\rho}}{dt} + \frac{d\rho_1}{dt} + \vec{\nabla}\left[(\bar{\rho}+\rho_1)(\bar{\vec{v}}+\vec{v}_1)\right] = 0$$

$$\Leftrightarrow \frac{d\rho_1}{dt} + \bar{\rho}\,\vec{\nabla}\vec{v}_1 + (\vec{\nabla}\rho_1)\bar{\vec{v}} + (\vec{\nabla}\bar{\vec{v}})\rho_1 = 0$$

$$\Leftrightarrow \boxed{\frac{d\rho_1}{dt} + \bar{\rho}\,\vec{\nabla}\vec{v}_1 + (\vec{\nabla}\rho_1)\bar{\vec{v}} + 3\frac{\dot{R}}{R}\rho_1 = 0} \qquad (1')$$

$$(2) \Rightarrow \frac{\partial \vec{v}_1}{\partial t} + (\vec{v}_1\cdot\vec{\nabla})\bar{\vec{v}} + (\bar{\vec{v}}\cdot\vec{\nabla})\vec{v}_1 + \frac{1}{\bar{\rho}}\vec{\nabla}p_1 + \vec{\nabla}\phi_1 = 0$$

$$\Leftrightarrow \boxed{\frac{\partial \vec{v}_1}{\partial t} + \frac{\dot{R}}{R}\vec{v}_1 + \frac{\dot{R}}{R}(\vec{r}\cdot\vec{\nabla})\vec{v}_1 + \frac{v_s^2}{\bar{\rho}}\vec{\nabla}\rho_1 + \vec{\nabla}\phi_1 = 0} \quad (2$$

Speed of sound:

$$v_s^2 = \left.\frac{\partial p}{\partial \rho}\right|_{\text{adiabatic}} \approx \frac{p_1}{\rho_1} = v$$

$$(3) \Rightarrow \boxed{\Delta\phi_1 = 4\pi G\rho_1} \qquad (3')$$

[due to superposition principle (linearity

Solution: 1) Define $\delta = \frac{\rho_1}{\rho_0} \Rightarrow \dot{\rho}_1 = \dot{\delta}\bar{\rho} + \delta\dot{\bar{\rho}} = \dot{\delta}\bar{\rho} - 3H\delta\bar{\rho}$

$$(1') \Rightarrow \boxed{\dot{\delta} + \vec{\nabla}\vec{v}_1 + (\vec{\nabla}\delta)\bar{\vec{v}} = 0}$$

2) Use comoving coordinates : $\vec{r}\,' \equiv \frac{\vec{r}}{R(t)}$

$$\hookrightarrow \vec{\nabla}' = R\vec{\nabla} \quad ; \quad \left.\frac{\partial}{\partial t}\right|_{\vec{r}\,'=\text{const}} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}=\text{const}} + \frac{1}{R}\bar{\vec{v}}\,\vec{\nabla}'$$

$$(*) \Rightarrow \left.\frac{\partial\delta}{\partial t}\right|_{\vec{r}\,'=\text{const}} \cancel{- \frac{1}{R}\bar{\vec{v}}\,\vec{\nabla}'\delta} + \frac{1}{R}\vec{\nabla}'\vec{v}_1 \cancel{+ \frac{1}{R}\bar{\vec{v}}}\,\vec{\nabla}'\delta = 0 \qquad \left|\frac{\partial}{\partial t}\right.$$

$$\Rightarrow \left.\frac{\partial^2\delta}{\partial t^2}\right|_{\vec{r}\,'=\text{const}} + \frac{1}{R}\vec{\nabla}'\Big[\cancel{\frac{1}{R}\bar{\vec{v}}\,\vec{\nabla}'\vec{v}_1} - \frac{\dot{R}}{R}\vec{v}_1 - \cancel{\frac{\dot{R}}{R^2}(\vec{r}\cdot\vec{\nabla}')\vec{v}_1}$$

$$- \frac{1}{R}v_s^2\,\vec{\nabla}'\delta - \frac{1}{R}\vec{\nabla}'\phi_1\Big] - \frac{\dot{R}}{R^2}\vec{\nabla}'\vec{v}_1 = 0$$

$$\Leftrightarrow \left.\frac{\partial^2\delta}{\partial t^2}\right|_{\vec{r}\,'=\text{const}} \underbrace{-2H\cdot\frac{1}{R}\vec{\nabla}'\vec{v}_1}_{\overset{(*)}{=}2H\dot{\delta}} - \frac{1}{R^2}v_s^2\,\Delta'\delta - \frac{1}{R^2}\Delta'\phi_1 = 0$$

$$\Leftrightarrow \boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{v_s^2}{R^2}\Delta'\delta - 4\pi G\bar{\rho}\,\delta = 0}$$

3) Fourier transform:

$$\delta(\vec{x}', t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}'} \delta_k(t)$$

$$\hookrightarrow \boxed{\ddot{\delta}_k + 2H\dot{\delta}_k + \frac{v_s^2}{R^2}\Big(k^2 - \underbrace{R^2\frac{4\pi G\bar{\rho}}{v_s^2}}_{=(\text{Jeans wave number } k_J)^2}\Big)\delta_k = 0} \qquad (*)$$

Limiting cases

i) No expansion $(H = 0, R = 1)$

$$\hookrightarrow \delta \sim \exp\left[\pm i\omega t \mp i\vec{k}\vec{x}\right]; \quad \omega = \sqrt{v_s^2(k^2 - k_J^2)}$$

For $k > k_J$: Propagating sound waves

For $k < k_J$: Exponentially growing and decaying modes

ii) Matter-dominated Universe $(v_s = 0)$, no curvature (no k-term in Friedmann eq

$$\hookrightarrow R \sim t^{\frac{2}{3(1+\omega)}} \quad ; \quad \dot{R} = \frac{2}{3(1+\omega)t} R$$

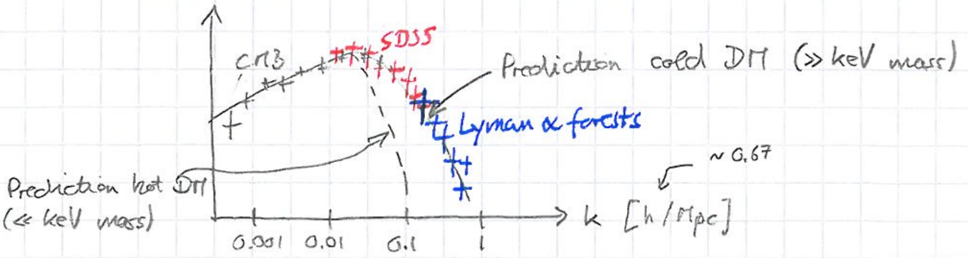
$$(*) \Rightarrow \ddot{\delta}_k + \frac{4}{3(1+\omega)t}\dot{\delta}_k - \underbrace{4\pi G\bar{\rho}}_{=+\frac{3}{2}(\dot{R}/R)^2 = +\frac{2}{3}\frac{1}{(1+\omega)^2t^2}} \cdot \delta_k \qquad \left(\text{Friedmann eq. } \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\bar{\rho}}{3}\right)$$

$$\overset{w=0}{\Longleftrightarrow} \ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k - \frac{2}{3t^2}\delta_k = 0$$

$$\Rightarrow \delta_k = c_1(\vec{k})\, t^{2/3} + c_2(\vec{k})\, t^{-1} \xrightarrow{\text{large } t} \sim t^{2/3}$$

Comparing with data

- Start from initial density perturbations (CMB anisotropies or theory — quantum fluctuations during inflation)
- Evolve in time
- Compare with observed distributions of galaxies (e.g. Sloan Digital Sky Survey). Observable: Matter power spectrum $P(\vec{k}) = |\delta_k|^2$

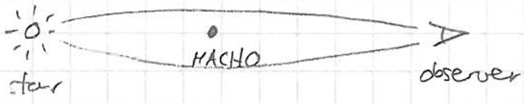


Conclusion: DM should be in the form of heavy ($\gg$ keV) particles.

- MACHO (Massive Compact Halo Object) searches

Question: Is DM in the form of Jupiter-like objects, white dwarfs etc.?

Method: Gravitational microlensing



→ transient light amplification

Result: e.g. EROS (Experience de Recherche d'Objets Sombres) 2006:

$6 \cdot 10^{-8} M_\odot < m_{MACHO} < 15\, M_\odot$ ruled out as 100% DM

5.2 The DM abundance : Freeze-out

see Kolb-Turner]

DM (assumed self-conjugate here)

Some SM particle + antiparticle (need not be a fermion)

Assume feeble DM - SM interaction $\chi\chi \longleftrightarrow \bar{f}f$

Very early: DM in thermal equilibrium

[Particle densities so high that even suppressed interactions are probable

Number density

$$n_{\chi,eq} = \frac{\overset{\text{number}}{N_{eq,\chi}}}{\underset{\text{volume}}{V}} = \frac{N_{eq,\chi}}{R^3(t)\cdot V_0}$$

$$= g \cdot \int \frac{d^3p}{(2\pi)^3} \exp\left[-\left(E(p) - \mu\right)/T\right]$$

g: # of internal degrees of freedom (e.g. 2 for spin $\frac{1}{2}$)

μ: chemical potential (usually negligible)

$$= \frac{g}{2\pi^2} \int dp \; p^2 \exp\left[-\left(E(p) - \mu\right)/T\right]$$

$$E = \sqrt{p^2 + m^2}$$

$$\frac{dE}{dp} = \frac{p}{E}$$

$$= \frac{g}{2\pi^2} \int dE \; E\sqrt{E^2 - m^2} \;\; \exp\left[-(E - \mu)/T\right]$$

Later: T, ρ decrease $\longrightarrow$ interactions freeze out

[scattering / annihilation probability reduced due to lower density, as T drops, fewer and fewer particles have sufficient energy to initiate the backwards reaction

Governed by Boltzmann equation :

rate

subdominant annihilation chan / elastic scatt

$$\boxed{\frac{dN_\chi}{dt} \doteq \Gamma(\bar{f}f \to \chi\chi) - \Gamma(\chi\chi \to \bar{f}f) + \Gamma_{\text{other}}}$$

[Computation of Γ :

S-matrix : $S \equiv \exp\left[-i \int d^4x \, \underset{\text{Hamiltonian}}{\mathcal{H}}\right] \simeq 1 - i\int d^4x \, \mathcal{H} - \frac{1}{2}\left(\int d^4x \, \mathcal{H}\right)^2 + \dots$

Matrix element:

$$(2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \cdot \mathcal{M}(\chi\chi \to \bar{f}f) = \langle \bar{f}(p_1) \, f(p_2) \, | \, S \, | \, \chi(k_1) \, \chi(k_2) \rangle$$

Transition probability :

$$|\mathcal{M}|^2 \cdot \left[\delta^{(4)}(p_1 + p_2 - k_1 - k_2)\right]^2 \cdot (2\pi)^8$$

Use $$2\pi\,\delta(\Delta E) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i\Delta E t}\, dt = \lim_{T\to\infty} \frac{2}{\Delta E} \sin\frac{\Delta E T}{2}$$

$$\left[2\pi\,\delta(\Delta E)\right]^2 = 2\pi\,\delta(\Delta E) \cdot \lim_{T\to\infty} \frac{2}{\Delta E} \sin\frac{\Delta E T}{2}$$

$$= \lim_{T\to\infty} 2\pi\,\delta(\Delta E) \cdot T$$

=> Transition rate [probability per time]

$$\Gamma = \frac{1}{\cancel{T}} (2\pi)^4 \cdot \cancel{T} \cdot V\, |\mathcal{M}|^2\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)$$

Continuum of final states and of initial states

Change normalization to Lorentz invariant one

$$\hookrightarrow \Gamma = V \int \underbrace{\frac{d^3p_1}{(2\pi)^3 2E(p_1)}}_{\equiv d\Pi(p_1)} \frac{d^3p_2}{(2\pi)^3 2E(p_2)} \frac{d^3k_1}{(2\pi)^3 2E(k_1)} \frac{d^3k_2}{(2\pi)^3 2E(k_2)}\, \phi_\chi(\vec{k}_1)\, \phi_\chi(\vec{k}_2)$$

i.s. phase space densities

Identical i.s. particles ?

$$\cdot\, (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|^2} \qquad (\square)$$

Note: For final state bosons, include factor $(1+\phi_f(\vec{p}_1))(1+\phi_f(\vec{p}_2))$,
for final state fermions $(1-\phi_f(\vec{p}_1))(1-\phi_f(\vec{p}_2))$
For low densities, $f \ll 1$

Note: $\overline{|\mathcal{M}|^2}$ includes factor $\frac{1}{2}$ for identical i.s. particles,
spin averaging factors, etc

$$\Rightarrow \frac{dN_\chi}{dt} = \frac{d(n_\chi V)}{dt} = V \frac{dn_\chi}{dt} + n_\chi \frac{d(R^3 V_0)}{dt} = V\frac{dn_\chi}{dt} + 3 n_\chi H V$$

$$\Rightarrow \boxed{\frac{dn_\chi}{dt} + 3 n_\chi H = \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2) \cdot \left[\phi_f(p_1)\phi_f(p_2) - \phi_\chi(k_1)\phi_\chi(k_2)\right] (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|^2} + \Gamma_{\text{other}}} \qquad (*)$$

assume $\mathcal{M}(\chi\chi \to \bar{f}f)$
$= \mathcal{M}(\bar{f}f \to \chi\chi)$
(T invariance + CP invarianc

- Assume $f, \bar{f}$ in thermal equilibrium

$$\hookrightarrow \quad \phi_{\bar{f}}(p_{1,2}) = \exp\left[-E(p_{1,2})/T\right] \equiv \phi_{\bar{f},eq}(p_{1,2})$$

Since $E(p_1) + E(p_2) = E(k_1) + E(k_2)$

$$\hookrightarrow \quad \phi_{\bar{f},eq}(p_1) \cdot \phi_{f,eq}(p_2) = \exp\left[-\left(E(p_1) + E(p_2)\right)/T\right]$$

$$= \exp\left[-\left(E(k_1) + E(k_2)\right)/T\right]$$

$$= \phi_{\chi,eq}(k_1) \cdot \phi_{\chi,eq}(k_2)$$

- Use $\sigma_{\chi\chi\to\bar{f}f} = \dfrac{\text{Rate per target particle}}{\text{Flux of i.s. particles}} = \dfrac{\text{Rate per target particle}}{\text{Density of i.s. particles} \cdot \text{Velocity}}$

Take (□), divide by $N = n_\chi \cdot V$ to obtain rate per target WIMP

$$\hookrightarrow \quad \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} = \frac{1}{n_\chi^2} \int d\Pi(p_1)\, d\Pi(p_2)\, (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|}^2$$

- Assume that after chemical decoupling (cessation of $\chi\chi \leftrightarrow \bar{f}f$), χ still stays in kinetic equilibrium ($\chi f \leftrightarrow \chi f$ still fast)

$$\Rightarrow \quad \phi_\chi = \frac{n_\chi}{n_{\chi,eq}}\, \phi_{\chi,eq}$$

- Thermally averaged x-sec

$$\langle \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} \rangle \equiv \frac{1}{n_\chi^2} \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2)\, \phi_\chi(k_1)\, \phi_\chi(k_2)$$

$$\cdot (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|}^2 \qquad (□□)$$

$$= \frac{1}{n_{\chi,eq}^2} \int d\Pi(p_1)\, d\Pi(p_2)\, d\Pi(k_1)\, d\Pi(k_2)\, e^{-[E_\chi(k_1) + E_\chi(k_2)/T]}$$

$$\cdot (2\pi)^4\, \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, \overline{|\mathcal{M}|}^2$$

$$= \langle \sigma_{\bar{f}f\to\chi\chi} \cdot v_{rel} \rangle$$

$\Rightarrow$ Rewrite $(*)$

$$\boxed{\begin{aligned} \frac{dn_\chi}{dt} + 3 n_\chi H &= -\langle \sigma_{\chi\chi\to\bar{f}f} \cdot v_{rel} \rangle \left[n_\chi^2 - n_{\chi,eq}^2 \right] + \Gamma_{other} \\ &= -\langle \sigma_{ann} \cdot v_{rel} \rangle \left[n_\chi^2 - n_{\chi,eq}^2 \right] \end{aligned}} \qquad (**)$$

$\uparrow$ total annihilation x-sec

- Detailed balance: In thermal equilibrium,

$$\Gamma^{eq}(\bar{f}f \to \chi\chi) = \Gamma^{eq}(\chi\chi \to \bar{f}f)$$

Alternative: Write out Γ according to (II) above as (schematically)

$$\Gamma(\chi\chi\to\bar{f}f) = \int d\Phi_\chi \, d\underbrace{\overset{\text{phase space}}{\Phi_f}}\, \underset{\text{distribution function}}{f(\chi)}\, \overline{|\mathcal{M}|^2} \cdot \delta^{(4)}(\Sigma p_\chi - \Sigma p_f)$$

and use

$$f^{eq}(\chi) = f^{eq}(f) \; ; \quad f(\chi) = \frac{n_\chi}{n_{\chi,eq}} f^{eq}(\chi) \; ;$$

$$\mathcal{M}(\chi\chi \to \bar{f}f) = \mathcal{M}(\bar{f}f \to \chi\chi)$$

$\Gamma(\bar{f}f \to \chi\chi)$ determined by $n_f = n_{f,eq}$

$$\hookrightarrow \quad \Gamma(\bar{f}f \to \chi\chi) = \Gamma^{eq}(\chi\chi \to \bar{f}f)$$

- Use cross section $= \dfrac{\text{total rate}}{(\text{\# of target particles})\cdot(\text{flux of beam particles})}$

$$\Gamma(\chi\chi \to \bar{f}f) = \underset{\text{averaging over } \vec{p}\text{-distrib.}}{\Big\langle} \sigma(\chi\chi \to \bar{f}f) \cdot (n_\chi \cdot \underset{\text{volume}}{V}) \cdot (n_\chi \cdot \underset{\text{rel. velocity of two } \chi \text{ particles}}{v_{rel}}) \Big\rangle$$

$$\Rightarrow \quad \frac{d(n_\chi \overset{\equiv V_0 R^3}{V})}{dt} = V \langle \sigma(\chi\chi \to \bar{f}f)\, v_{rel} \rangle (n_{\chi,eq}^2 - n_\chi^2)$$

$$\boxed{\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma(\chi\chi \to \bar{f}f)\, v_{rel} \rangle (n_\chi^2 - n_{\chi,eq}^2)} \quad (**)$$

Useful definition : $\boxed{Y = \frac{n}{s}}$

where $s =$ <u>entropy density</u>.

Note : $s \cdot R^3 = \underbrace{\text{const}}_{\equiv s_0}$ (Entropy in comoving volume element conserved for adiabatic expansion)

$$\hookrightarrow \dot{Y} = \frac{\dot{n}}{s} - Y\frac{\dot{s}}{s} = \frac{\dot{n}}{s} + 3\frac{\dot{R}}{R}Y$$

$\Rightarrow$ (**) becomes $\boxed{\frac{dY}{dt} = -\langle\sigma_{ann} v_{rel}\rangle\, s\left[Y^2 - Y_{eq}^2\right]}$ (omitting index χ)

[show numerical solution already here?]

<u>Solving the Boltzmann equation</u>

More useful than t : $x = \frac{m_\chi}{T}$

To relate x and t :

see expression for n_{eq}

fermions

$$\rho \overset{\text{rad. dom.}}{=} \frac{\rho_0}{R^4} \overset{\text{rad. dom.}}{=} \frac{\rho_0}{(R_0\sqrt{t})^4} = \sum_{\substack{i=\text{rel.}\\ \text{species}}} \frac{g_i}{2\pi^2}\int_0^\infty \frac{E}{\exp[E/T]\pm 1}\,E^2 dE$$

bosons

$$= \sum_{\substack{i=\text{rel.}\\ \text{bosons}}} \frac{\pi^2}{30} g_i T^4 + \sum_{\substack{i=\text{rel.}\\ \text{fermions}}} \frac{7}{8}\cdot\frac{\pi^2}{30} g_i T^4$$

$$\equiv g^* \frac{\pi^2}{30} T^4 \qquad \left[g^* = \sum_{\text{bosons}} g_i + \frac{7}{8}\sum_{\text{fermions}} g_i\right.$$

$\Rightarrow$ • $R \sim \frac{1}{T} \sim x \Rightarrow H = \frac{\dot{R}}{R} = \frac{\dot{x}}{x}$

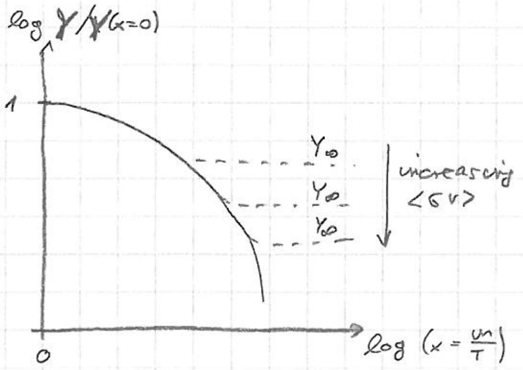
• $H \sim \frac{1}{t} \sim T^2 \sim \frac{m^2}{x^2}$

deduced from measured parameters (probably h_0)

$$\Rightarrow \frac{dY}{dt} = \frac{dY}{dx}\cdot x H(x) = \frac{dY}{dx}\frac{H(m)}{x}$$

$$\Rightarrow \boxed{\frac{dY}{dx} = -\langle\sigma_{ann} v_{rel}\rangle \frac{x s}{H(m)}\left(Y^2 - Y_{eq}^2\right)}$$

Numerical solution



survival of the weakest

Freeze-out happens at $x = x_F \simeq 20$

↳ Yield at late times (neglect Y_{eq} compared to Y):

$$\frac{dY}{dx} = - \langle \sigma_{ann} v_{rel} \rangle \frac{x\, s}{H(m)} Y^2$$

$$\int_{Y(x_F)}^{Y(\infty)} \frac{1}{Y^2}\, dY = - \underbrace{\langle \sigma_{ann} v_{rel} \rangle}_{\text{assumed v- and x-independent}} \frac{1}{H(m)} \int dx \; x \cdot s$$

$$-\frac{1}{Y(\infty)} + \underbrace{\frac{1}{Y(x_F)}}_{\text{neglect}} = - \langle \sigma_{ann} v_{rel} \rangle \cdot \text{const}$$

Note: $s \sim \frac{1}{R^3} \sim T^3 \sim \frac{m^3}{x^3}$

↳ $\int dx\, x s \sim \int dx \frac{m^3}{x^2} \sim \frac{m^3}{x_F}$

$\frac{1}{H(m)} \sim \frac{1}{m^2}$ $\Rightarrow \text{const} \sim m$

$$\boxed{Y(\infty) = \frac{\text{const}}{\langle \sigma_{ann} v_{rel} \rangle}}$$

$$\Omega h^2 = \frac{\rho}{\rho_c} h^2 = \frac{m_\chi Y(\infty)\, s(t_0)}{\rho_c(t_0)} h^2 = \frac{\text{const}}{\langle \sigma_{ann} v_{rel} \rangle}$$

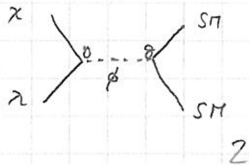
H_0 in units of 100 km s^{-1} Mpc^{-1}

Numerically, for $m_\chi \gtrsim 10$ GeV:

$$\boxed{\Omega h^2 \simeq \frac{25 \times 10^{-27} \frac{cm^3}{sec}}{\langle \sigma_{ann} v_{rel} \rangle}}$$

Planck CMB data: $\Omega h^2 = 0.1196 \pm 0.0031$

Assume annihilations mediated by particle ϕ, $m_\phi \sim m_\chi \sim 100$ GeV coupling $\frac{g^2}{4\pi} \sim 0.01$



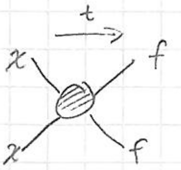
$\Rightarrow \langle \sigma_{ann} v_{rel} \rangle \sim \text{few} \times 10^{-26} \frac{cm^3}{sec}$

WIMP miracle

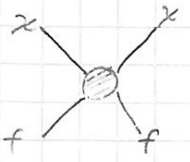
Strong motivation for ewk scale DM!

5.3 Direct DM detection

Freeze-out requires



Turn diagram around: DM - SM scattering



$\Rightarrow$ Galactic WIMPs detectable by scattering on SM particles (in particular nuclei for kinematic reasons)

Toy model: $\mathcal{L} \supset \sum_q \frac{m_q}{\Lambda^3} (\overline{\chi}\chi)(\overline{q}q)$

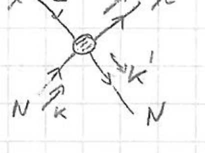
DM fermion

SM quark

cutoff scale

↳ 

$$= i\frac{m_q}{\Lambda^3}(\overline{u}_\chi u_\chi)(\overline{u}_q u_q)$$



$$= i\sum_q \frac{m_q}{\Lambda^3} \langle N|\overline{q}q|N\rangle [\overline{u}_\chi(p')u_\chi(p)][\overline{u}_N(k')u_N(k)] \equiv i\mathcal{M}$$

factorize spinor part (possible in non-rel. limit)

F.S. spin sum, I.S. spin avg.:

$$\frac{1}{4}\overline{|\mathcal{M}|^2} = \frac{1}{4\Lambda^6}\left[\sum_q m_q \langle N|\overline{q}q|N\rangle\right]^2 tr\left[(\not{p}' + m_\chi)(\not{p} + m_\chi)\right]$$

$$\cdot \underbrace{tr\left[(\not{k}' + m_N)(\not{k} + m_N)\right]}$$

$$= tr(\not{k}_1 \not{k}_2) + 4m_N^2$$
$$= 4(k_1 k_2 + m_N^2)$$
$$\simeq 8m_N^2 \quad \text{for non-relativistic process}$$

$$= \frac{1}{\Lambda^6}\left[\sum_q m_q \langle N|\overline{q}q|N\rangle\right]^2 \cdot 16\, m_N^2 m_\chi^2$$

$$\sigma_{\chi N} = \frac{\Gamma}{N_N j} \overset{n.r.}{=} \frac{\Gamma}{n_N \cdot V \cdot n_\chi \cdot v_\chi}$$

number of target nucleons

flux of incoming DM particles

see p. 24, (□)

note: ϕ_χ is normalized to $\int \frac{d^3p}{(2\pi)^3} \phi_\chi = m_\chi$

$$= \int \frac{d^3p'}{(2\pi)^3 \cdot 2E_{p'}} \frac{d^3k'}{(2\pi)^3 \cdot 2E_{k'}} (2\pi)^4 \delta^{(4)}(p + k \cdot p' - k') \overline{|\mathcal{M}|^2}$$

$$\cdot \frac{1}{2E_p} \cdot \frac{1}{2E_k} \cdot \frac{1}{v_\chi}$$

$$\cong \frac{1}{4 m_\chi m_N \; v_\chi} \cdot \frac{1}{(2\pi)^2} \int \frac{d^3k'}{4E_{k'}E_{p'}} \, \delta\left(E_p + E_k - E_{p'} - E_{k'}\right) \overline{|\mathcal{M}|^2}$$

Note: $d^3k' = d\Omega \, k'^2 \, dk'$

Transform to c.m. system; use $\mathcal{M}$ independent of k', p':

$$\sigma_{\text{CMS},\chi N} = \frac{4\pi}{16\pi^2 m_\chi m_N \underset{\uparrow \text{ rel. velocity in c.m.s}}{v_{\chi N}}} \int \frac{dk' k'^2}{4E_{k'}E_{p'}} \, \delta\left(E_p + E_k - \sqrt{k'^2 + m_\chi^2} - \sqrt{k'^2 + m_N^2}\right) \overline{|\mathcal{M}|}$$

$$\left[= \frac{1}{4\pi \, m_\chi m_N v_{\chi N}} \int \frac{dk' \, k'^2}{4E_{k'}E_{p'}} \left(\frac{k'}{E_{k'}} + \frac{k'}{E_{p'}}\right)^{-1} \delta(k' - \ldots) \overline{|\mathcal{M}|^2} \right.$$

$$\left. \delta(f(x)) = \frac{1}{|f'(x)|} \delta(x) \right] \quad = \frac{1}{4\pi \, m_\chi m_N v_{\chi N}} \, \frac{k'}{4(E_{k'} + E_{p'})} \, \overline{|\mathcal{M}|^2}$$

$$\left. k' \overset{\text{c.m.s.}}{=} \frac{m_N m_\chi v_{\chi N}}{m_N + m_\chi} \right] \quad = \frac{1}{16\pi (m_N + m_\chi)^2} \, \overline{|\mathcal{M}|^2}$$

$$= \frac{m_\chi^2 \cdot m_N^2}{\pi \Lambda^6 (m_N + m_\chi)^2} \left[\sum_q m_q \langle N | \bar{q} q | N \rangle\right]^2$$

$$\left[\begin{aligned} k' &= m_\chi v_\chi = m_N v_N \\ &= m_\chi (v_{\chi N} - v_N) \\ &= m_\chi v_{\chi N} - \frac{m_\chi}{m_N} k' \\ \Rightarrow k' &= m_\chi v_{\chi N} \left(1 + \frac{m_\chi}{m_N}\right)^{-1} \\ &= \frac{m_\chi m_N v_{\chi N}}{m_\chi + m_N} \end{aligned} \right.$$

Or, use e.g Peskin / Schröder eq. 4.84:

$$\left.\frac{d\sigma_{\chi N}}{d\Omega}\right|_{\text{CMS}} = \frac{1}{4 m_\chi m_N \underset{\uparrow \text{ rel velocity in CMS}}{v_{\chi N}}} \, \frac{k'}{(2\pi)^2 \cdot 4 E_{cm}} \, \overline{|\mathcal{M}|^2}$$

$$\left. k' \overset{\text{c.ms}}{=} \frac{m_N m_\chi v_{\chi N}}{m_N + m_\chi} \right] \quad \sigma_{\chi N} = \frac{1}{16\pi (m_N + m_\chi)^2} \, \overline{|\mathcal{M}|^2}$$

$$\boxed{\underset{\uparrow = 4\pi \frac{d\sigma}{d\Omega}}{\sigma_{\chi N}} = \frac{m_\chi^2 \, m_N^2}{\pi \Lambda^6 (m_N + m_\chi)^2} \left(\sum_q m_q \langle N | \bar{q} q | N \rangle\right)^2}$$

Also useful: $\frac{d\sigma}{dE_r}$

$\uparrow$ nuclear recoil energy

$$E_r = E_{k'} - m_N = \frac{1}{2} m_N v_N'^2$$

To calculate, note

$$\begin{aligned} E_r &= \frac{1}{2} m_N v_N'^2 \\ &= \frac{1}{2} m_N \left(\frac{m_\chi \vec{v}_\chi}{m_\chi + m_N} + \vec{v}'_{N,\mathrm{CMS}} \right)^2 \\ &= \frac{1}{2} m_N \left[\frac{m_\chi^2 v_\chi^2}{(m_\chi + m_N)^2} + v'^2_{N,\mathrm{CMS}} + \frac{2 m_\chi v_\chi v'_{N,\mathrm{CMS}} \cos\theta}{m_\chi + m_N} \right] \end{aligned}$$

$$\begin{aligned} \frac{dE_r}{d(\cos\theta)} &= \underbrace{\frac{m_\chi m_N}{m_\chi + m_N}}_{\equiv \mu_N} v_\chi v'_{N,\mathrm{CMS}} \\ &= \mu_N v_\chi \frac{m_\chi v_\chi}{m_\chi + m_N} \end{aligned}$$

σ indep. of Ω

$$\Rightarrow \frac{d\sigma}{dE_r} = \int d\phi \underbrace{\frac{d\sigma}{d\Omega}}_{d\Omega = d\phi\, d(\cos\theta)} \frac{d(\cos\theta)}{dE_r} = 2\pi \frac{d\sigma}{d\Omega} \frac{m_N}{\mu_N^2 v_\chi^2} \overset{\swarrow}{=} \frac{\sigma}{2} \frac{m_N}{\mu_N^2 v_\chi^2}$$

- Count rate per time, energy, target mass:

DM number density

Target number density

for hypothetical hydrogen target

$$\boxed{\frac{dR_{\chi N}}{dE_r} = \int_{v_{\mathrm{min}}(E_r)}^{\infty} dv_\chi\, v_\chi^2\, d\Omega_{v_\chi}\, f_\oplus(\vec{v}_\chi)\, n_\chi v_\chi \frac{n_N}{\rho_N} \frac{d\sigma_{\chi N}}{dE_r}}$$

$\uparrow$ integral over DM velocity distribution

$\uparrow$ DM velocity distribution in Earth frame

DM density $\simeq 0.3$ GeV/cm^3

$$\boxed{\frac{dR_{\chi N}}{dE_r} = \frac{\rho_\chi}{m_\chi} \frac{\sigma_{\chi N}}{2\mu_N^2} \int_{v_{\mathrm{min}}(E_r)}^{\infty} dv_\chi\, d\Omega_{v_\chi}\, v_\chi f_\oplus(\vec{v}_\chi)}$$

- Here: v_{min} = minimal velocity to achieve E_r. Derivation: head-on collision

$\chi \longrightarrow \overset{N}{\bullet} \dashrightarrow$

$$\text{(i)} \quad m_\chi v_\chi = m_\chi v_\chi' + m_N v_N' = m_\chi v_\chi' + m_N \sqrt{\frac{2E_r}{m_N}}$$

$$\text{(ii)} \quad \frac{1}{2} m_\chi v_\chi^2 = \frac{1}{2} m_\chi v_\chi'^2 + E_r$$

$$\text{(ii)} \Rightarrow \frac{1}{2} m_\chi v_\chi^2 = \frac{1}{2} m_\chi \frac{1}{m_\chi^2} \left(m_\chi v_\chi - m_N \sqrt{\frac{2E_r}{m_N}} \right)^2 + E_r$$

$$\Leftrightarrow \quad 0 = \frac{1}{\cancel{2}m_\chi} \cdot \cancel{2}m_N E_r - \frac{\overset{1}{\cancel{2}}}{2\cancel{m_\chi}} \cancel{m_\chi} v_\chi \sqrt{2m_N E_r} + E_r$$

$$\Leftrightarrow \quad v_\chi = \left(1 + \frac{m_N}{m_\chi}\right)\sqrt{E_r} \cdot \frac{1}{\sqrt{2m_N}}$$

$$\boxed{v_{min}(E_r) = \sqrt{\frac{m_N E_r}{2\mu_N^2}}}$$

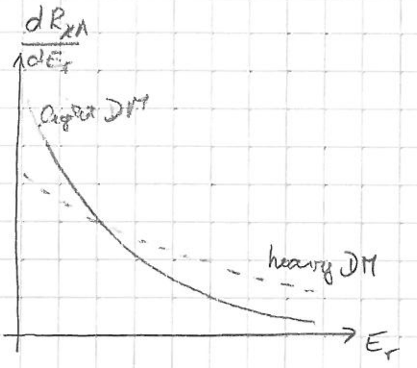
- Count rate for nuclear (non-hydrogen) target

nuclear form factor accounts for loss of coherence at large momentum transf

$$\boxed{\frac{dR_{\chi A}}{dE_r} = \frac{\rho_\chi}{m_\chi}\left(\sigma_{\chi N} \cdot \frac{\mu_A^2}{\mu_N^2}\right) \cdot \frac{1}{2\mu_A^2} \cdot A^2 \; F^2\left(\sqrt{2m_N E_r}\right) \int_{v_{min}(E_r)}^{\infty} dv_\chi \, d\Omega_{v_\chi} \, v_\chi f_\oplus(\vec{v}_\chi)}$$

modify kinematic factors

matrix element larger by factor A (coherent scattering on all nucleons)



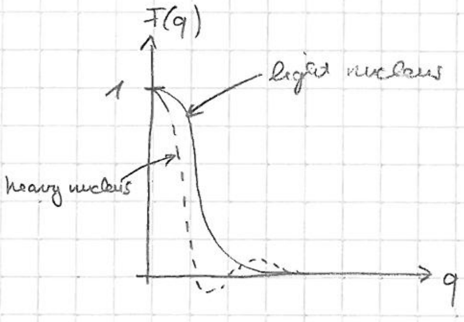
- The nuclear form factor:

Convenient parametrization:

$$F(q) = 3\, e^{-q^2 s^2/2} \frac{\sin qr - qr \cos qr}{(qr)^3}$$

$$s \equiv 1 \text{ fm}$$

$$r = \sqrt{(1.2 \cdot A^{1/3})^2 - 5s^2}$$



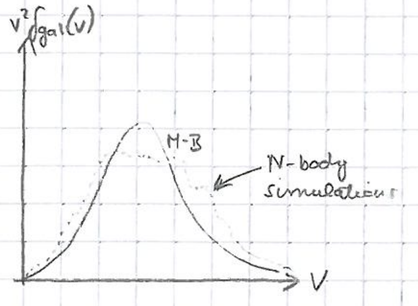
- The DM velocity distribution

Assumptions:
- isotropic (in the galactic frame)
- radially symmetric

Common assumption: Maxwell-Boltzmann distribution w/ cutoff

$$f_{gal}(\vec{v}) \sim \begin{cases} \exp[-|\vec{v}|^2/\bar{v}^2] - \exp[v_{esc}^2/\bar{v}^2] & \text{for } v < v_{esc} \\ 0 & \text{for } v \geq v_{esc} \end{cases}$$

$\bar{v}$ = velocity dispersion ~ 220 km/s
v_{esc} = escape velocity ~ 600 km/s

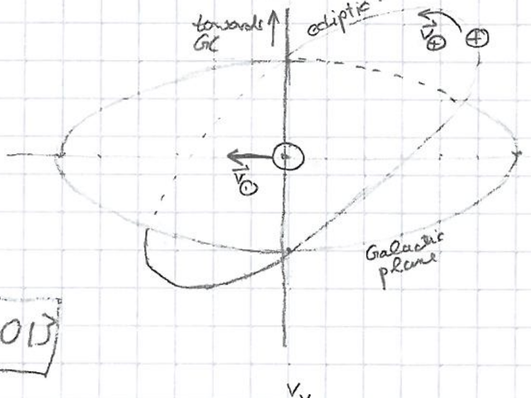


e.g. 120.2328

In the Earth's rest frame:

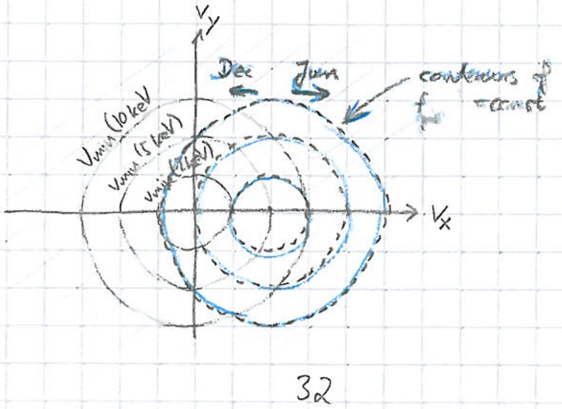
$$f_\oplus(\vec{v}) = f_{gal}(\underbrace{\vec{v}}_{\text{DM velocity in Earth frame}} + \overset{\text{motion of Sun rel. to GC } (\sim 220\,\text{km/s})}{\vec{v}_\odot} + \underbrace{\vec{v}_\oplus(t)}_{\text{velocity of Earth around Sun; } |\vec{v}_\oplus| \sim 30\,\text{km/s.}})$$

$f_\oplus(\vec{v})$ is non-isotropic and varies throughout the year

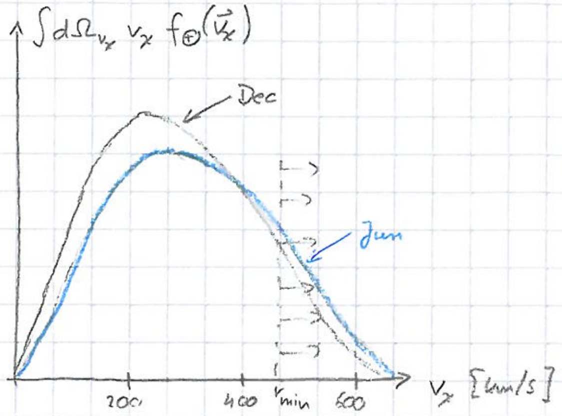


see e.g. Thomas' ISAPP 2011 lecture

end 06.06.2013



Remember: $\frac{dR_{\chi A}}{dE_r} \sim \int_{v_{min}}^{\infty} dv_\chi \; d\Omega_{v_\chi} \; v_\chi \, f_\oplus(\vec{v}_\chi)$



⇒ Expect annual modulation of count rate
Extremal around Jun 2 & Dec 2

- Direct detection experiments

a) Xenon (@ MPIK!) 1207.5988

[Show Xe-100 drawing and explain]

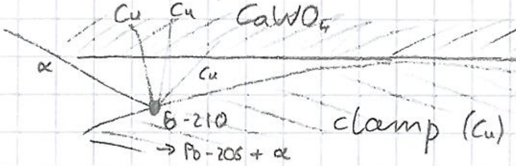
[Show Xe-100 S1-S2 plot, Xe-100 exclusion curve and explain]

- Two-phase xenon detector
- S1 signal: scintillation
  S2 signal: ionization
- S1/S2 different for nuclear recoils (= signal) and electron recoils (= background)

b) CRESST 1109.0702

[Show CRESST cryostat, CRESST module drawings, scintillation-vs-heat plot, and result (from ~~Xe100~~ plot)]

- $CaWO_4$ crystals
- Scintillation + phonon signals
- Superconducting phase transition thermometers → Cryogenic
  (use strong R(T) gradient at T_c)
- Excess events seen
- Possible explanation (1203.1576)

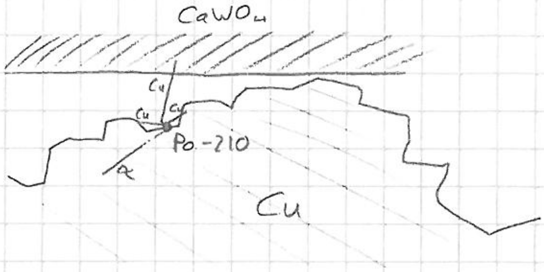


radioactive impurity (unavoidable)

$^{210}Po \to {}^{206}Pb + \alpha$

"sputtering" of secondary Cu atoms from clamp material

More realistic model: Surface roughness (not taken into account by CRESST)



Fewer sputtered Cu atoms reach detector
↳ lower event energy
→ larger BG for DM search

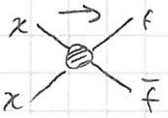
c) DAMA

[Show DAMA drawing, annual modulation plots, allowed regions (use Xe100 plot)]

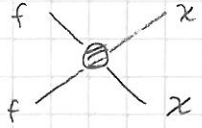
- very clean NaI(Tl) scintillator detectors
- no signal / BG discrimination
- use annual modulation as signature

5.4 DM production at colliders

Freeze-out requires



Flip diagram:



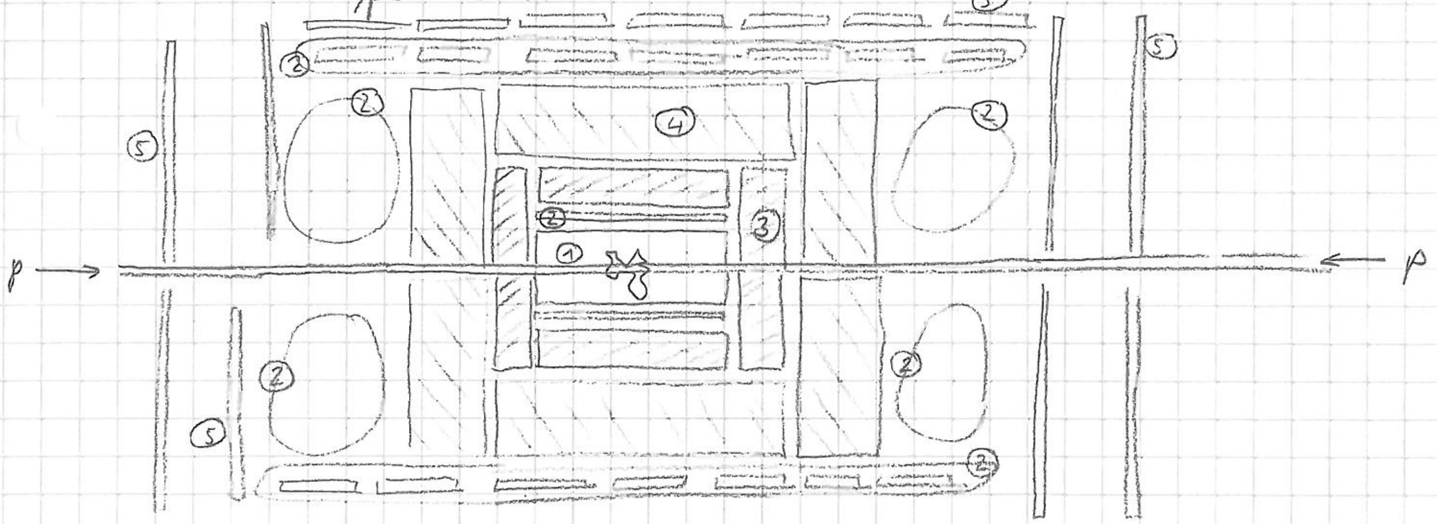
↳ DM production in particle collisions

5.4.1 Particle colliders and detectors

- Lepton colliders: $e^+ \longrightarrow \;\longleftarrow e^-$, e.g. LEP, KEKB
  - "Clean" collisions (no BG from spectator quarks, i.s. QCD, etc.)
  - Energy limited by synchrotron losses (in ring accelerators) ($P \sim E^4/m_e^4 r^2$, Jackson 14.31)
  - Beams polarisable

- Hadron colliders: $p \longrightarrow \;\longleftarrow p$ or $\bar{p}$, e.g. Tevatron, LHC
  - "Dirty" environment (lots of ISR, pile-up)
  - Initial parton momenta not known
  - Higher energies possible

- Typical detector



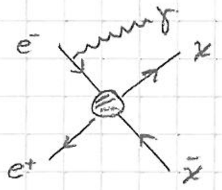
① Inner detector —— tracking
② Magnets
③ El.-mag. calorimeter (small, typically high-Z material)
④ Hadronic calorimeter
⑤ Muon chambers

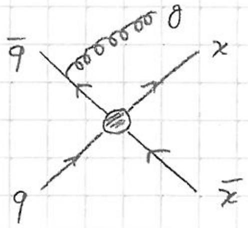
- Can only detect particles with em or strong interactions
- Can distinguish e, μ, γ, jet (to some degree: b-quarks, τ leptons)

5.4.2 Missing energy signatures of DM

Problem: No visible f.s. particles in $f\bar{f} \to \chi\chi$

One solution: ISR



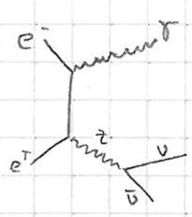


Only representative diagrams (one of many) shown!

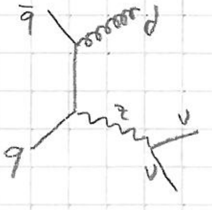
Signature: γ + Missing energy

jet + Missing transverse energy
(visible transverse $\vec{p}_T$ not balanced
$p_\parallel$ not known in hadron collider)

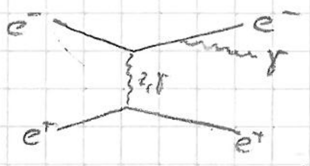
SM BG:



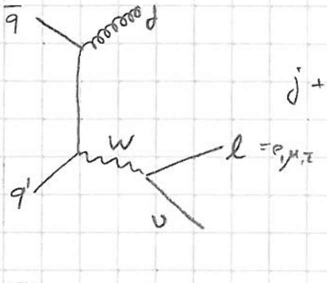
$\gamma + (Z \to \nu\bar{\nu})$



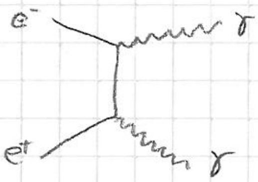
$j + (Z \to \nu\bar{\nu})$



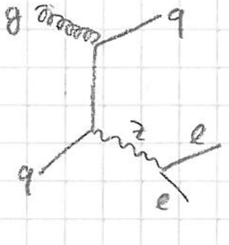
$e^+e^-\gamma$ (leptons missed)



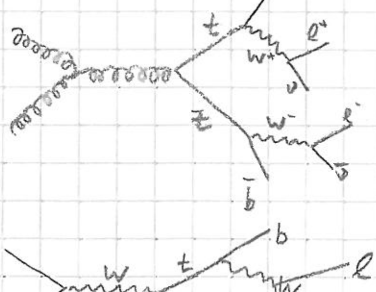
$j + (W \to \ell\nu)$
(ℓ missed)



$\gamma\gamma$ (one γ missed)

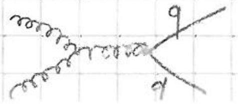


$j + (Z \to \ell\ell)$
(both leptons missed)



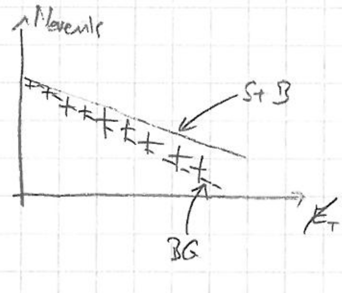
$t\bar{t}$ (leptons and extra jets missed)

single top



QCD multijet
(jets mis-reconstructed)

Strategy:
- Use generic DM model, e.g. $\mathcal{L} \supset \bar{\chi}\gamma_\mu\chi \;\bar{q}\gamma^\mu q$
- Simulate signal events
- Simulate all SM BGs (or determine from data)
- Develop cuts to enhance $S/\sqrt{B}$
- Determine systematic uncertainties in data and prediction
- Compare data with S+B prediction



E.g. CL_s upper limit on signal x-sec σ: 1204.3851

- Define test statistic log-likelihood ratio

$$LLR(\{N_i^{obs}\}|\sigma) = -2\log\frac{\mathcal{L}(\{N_i^{obs}\}|\sigma)}{\mathcal{L}(\{N_i^{obs}\}|\sigma^{\text{best fit}})}$$

predicted # of events in i-th bin

observed # of event in i-th bin

$$\mathcal{L} = \text{Likelihood} = \prod_{i=\text{bins}} \underbrace{\frac{1}{N_i^{obs}!}\, e^{-N_i^{th}(\sigma)} \cdot \left[N_i^{th}(\sigma)\right]^{N_i^{obs}}}_{\text{Poisson distribution}}$$

probability of data $\{N_i^{obs}\}$ given theory parameter σ

- Determine PDF of LLR for each σ: $f(LLR|\sigma^{true}, \sigma^{test})$ e.g. by MC simulation; analytic approximations exist, see e.g. 1204.3851

- Define $$CL_{S+B}(\{N_i^{obs}\}|\sigma^{test}) \equiv \int_{LLR(\{N_i^{obs}\}|\sigma^{test})}^{\infty} f(LLR'|\sigma^{test},\sigma^{test})\, dLLR'$$

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If $\sigma^{true} = \sigma^{test}$ this is the probability that an experiment yields an LLR at least as "extreme" as the observed one (which is evaluated for x-sec σ^{test})

$$1 - CL_B(\{N_i^{obs}\}|\sigma) \equiv \int_{LLR(\{N_i^{obs}\}|\sigma^{test})}^{\infty} f(LLR'|0,\sigma^{test})\, dLLR'$$

If $\sigma^{true} = 0$, this is the probability that an experiment yields an LLR at least as "extreme" as the observed one (which is evaluated for x-sec σ^{test})

$$CL_s(\{N_i^{obs}\}|\sigma) \equiv \frac{CL_{S+B}(\{N_i^{obs}\}|\sigma)}{1 - CL_B(\{N_i^{obs}\}|\sigma)}$$

Requirement $CL_s(\{N_i^{obs}\}|\sigma) > \alpha$ defines confidence interval at CL $1-\alpha$ (conservative CL!)

Compared to simply using CL_{S+B} (or simply $\mathcal{L}(\{N_i^{obs}\}|\sigma)$) : Robust w.r.t. unknown systematics.

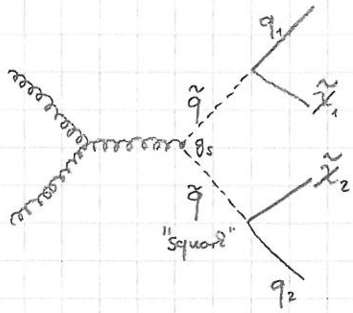
CL_s is large (σ allowed) if
- data well-described by signal <u>and</u> poorly by BG

[Show results from ATLAS or CMS monojet search]

5.4.3 Cascade decays

Idea: New particles with large production x-sec can decay to DM

Example: MSSM



- Large x-sec $(\sim \alpha_s^2)$
- Signature eg jets + MET

Mass reconstruction: The m_{T2} method hep-ph/0304226

Goal: Determine $m_{\tilde{q}}$ and $m_{\tilde{\chi}}$

Problem: Incomplete kinematical information

$$m_{\tilde{q}}^2 = (p_{q1} + p_{\chi 1})^2 = \underbrace{m_{q1}^2}_{\approx 0} + m_\chi^2 + 2(E_{q1} E_{\chi 1} - \vec{p}_{q1} \vec{p}_{\chi 1})$$

$$= (p_{q2} + p_{\chi 2})^2$$

We measure: $E_{q1}, \vec{p}_{q1}, E_{q2}, \vec{p}_{q2}, \underbrace{\vec{p}_{\chi 1,T}}_{= \begin{pmatrix} p_{\chi 1,x} \\ p_{\chi 1,y} \end{pmatrix}} + \vec{p}_{\chi 2,T} = p_{T,\text{miss}}$

We can't measure: $E_{\chi 1}, E_{\chi 2}, \vec{p}_{\chi 1,T} - \vec{p}_{\chi 2,T}, p_{\chi 1,z}, p_{\chi 2,z}$

Transverse mass: $$m_T^2(p_q, p_\chi) \equiv \left[\begin{pmatrix} \sqrt{m_q^2 + \vec{p}_{q,T}^{\,2}} \\ \vec{p}_{q,T} \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{m_\chi^2 + \vec{p}_{\chi,T}^{\,2}} \\ \vec{p}_{\chi,T} \\ 0 \end{pmatrix} \right]^2$$

$$= m_q^2 + m_\chi^2 + 2(\underbrace{E_{q,T}}_{= \sqrt{m_q^2 + \vec{p}_{qT}^{\,2}}} E_{\chi,T} - \vec{p}_{qT} \vec{p}_{\chi T})$$

For $|\vec{p}_q| = |\vec{p}_{qT}|, |\vec{p}_\chi| = |\vec{p}_{\chi,T}|$: $m_T(p_q, p_\chi) = m_{\tilde{q}}$

Otherwise : $m_T(p_q, p_\chi) \leq m_{\tilde{q}}$

[Exercise : Proof !]

Invariant mass:

$$m^2 = m_q^2 + m_\chi^2 + 2\left(E_q E_\chi - \vec{p}_q \vec{p}_\chi\right)$$

Transverse mass

$$m_T^2 = m_q^2 + m_\chi^2 + 2\left(E_{qT} E_{\chi T} - \vec{p}_{qT} \vec{p}_{\chi T}\right)$$

$$\Rightarrow m_T^2 > m^2 \iff E_q E_\chi - p_{qz} p_{\chi z} < E_{qT} E_{\chi T}$$

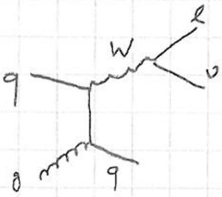
<u>Proof</u>: Suppose $E_q E_\chi < E_{qT} E_{\chi T} + p_{qz} p_{\chi z}$

$$\Rightarrow (m_q^2 + p_q^2)(m_\chi^2 + p_\chi^2) < (m_q^2 + p_{qT}^2)(m_\chi^2 + p_{\chi T}^2) + p_{qz}^2 p_{\chi z}^2 + 2\sqrt{m_q^2 + p_{qT}^2}\sqrt{m_\chi^2 + p_{\chi T}^2}\; p_{qz} p_{\chi z}$$

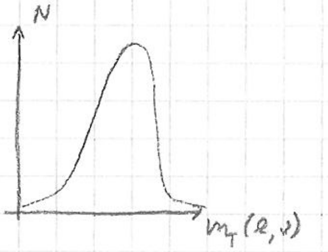
$$\iff p_{qz}^2(m_\chi^2 + p_{\chi T}^2) + p_{\chi z}^2(m_q^2 + p_{qT}^2) < 2\sqrt{m_q^2 + p_{qT}^2}\sqrt{m_\chi^2 + p_{\chi T}^2}\; p_{qz} p_{\chi z}$$

$$\iff \left(p_{qz}\sqrt{m_\chi^2 + p_{\chi T}^2} - p_{\chi z}\sqrt{m_q^2 + p_{qT}^2}\right)^2 < 0 \quad ↯$$

□



Useful e.g. for measuring W mass in
Plot m_T distribution, W mass is upper endpoint.



<u>Here</u>: Don't know $\vec{p}_{\chi 1,T}$, $\vec{p}_{\chi 2,T}$ separately

<u>Solution</u>:

$$m_{T2}^2 \equiv \min_{\vec{p}_{\chi 1,T}, \vec{p}_{\chi 2,T} \,|\, \vec{p}_{\chi 1 T} + \vec{p}_{\chi 2,T} = \vec{p}_{T,\text{miss}}^{\,\text{obs}}} \max\left[m_T^2(p_{q1}, p_{\chi 1})\,;\; m_T^2(p_{q2}, p_{\chi 2})\right]$$

- $m_{T2} \leq m_{\tilde{q}}$ because $m_T \leq m_{\tilde{q}}$
- $m_{T2} = m_{\tilde{q}}$ for specific configurations [Homework: find one!]

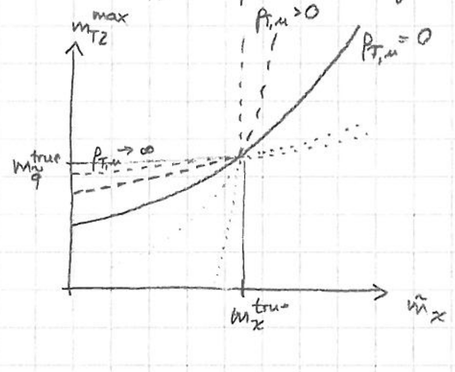
  [e.g. $\tilde{q}\tilde{q}$ production at rest, both decays in the transverse plane
- Need m_χ to compute m_{T2}.

0711.4008
0810.5576

<u>Determining m_χ:</u>

- Define $m_{T2}(\tilde{m}_\chi)$ = value of m_{T2} replacing true m_χ by test value $\tilde{m}_\chi$
- Consider events with $\underbrace{\vec{p}_{\tilde{q}1T} + \vec{p}_{\tilde{q}2,T}}_{= -\vec{P}_{T,u} \text{ ("upstream momentum")}} \neq 0$

- Plot m_{T2} endpoint as function of $\tilde{m}_\chi$



Kink gives m_χ^{true} and $m_{\tilde{q}}^{true}$

Explanation: Two extremal configurations

$$\left.\begin{array}{l} \bullet\ \vec{p}_{\tilde{q}1} \upuparrows \vec{p}_{\tilde{q}2} \upuparrows \vec{p}_{T,u} \\ \bullet\ \vec{p}_{\tilde{q}1} \upuparrows \vec{p}_{\tilde{q}2} \uparrow\downarrow \vec{p}_{T,u} \end{array}\right\} \text{and all } \vec{p} \text{ in transverse plane}$$

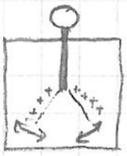
Both extremal configurations have to give same m_{T2} at $m_\chi = m_\chi^{true}$, but different functional dependence on $\tilde{m}_\chi$.

6. High energy cosmic rays

6.1 Experimental evidence

around 1800 : anomalous blackening of photographic plates (attributed to environmental radioactivity)

~1910 : Theodor Wulf, Domenico Pacini, Albert Gockel, ...
Electrometer experiments



measure discharge rate in various location (land, sea, underwater, in caves, on the Eiffel tower, ...)

→ radiation seemed to <u>increase</u> with altitude
→ extraterrestrial origin

Results not widely accepted

1912 : Victor Hess : balloon flights with electrometers
→ evidence for cosmic rays

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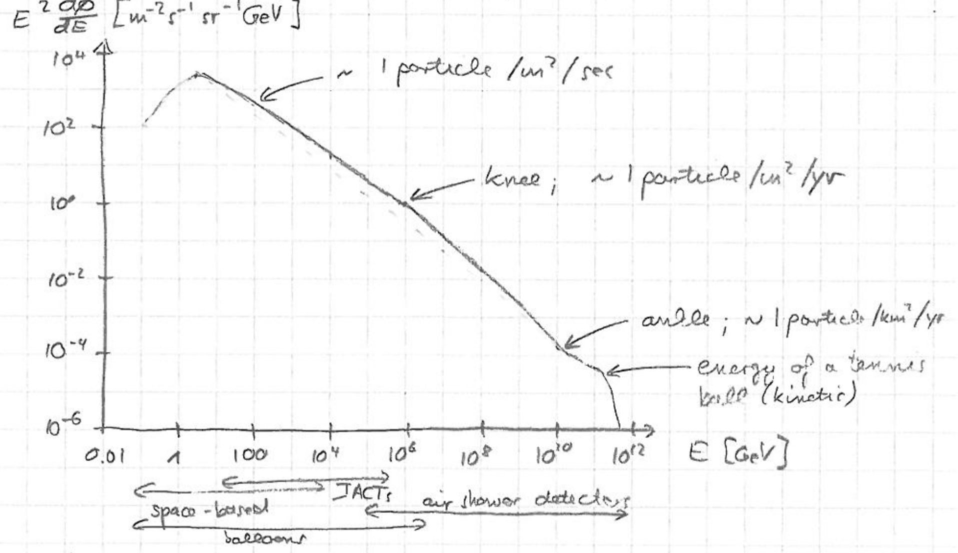
today : [show CR spectrum plot]

CR spectrum and composition well measured, but lots of open questions, especially at high E



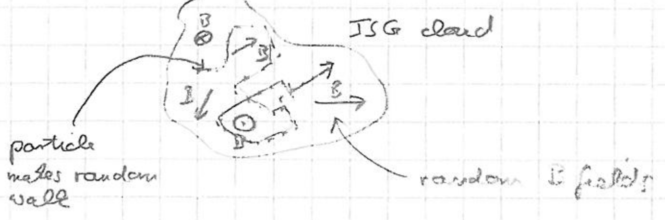
Many tools to study :
- space-based experiments (PAMELA, AMS-02, Fermi): E≲
- Imaging Air Čerenkov telescopes (HESS, MAGIC, VERITAS) (50 GeV – 50 TeV)
- Air shower detectors (Auger etc.)

[show pictures for each type of exp.]

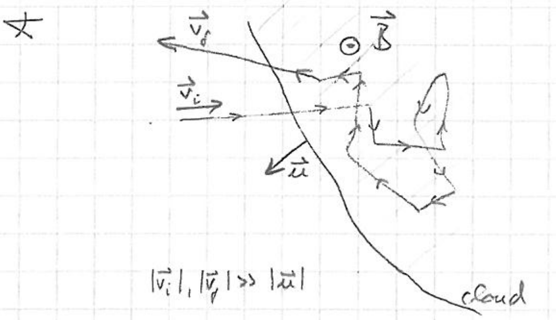
6.2 Cosmic ray acceleration: The Fermi mechanism (Fermi 1949)

Question: How can particles reach 10^{19} eV?

Consider dilute (~ 1 particle / cm^3) interstellar gas with magnetic fields
→ interactions only with em fields, no particle collisions



Moving magnetic cloud:



$|\vec{v}_i|, |\vec{v}_f| \gg |\vec{u}|$

- non-relativistic toy model

Observer frame S: $\vec{v}_i$, $\vec{v}_f$

Cloud frame S': $\vec{v}_i{}' = \vec{v}_i - \vec{u}$; $\vec{v}_f{}' = \vec{v}_f - \vec{u}$

$$\boxed{|\vec{v}_i{}'| = |\vec{v}_f{}'|}$$

$$\Rightarrow \Delta E = \frac{1}{2} m \left(v_f^2 - v_i^2 \right)$$

$$= \frac{1}{2} m \left(\vec{v}_f{}'^2 + 2 \vec{v}_f{}' \vec{u} + \vec{u}^2 - \vec{v}_i{}'^2 - 2 \vec{v}_i{}' \vec{u} - \vec{u}^2 \right)$$

$$= m \left(\vec{v}_f{}' - \vec{v}_i{}' \right) \vec{u}$$

For head-on collision: $\vec{v}_i \uparrow\downarrow \vec{u}$; $\vec{v}_i \uparrow\downarrow \vec{v}_f$; $\vec{v}_i{}' = -\vec{v}_f{}'$
w/ reflection

$$\vec{v}_f = \vec{v}_f{}' + \vec{u}$$
$$= -\vec{v}_i{}' + \vec{u}$$
$$= -\vec{v}_i + 2\vec{u}$$

$$\Rightarrow \boxed{\Delta E = 2m \left(\vec{u}^2 - \underbrace{\vec{v}_i \vec{u}}_{<0} \right) > 0}$$

For rear-end collision: $\vec{v}_i \uparrow\uparrow \vec{u}$; $\vec{v}_i \uparrow\downarrow \vec{v}_f$; $\vec{v}_i{}' = -\vec{v}_f{}'$
w/ reflection

$$\boxed{\Delta E = 2m \left(\vec{u}^2 - \underbrace{\vec{v}_i \vec{u}}_{>0} \right) < 0}$$

Energy <u>gain</u> in head-on collision is larger than energy loss in rear-end collision

↳ On average: Energy gain

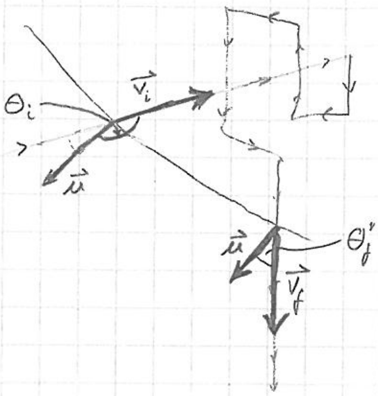
High E after many collisions

- Relativistic model

$$E_i' = E_i\gamma - \underset{\simeq E_i \text{ for rel. particles}}{\mu\gamma p_i} \cos\Theta_i$$

$$\simeq E_i \gamma \,(1 - \mu\cos\Theta_i)$$

$$\gamma = \frac{1}{\sqrt{1-\mu^2}}\,,$$



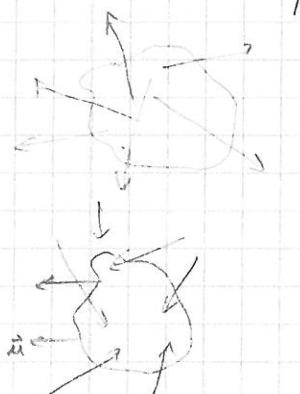
$$E_f \simeq E_f'\gamma\,(1 + \mu\cos\Theta_f')$$

Scattering in the cloud is elastic $\Rightarrow \boxed{E_i' = E_f'}$

$$\hookrightarrow E_f = \gamma^2 E_i\,(1-\mu\cos\Theta_i)(1+\mu\cos\Theta_f')$$

$$\left\langle\frac{\Delta E}{E}\right\rangle = \left\langle \gamma^2\left[\underbrace{1-\frac{1}{\gamma^2}}_{=\mu^2} + \mu\,(\cos\Theta_f' - \cos\Theta_i) - \mu^2\cos\Theta_i\cos\Theta_f'\right]\right\rangle$$

Need average angles:



can reach cloud from any direction under any angle

- By assumption: $\langle \cos\Theta_f'\rangle = 0$ (isotropization in cloud, particles can leave cloud in any direction under any angle)
- Number of particles reaching the cloud per time δt: (assume Θ_i isotropically distributed)

$$dN \propto (1 - \mu\cos\Theta_i)\,\delta t$$

$$\Rightarrow \langle\cos\Theta_i\rangle = \frac{\int_{-1}^{1} d(\cos\Theta_i)\,\cos\Theta_i\,(1-\mu\cos\Theta_i)}{\int_{-1}^{1} d\cos\Theta_i\,(1-\mu\cos\Theta_i)}$$

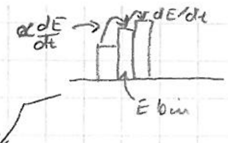
$$= -\frac{\frac{2}{3}\mu}{2} = -\frac{\mu}{3}$$

$$\Rightarrow \left\langle\frac{\Delta E}{E}\right\rangle = \frac{1}{1-\mu^2}\left[\mu^2 + \frac{\mu^2}{3}\right]$$

$$= \frac{4}{3}\frac{\mu^2}{1-\mu^2}$$

$$\boxed{\left\langle\frac{\Delta E}{E}\right\rangle \simeq \frac{4}{3}\mu^2 > 0}$$

- Energy gain on average
- 2nd order in μ → 2nd order Fermi acceleration
- Typically $\mu \sim 10\,\frac{km}{s} \Rightarrow \frac{\Delta E}{E} \sim 10^{-8}$
- Distance between clouds ~ ly
  $\Rightarrow \tau_{acc} \sim 10^8$ yrs (Duration of acceleration process)
- At low E: energy loss import (Coulomb) → need different mechanism at low E



- Final spectrum:

Diffusion eq: $$0 = \frac{\partial N(E)}{\partial t} = -\frac{N(E)}{\tau_{esc}} - \frac{\partial}{\partial E}\left(\frac{dE}{dt} N(E)\right)$$

τ_{esc}: escape time from acceleration region

$$\simeq -\frac{N(E)}{\tau_{esc}} - \frac{\partial}{\partial E}\left(\frac{E}{\tau_{acc}} N(E)\right)$$

$$\Rightarrow -\frac{N}{\tau_{esc}} - \frac{N}{\tau_{acc}} = \frac{E}{\tau_{acc}} \frac{dN}{dE}$$

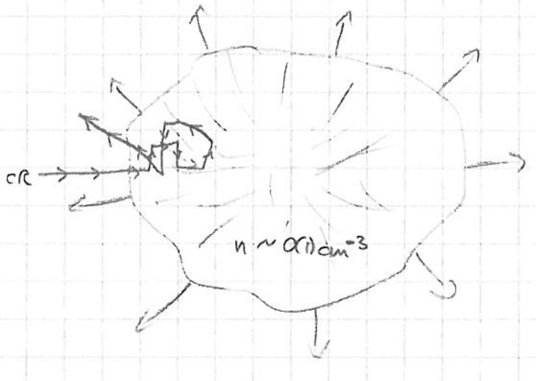
$$\frac{dN}{dE} = -\frac{N}{E}\left(1 + \frac{\tau_{acc}}{\tau_{esc}}\right)$$

$$\boxed{N \propto E^{-\alpha} \quad \text{with} \quad \alpha \simeq 1 + \frac{\tau_{acc}}{\tau_{esc}}}$$

Power law index α depends on
- velocity of clouds ($\tau_{acc} \sim 1/u^2$)
- density and size of acceleration region (τ_{esc})

5.3 Diffusive shock acceleration

Consider shock wave rather than random clouds, e.g. from supernova remnant



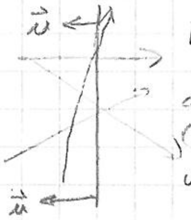
No rear-end collisions, only head-on!

$$\frac{\Delta E}{E} = \underset{\simeq 1}{\gamma^2} \overset{= v_{upstream} - v_{downstream} \; (\neq v_{shock})}{u} (\cos\theta_f' - \cos\theta_i) + \mathcal{O}(u^2) \qquad \text{(see above)}$$

Particles reaching cloud per time per solid angle

$$dN \propto \cos\theta_i \, d\Omega \, dt$$

$$\Rightarrow \langle \cos\theta_i \rangle \overset{u \simeq 0}{=} \frac{\int_{-1}^{0} d\cos\theta_i \, \cos^2\theta_i}{\int_{-1}^{0} d\cos\theta_i \, \cos\theta_i} = -\frac{2}{3}$$



particles can reach shock from only one direction (relative to $\vec{u}$), but under any angle

By similar arguments:

$$\langle \cos \theta_j' \rangle = + \frac{2}{3}$$

$$\Rightarrow \boxed{\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3} u > 0}$$

- much larger gain than for 2nd order Fermi acceleration
- $\propto u$ → "1st order Fermi acceleration"

end 27.06.2013

After n acceleration cycles:

$$E_n = (\underbrace{1 + \tfrac{4}{3} u}_{\equiv k})^n E_0$$

To produce particle with energy E, need $\frac{\log E/E_0}{\log(1+k)}$ cycles

Include escape probability P_{esc}

$$\hookrightarrow N(\geq E) = N_0 \, (1 - P_{esc})^{\frac{\log E/E_0}{\log(1+k)}}$$

$$= N_0 \left(\frac{E}{E_0}\right)^{\frac{\log(1-P_{esc})}{\log(1+k)}}$$

Expand $\log(1-P_{esc}) \simeq -P_{esc}$; $\log(1+k) \simeq k$

$$\hookrightarrow N(\geq E) \simeq N_0 \left(\frac{E}{E_0}\right)^{-P_{esc}/k}$$

$$\boxed{N(E) = \frac{dN(\geq E)}{dE} \propto \left(\frac{E}{E_0}\right)^{-\alpha} \quad \text{with} \quad \alpha = \frac{P_{esc}}{k} + 1}$$

see slides by Régis Terrier

Using hydrodynamic conservation laws, one can show that, for strong shocks ($v_{upstream} \gg v_{sound}$) : $\underline{\underline{\alpha = \mp 2}}$.

6.4 The GZK cutoff (Greisen - Zatsepin - Kuzmin)

Consider $\gamma + p \rightarrow p/n + \pi^0/\pi^+$ (large x-sec !)

in collisions of UHECR with CMB photons ($E_\gamma \sim 2 \cdot 10^{-4}$ eV)

Energetics: Threshold (in cms) : $m_\pi \sim 135$ MeV

$$\hookrightarrow \quad E_{cms} = \sqrt{2\, E_{CR}\, E_\gamma\, (1-\cos\Theta) + m_p^2} \overset{!}{=} m_\pi + m_p$$

$$\overset{\Theta=0}{\Longleftrightarrow} \quad \boxed{E_{GZK} \sim 10^{20}\, eV \simeq 15\, J}$$

CR with $E > E_{GZK}$ loose energy <u>very</u> quickly.

↳ Do not expect events above E_{GZK}

6.5 The Hillas plot

<u>Question:</u> Max. energy an accelerator can magnetically contain

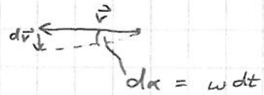
$$E^{max} \simeq \underbrace{Ze}_{\text{CR charge}} \cdot \underset{\substack{\uparrow \\ \text{accelerator size}}}{R} \cdot \underbrace{B}_{\text{avg. magnetic field}} \cdot \underset{\substack{\uparrow \\ \text{CR velocity}}}{c^{-1}}$$

based on expression for relativistic gyroradius:

$$\frac{d\overset{\swarrow = \gamma m v}{p^\mu}}{d\tau} = Ze\, F_{\mu\nu}\, u^\mu \qquad \left[\text{Jackson 12.1 and 12.2}\right]$$

$$\tau = \frac{t}{\gamma} \qquad u = \frac{dx^\mu}{d\tau}$$

$$\Longleftrightarrow \frac{d\vec{v}}{dt} = \vec{v} \times \underbrace{\vec{\omega}_B}_{\equiv \frac{Ze\vec{B}}{\gamma m}} = \text{angular frequency}$$

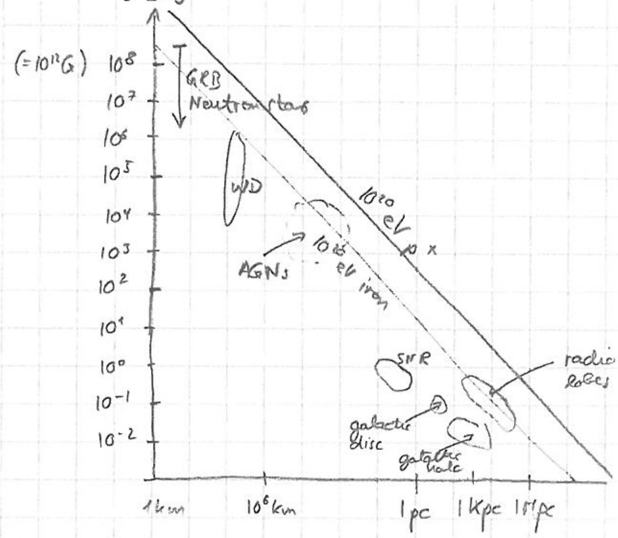


$$\omega_B = \frac{2\pi}{\underset{\nwarrow \text{revolution time}}{T}} = \frac{2\pi|\vec{v}|}{2\pi R}$$

$$\Rightarrow \quad Ze \cdot B \cdot R = E \cdot v$$

UHE CR can only be produced in accelerators with large B field and large R

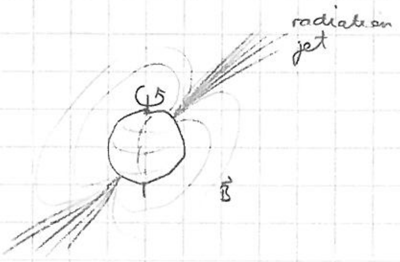
↳ Hillas plot [show!]



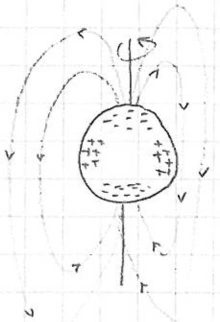
6.6 Pulsars as cosmic e^+e^- accelerators and γ ray sources

Pulsar = fast rotating neutron star with large B-field

f can be ~ kHz



Field configuration:



- charge separation ⇒ strong $\vec{E}$-fields
- plasma outside the star cancels E-field components parallel to B-field if dense enough (particles move only along B field lines)
- acceleration in regions of low plasma density → requires open field lines

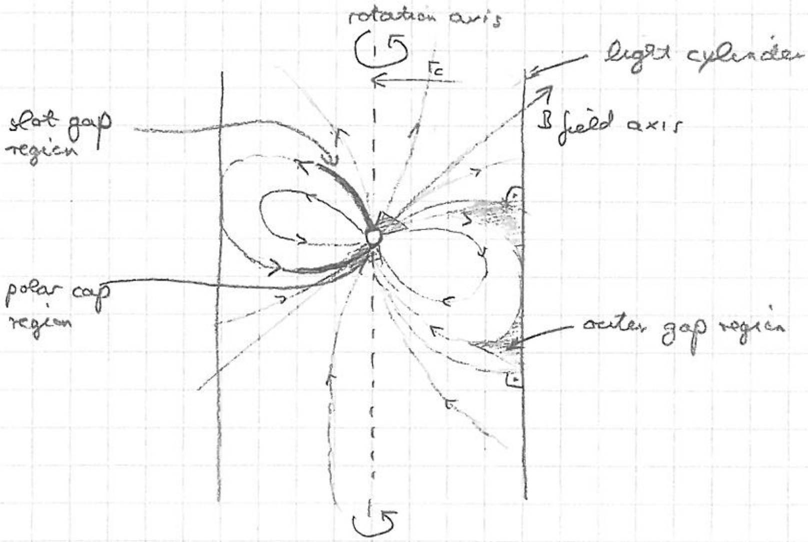
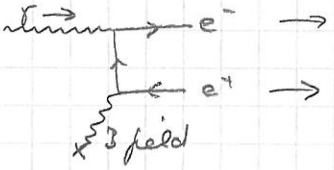
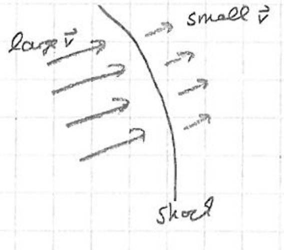


Image from
Science 322 (2008) 1221

- e^- from pulsar surface or surrounding plasma can easily reach $E \sim 100$ GeV
- γ emission by curvature radiation (= synchrotron emission due to propagation along curved B field lines)
- e^+e^- pair production via



- Pulsar wind nebulae:
  Relativistic particles from pulsar stream outward, form standing shock wave => • 1st order Fermi acceleration
  - magnetic flux reconnection

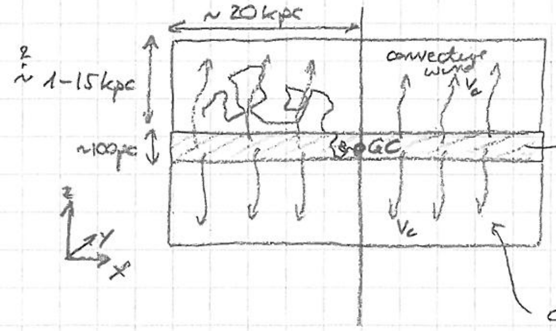


  - other mechanisms...

- Many details <u>not understood</u>

6.7 Cosmic ray transport

How do high-E charged particles travel from their sources to the Earth?



galactic disc: gas, stars — CR sources

confinement layer: turbulent B-fields

Cirelli et al.: 1012.4515

Salati, Cargèse 2009 lectures

Master equation (diffusion - loss eq.)

$$\underbrace{\frac{\partial f(t,\vec{x},E)}{\partial t}}_{=0 \text{ in steady state}} - K \cdot \Delta f + \frac{\partial}{\partial E}\left[b(E) \cdot f \right]$$

density of particles per unit energy

diffusion coefficient (assumed x-indep.)

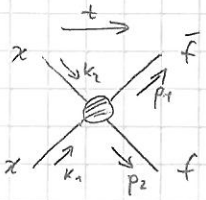
loss coefficient function

$$+ \underbrace{\frac{\partial}{\partial z}\left(v_c f \right) \cdot \text{sgn}\, z}_{\text{convective wind}} = \underbrace{Q\left(t, \vec{x}, E\right)}_{\text{source term}}$$

[Flux $J = K \cdot \vec{\nabla} f \Rightarrow \frac{\partial f}{\partial t} = \vec{\nabla} J + \ldots = K \Delta f + \ldots$

6.7b) Indirect DM detection

Freeze-out requires



This can still happen today.

Remember (□), p.24: Rate per volume element

$$\frac{\Gamma}{V} = \left(\tfrac{1}{2}\right) \int \underbrace{d\Pi(p_1)}_{= \frac{d^3 p_1}{(2\pi)^3 \cdot 2E(p_1)}} \, d\Pi(p_2) \, d\Pi(k_1) \, d\Pi(k_2) \, \phi_\chi(k_1) \, \phi_\chi(k_2)$$

only for self-conjugate DM to avoid double counting

$$\cdot (2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \, \overline{|\mathcal{M}|^2}$$

$$\overset{(\square\square)\,p.25}{=} n_\chi^2 \langle \sigma_{\chi\chi \to f\bar{f}} \, v_{rel} \rangle$$ (or → p. 25a)

Strong dependence on DM number density

6.7.1 DM distribution in the Milky Way

Nare, N-body simulations
www.usm.uni-muenchen.de/..

from N-body simulations (linearized method from sec. 5.1 not applicable)

Idea: Trace evolution of many ($10^8 - 10^9$) "particles" with mass $m \sim 10^3 M_\odot - 10^4 M_\odot$ 0803.0898

Force =

$$\underset{\uparrow \text{ force on } i\text{-th particle}}{\vec{F}_i} = m_i \sum_{j \neq i} \frac{G m_j (\vec{x}_i - \vec{x}_j)}{(|\vec{x}_i - \vec{x}_j|^2 + \varepsilon^2)^{3/2}} \qquad (*)$$

regulator to avoid divergence

Time step:

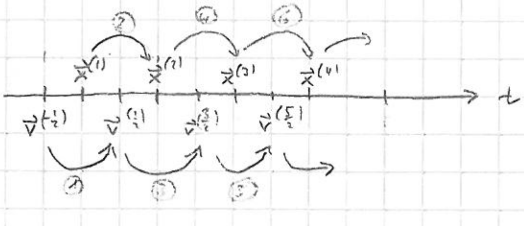
$$\boxed{\begin{aligned} \vec{x}_i^{(n+1)} &\equiv \vec{x}_i^{(n)} + \vec{v}_i^{(n+\frac{1}{2})} \Delta t \\ \vec{v}_i^{(n+\frac{1}{2})} &\equiv \vec{v}_i^{(n-\frac{1}{2})} + \frac{1}{m} \vec{F}_i(\vec{x}_1^{(n)}, \ldots \vec{x}_N^{(n)}) \Delta t \end{aligned}}$$

at $t = t^{(n)}$

at $t = t^{(n+\frac{1}{2})}$

particle mass

[Note: For $\Delta t \to 0$: $\dot{\vec{x}} = \vec{v}$; $\dot{\vec{v}} = \vec{F}/m$]



"Leapfrog scheme"
makes algorithm T-invariant
(to order $(\Delta t)^2$)

[In both time directions, first compute next $\vec{v}$, then use it to compute next $\vec{x}$.

Accuracy: Assume exact solution is $\vec{X}(t)$, $\vec{V}(t)$

Discretisation error in n-th step

$$\delta^{(n)} \equiv \frac{1}{m}\vec{F}\left(\vec{X}(t^{(n)})\right) - \frac{1}{\Delta t}\left[\vec{V}(t^{(n+\frac{1}{2})}) - \vec{V}(t^{(n-\frac{1}{2})})\right]$$

$$= \frac{1}{m}\vec{F}\left(X(t^{(n)})\right) - \frac{1}{(\Delta t)^2}\left[X(t^{(n+1)}) - 2X(t^{(n)}) + X(t^{(n-1)})\right]($$

Taylor-expand:

$$X(t^{(n\pm 1)}) = X(t^{(n)}) \pm \Delta t\, \dot{X}(t^{(n)}) + \frac{1}{2}(\Delta t)^2 \underbrace{\ddot{X}(t^{(n)})}_{=\frac{1}{m}\vec{F}(X(t^{(n)}))} + \ldots$$

$$\pm \frac{1}{6}(\Delta t)^3 \dddot{X}(t^{(n)}) + \frac{1}{24}(\Delta t)^4 \frac{d^4X(t^{(n)})}{dt^4}$$

$$\Rightarrow \boxed{\delta^{(n)} = \frac{(\Delta t)^2}{12}\frac{d^4X(t^{(n)})}{dt^4}}$$

plug into (†) for $X(t^{(n+1)})$ and $X(t^{(n-1)})$ $\rightarrow$

$\Rightarrow$ Method is 2nd order accurate

Numerical tricks: Noak and Scholapedia

Main bottleneck: Sum in (*) $\sim O(N^2)$

- Tree methods:
  - Divide cubic volume into 8 sub-cubes
  - If sub-cube contains <2 particles $\rightarrow$ done, continue with next sub-cube
  - If sub-cube contains ≥ 2 particles $\rightarrow$ sub-divide again
  - and so on

  $\Rightarrow$ Each tree node = large pseudo-particle

  When computing $\vec{F}$:
  - For small $|\vec{x}_i - \vec{x}_j|$: exact evaluation (use tree "leaves")
  - For large $|\vec{x}_i - \vec{x}_j|$: use larger and larger pseudo-particles (higher tree nodes)

  $\hookrightarrow$ $O(N \log N)$ operations

- Mesh methods:
  - Compute $\vec{F}$ on discrete grid (solve Poisson eq. with FFT) then use this force field instead of (*)
  - Adaptive methods: Use smaller grid spacing in "interesting" regions; adapt dynamically

- Hybrid methods: (*) or tree method on small scales (more accurate) mesh method on large scale (faster)

Results

[Show Aquarius video www.mpa-garching.mpg.de/aquarius/Aq-A-2-evolv.mp4]

[Show Aquarius halo profiles]

[r^{-2}: radius where $\rho(r)$ has "isothermal"
value -2

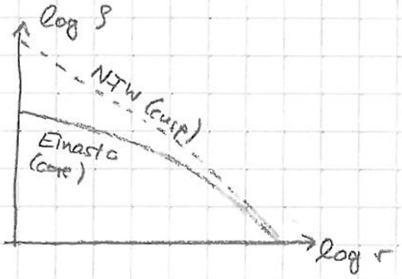
Phenomenological fits:

Navarro-Frenk-White (NFW):
(divergent @ $r=0$ ⇒ "cusp")

$$\boxed{\rho^{NFW}(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}}$$

Einasto:
(finite @ $r=0$ ⇒ "core")

$$\boxed{\rho^{Einasto}(r) = \rho_0 \exp\left[-\frac{2}{\alpha}\left[\left(\frac{r}{r_0}\right)^{\alpha}-1\right]\right]}$$

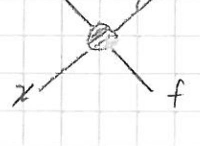


Complications

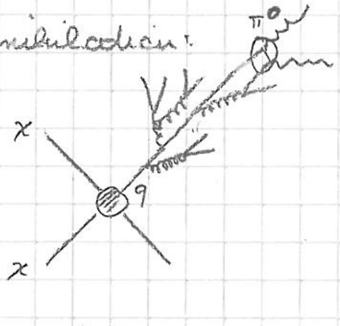
- Subhalos (dwarf galaxies)
- Effect of baryons

6.7.2 Gamma rays

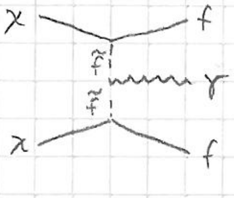
Prompt γ rays from DM annihilation:



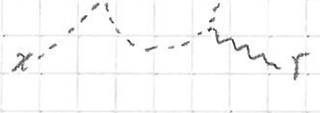
Final state radiation
(FSR)



π° decay
(for hadronic final state)



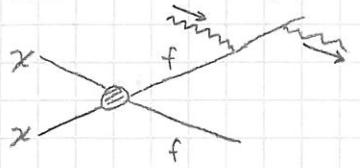
Internal bremsstrahlung
($\tilde{f}$ e.g. sfermion in SUSY)



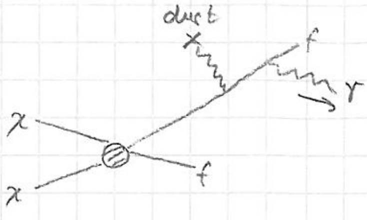
Direct annihilation
(only through loops)

⇒ $\boxed{\text{highly model-dependent}}$

Secondary γ rays:



Inverse Compton scattering (ICS)
on CMB + starlight



Bremsstrahlung
(less important)

Exercise 10 from S Profumo's TASI 2012 lecture ?

$\frac{1-\beta}{\sqrt{1-\beta^2}} = \left(\frac{1+\beta}{\sqrt{1-\beta^2}}\right)^{-1}$

Flux: $\left[m^{-2} s^{-1} sr^{-1} GeV^{-1}\right]$

line of sight in direction ψ from within $\Delta\Omega$

$$\boxed{\phi_\gamma = \frac{\Delta\Omega}{4\pi} \underbrace{\left[\frac{1}{\Delta\Omega}\int_{\Delta\Omega} d\Omega \int dl(\psi)\ \rho^2_{DM}(l,\psi)\right]}_{\text{"J-factor" (contains all astrophysics dependence)}} \frac{\langle\sigma v_{rel}\rangle}{(2)m_\chi^2} \frac{dN_\gamma}{dE_\gamma}}$$

avoid double counting for self-conjugate DM

$\Gamma = \langle\sigma v_{rel}\rangle \cdot \frac{\rho^2}{m_\chi^2}$

injection spectrum

$\boxed{\text{end } 04.07.2013}$

Places to look for γ rays from DM annihilation:

- Galactic Center (high DM density, lots of astrophysical BG sources)
- Dwarf galaxies (overdense regions in the DM halo with few baryonic objects)
- Diffuse γ rays from all over the sky (except center + disc of MW)
- galaxy clusters
- ....

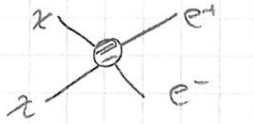
Results:

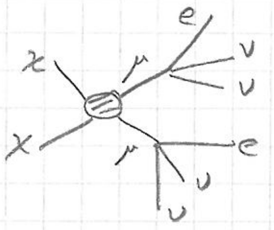
[show GC spectrum fit + morphology (fig. 2 in 1207.6047)

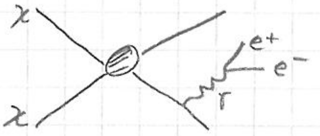
show dSph limits]

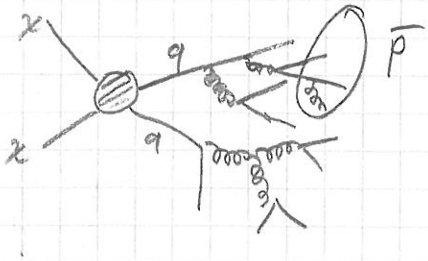
6.7.3 Charged cosmic rays

e.g. from









- e^+ and e^-
  - cannot travel too far → locally produced
  - BG from pulsars

  [show AMS-02 spectrum]

- $\bar{p}$, $\bar{D}$, ...
  - produced in hadronic DM annihilation

  [show PAMELA $\bar{p}$ spectrum]

7. Axions

7.1 The strong CP problem

Thesis of Wilco J. den Dunnen
(Amsterdam, 2008)

Consider QCD Lagrangian:

$$\mathcal{L}_{QCD} = i\,\overline{\Psi}\,\not{D}\,\Psi - \overline{\Psi_L} M \Psi_R - \overline{\Psi_R} M^{+} \Psi_L$$
$$- \frac{1}{4}\mathrm{tr}\, F_{\mu\nu}^{a} F^{\mu\nu,a} + \boxed{\frac{\theta g^2}{16\pi^2}\,\mathrm{tr}\, F_{\mu\nu}^{a}\,\tilde{F}^{\mu\nu,a}}$$

with

$$D_\mu = \partial_\mu - i g A_\mu^a \overset{\text{gauge group generators}}{t^a}$$

$$F_{\mu\nu} = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \overset{\text{gauge group structure constants}}{f^{abc}} A_\mu^b A_\nu^c\right) t^a$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma,a}\, t^a$$

Infinitesimal gauge transformation

$$\Psi(x) \rightarrow (1 + i\alpha^a(x)\, t^a)\,\Psi(x)$$

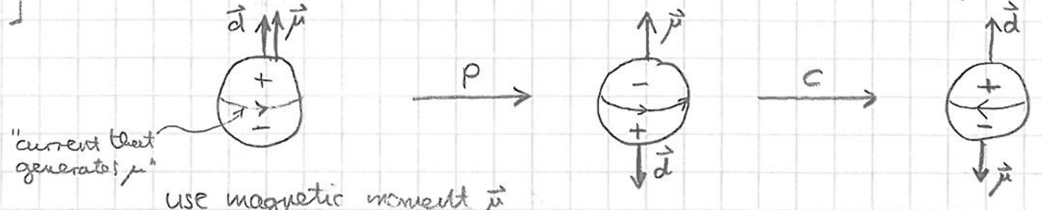
$$A_\mu^a(x) \rightarrow A_\mu^a(x) + \frac{1}{g}\partial_\mu \alpha^a(x) + f^{abc} A_\mu^b(x)\,\alpha^c(x)$$

Consequences of the extra term:

- P and CP odd: $F^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \neq 0$ only if μ, ν, ρ, σ are different → 3 spatial components, which flip sign under P. (C does not affect A_μ^a since it is a real field.)

see Wikipedia article on neutron EDM]

- Observable CP violation: Neutron electric dipole moment



use magnetic moment $\vec{\mu}$ to define the z axis (the only preferred direction in the neutron)

μ generated by circular currents, which are reversed under C

⇒ CP maps $\vec{\mu} \cdot \vec{d} \longrightarrow -\vec{\mu} \cdot \vec{d}$

we know $\vec{\mu} \neq 0$ ⇒ CP conservation only if $\vec{d} = 0$

Experimental searches constrain $|\vec{d}| < 0.29 \cdot 10^{-25}$ e cm

One can show (using heavy theoretical machinery):

$$d_n \cong \Theta \cdot 10^{-16} \, e \cdot cm$$

$$\Rightarrow \quad \boxed{\Theta \ll 10^{-10}}$$

Strong CP problem: Why is Θ so small?

Relation to axial $U(1)_A$ transformations

following Peskin/Schröder sec. 19.2

Consider transformation $\psi \to \psi' \equiv (1 + i\overset{\text{infinitesimal}}{\alpha}\gamma^5)\psi$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}(1 + i\alpha\gamma^5)$$

$$\Rightarrow \mathcal{L}_{QCD} \to \mathcal{L}'_{QCD} = i\bar{\psi}(1+i\alpha\gamma^5)\gamma^\mu D_\mu (1+i\alpha\gamma^5)\psi$$

$$-\left[M\bar{\psi}(1+i\alpha\gamma^5)\left(\tfrac{1+\gamma^5}{2}\right)(1+i\alpha\gamma^5)\psi + h.c.\right]$$

$$- \text{gauge kinetic term} - \Theta\text{-term}$$

$$= \mathcal{L}_{QCD} - \left[2i\alpha M \bar{\psi}_L \psi_R + h.c.\right] \quad (*)$$

⇒ Massless QCD ($M=0$) should be invariant ....

However: Quantum effects break this classical symmetry (→ <u>axial anomaly</u>)

Deeper reason: $S = \int d^4x \, \mathcal{L}_{QCD, M=0}$ invariant, but functional measure $\mathcal{D}$ in path integrals is not:

$$Z = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}A \exp\left[i \int d^4x \left(\bar{\psi} i \not{D} \psi + \frac{1}{4} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + \frac{\Theta g^2}{16\pi^2} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \right]$$

Only fermions affected by $U(1)$ $\Rightarrow$ Consider only

$$Z_\psi \equiv \int \mathcal{D}\psi \, \mathcal{D}\overline{\Psi} \; \exp\left[i \int d^4x \; \overline{\Psi} \, i\not{D} \, \psi \right]$$

Write $\psi, \overline{\Psi}$ in terms of eigenfunctions of $i\not{D}$

$$i\not{D} \, \phi_m = \lambda_m \phi_m$$

$$\hat{\phi}_m (i\overleftarrow{\not{D}}) \equiv -i (D_\mu \hat{\phi}_m) \gamma^\mu = \lambda_m \hat{\phi}_m$$

[Minus sign: $\phi^+ \overset{A=A^+}{\overset{\downarrow}{A}} = \phi^+ A^+ = (A\phi)^+$

$$\Rightarrow \psi(x) = \sum_m \underset{\uparrow}{a_m} \phi_m(x) \quad ; \quad \overline{\Psi}(x) = \sum_m \underset{\nearrow}{\hat{a}_m} \hat{\phi}_m(x)$$

anticommuting (Grassmann) coefficients

$$\Rightarrow \mathcal{D}\psi \, \mathcal{D}\overline{\Psi} = \prod_m da_m \, d\hat{a}_m$$

Apply axial transformation:

$$a_m' = \sum_n \int d^4x \; \phi_m^+(x) \left(1 + i\alpha\gamma^5\right) \phi_n(x) \, a_n$$

$$\equiv a_m + \sum_n \underbrace{C_{mn}}_{= \int d^4x \, \phi^+_m \, i\alpha\gamma^5 \phi_n} a_n$$

$$\Rightarrow \mathcal{D}\psi' = \left[\det(1 + C) \right]^{-1} \mathcal{D}\psi \qquad (\text{analogously for } \mathcal{D}\overline{\Psi})$$

[Consider $\int da_m \, f(a_m) = \int da_m \, (A + B a_m) = B$

Substitute $a_m \to (1 + c) a_m \equiv a_m'$

$\hookrightarrow \int da_m \left(A + \frac{B}{1+c} a_m'\right) = \int da_m' \, J \left(A + \frac{B}{1+c} a_m'\right) \overset{!}{=} B$

$\Rightarrow J = 1+c \quad ; \quad da_m' = \frac{1}{J} da_m$

$$\text{Jacobian:} \quad \det(1 + C) = \exp\left[\operatorname{tr} \log(1 + C) \right] \overset{C \text{ infinitesimal}}{=} \exp(\operatorname{tr} C)$$

$$= \exp\left[i\alpha \int d^4x \sum_n \phi_n^+ \gamma^5 \phi_n \right]$$

[$\neq \exp\left[i\alpha \int d^4x \; \operatorname{tr} \gamma^5 \right] = 1$

because need to regularize infinite sum.

$$\log \det (1+G) = \lim_{M\to\infty} i\alpha \int d^4x \sum_n \phi_n^\dagger(x)\gamma^5 \phi_n(x) e^{\lambda_n^2/M^2}$$

end 11.07.2013

Note: For free particles ($A_\mu = 0$) or large momentum:

$$\lambda_m^2 = k^2 = E_{\vec k}^2 - \vec k^2$$

After Wick rotation to Euclidean space: $E_{\vec k}^2 \to -E_{\vec k}^2$

$$\Rightarrow \lambda_m^2 < 0$$

$$= \lim_{M\to\infty} i\alpha \int d^4x \langle x | \operatorname{tr} \gamma^5 e^{(i\not D)^2/M^2} | x \rangle$$

trace over Dirac indices and gauge group indices

Rewrite $(i\not D)^2$:

$$(i\not D)^2 = -\left(\partial_\mu - igA_\mu^a t^a\right) \underbrace{\gamma^\mu\gamma^\nu}_{=\frac{1}{2}\left(\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{=2g^{\mu\nu}} + \underbrace{[\gamma^\mu,\gamma^\nu]}_{-2i\sigma^{\mu\nu}}\right)} \left(\partial_\nu - igA_\nu^b t^b\right)$$

$$\overset{\sigma^{\mu\nu}\text{ antisymm.}}{=} -D^2 + i\sigma^{\mu\nu}\left(-ig(\partial_\mu A_\nu^b)t^b - g^2 A_\mu^a A_\nu^b t^a t^b\right)$$

$$\overset{\sigma^{\mu\nu}\text{ antisymm.}}{=} -D^2 + \frac{1}{2}i\sigma^{\mu\nu}\left[-ig\left(\partial_\mu A_\nu^b - \partial_\nu A_\mu^b\right)t^b - g^2 A_\mu^a A_\nu^b [t^a, t^b]\right]$$

$$= -D^2 + \frac{g}{2}\sigma^{\mu\nu}\left[\partial_\mu A_\nu^b - \partial_\nu A_\mu^b + g f^{acb} A_\mu^a A_\nu^c\right] t^b$$

$$= -D^2 + \frac{g}{2}\sigma^{\mu\nu}F_{\mu\nu}$$

In the limit $M \to \infty$:

- expand $(i\not D)^2$ in A_μ^a (assume A_μ^a and its derivative bounded from above, while M is not)
- Derivative = momentum k, only large k relevant $\Rightarrow$ keep only term containing two derivatives in the exponent, expand the rest

no, k = fermion momentum

$$\log\det(1+G) = \lim_{M\to\infty} i\alpha\int d^4x \operatorname{tr}\left[\gamma^5 \frac{1}{2!}\left(\frac{g}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right]\langle x | e^{-\partial^2/M^2} | x\rangle$$

traces with γ^5 need at least 4 Dirac matrices to be non-zero

- $$\langle x | e^{-\partial^2/M^2} | x \rangle = \lim_{x\to y}\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} e^{k^2/M^2}$$

$$\underset{k^0 \to ik_E^0}{\overset{\text{Wick rotation}}{=}} i\int\frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2}$$

$$= \frac{iM^4}{16\pi^2}$$

- surface of unit 3-sphere: $2\pi^2$
- $\int_0^\infty x^3 e^{-x^2/M^2} dx = \frac{M^4}{2}$

$$\bullet \quad \mathrm{tr}\, \gamma^5 \sigma^{\mu\nu} \sigma^{\varrho\tau} = -\tfrac{1}{4} \cdot (-4i)\, \varepsilon^{\mu\nu\varrho\tau} - (\mu \leftrightarrow \nu) - (\varrho \leftrightarrow \tau)$$
$$+ (\mu \leftrightarrow \nu,\ \varrho \leftrightarrow \tau)$$
$$= 4i\varepsilon^{\mu\nu\varrho\tau}$$

$$\Rightarrow \log \det (1 + C) = \lim_{M \to \infty} i\alpha \int d^4x\, \frac{g^2}{8M^4}\, 4i\varepsilon^{\mu\nu\varrho\tau}\, \mathrm{tr}\, F_{\mu\nu} F_{\varrho\tau} \cdot \frac{iM^4}{16\pi^2}$$
$$= -i\alpha \int d^4x\, \frac{g^2}{16\pi^2}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Result:

$$\boxed{Z_\psi \longrightarrow Z'_\psi = \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' \exp\left[i \int d^4x\ \bar{\psi}' i\not{D} \psi' + i\alpha \frac{g^2}{8\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu,a} \right]}$$

In massless QCD, the Θ-term can be removed by a $U(1)_A$ rotation (→ absorbed into field redefinitions)

In massive QCD, Θ can be traded for complex phases in the mass matrix. See (*) and its generalization to non-infinitesimal transformations. What remains unchanged is

$$\Theta_{\text{eff}} \equiv \overset{\curvearrowleft}{N_f}\Theta + \arg \det M$$

factors $\mathcal{D}\psi \mathcal{D}\bar{\psi}$ for every quark flavor

Peccei Quinn
PRL 38 (1977) 1440
PRD 16 (1977) 1791

7.2 The Peccei – Quinn mechanism

Idea: Add a dynamical field that contributes to arg det M, and show that its vev is at $\langle a \rangle + \Theta_{\text{eff}} = 0$

Thesis of Wilco den Dunnen

Peccei, hep-ph/0607268

Toy model:

fermion
complex scalar

$$\mathcal{L} = -\tfrac{1}{4} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \not{D} \psi + y\, \phi\, \overline{\psi_L} \psi_R + y^* \phi^+ \overline{\psi_R} \psi_L$$

Yukawa coupling

$$- (\partial_\mu \phi)^+ (\partial^\mu \phi) - V(\phi^+, \phi) + \frac{\Theta g^2}{16\pi^2} \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Axial $U(1)_{PQ}$ transformations:

$$\psi_L \to e^{i\alpha} \psi_L \quad ; \quad \psi_R \to e^{-i\alpha} \psi_R \ ; \ \phi \to e^{2i\alpha} \phi$$

- as above: absorb θ into complex phase of y
- assume ϕ develops vev
- show that $\boxed{\arg\langle\phi\rangle + \arg y = 0 \mod 2\pi}$

<u>Outline of proof:</u> Goal: Compute effective potential (= scalar potential including quantum corrections)

Partition function $Z[J_\phi] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \mathcal{D}A \, e^{i\int d^4x \, (\mathcal{L} + J\phi)}$
(with "external source")

$$\Rightarrow \quad \langle\phi\rangle = -i \, \frac{\delta \log Z[J]}{\delta J}$$

Effective action: $\Gamma[\langle\phi\rangle] \equiv -i\log Z[J] - \int d^4y \, J(y) \langle\phi\rangle(y)$

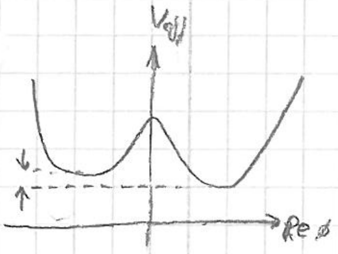
$$\Rightarrow \quad \frac{\delta\Gamma[\langle\phi\rangle]}{\delta\langle\phi\rangle(x)} = -i\int d^4y \, \frac{i \log Z[J]}{\delta J(y)} \frac{\delta J(y)}{\delta\langle\phi\rangle(x)}$$

$$- J(x) + \int d^4y \, \frac{\delta J(y)}{\delta\langle\phi\rangle(x)} \langle\phi\rangle(y)$$

$$= -J(x)$$

$\Rightarrow$ for zero external source, $\langle\phi\rangle$ is the minimum of $\Gamma[\langle\phi\rangle]$.

Effective potential: $V_{eff}[\langle\phi\rangle] \equiv -\frac{1}{V \cdot T} \, \Gamma[\langle\phi\rangle]$



Here: - Terms depending on $\arg\langle\phi\rangle$ have to come from Yukawa couplings
- Assume $y \ll 1$ and expand

$$\hookrightarrow \quad V_{eff}[\langle\phi\rangle] \sim V_{class}(|\langle\phi\rangle|^2) + y\langle\phi\rangle K + y^*\langle\phi\rangle^* K$$

real because the rest of $\mathcal{L}$ is symmetric under $\psi_L \leftrightarrow \psi_R$

$$\sim -\cos(\arg\langle\phi\rangle + \arg y) \cdot |y| \cdot |\langle\phi\rangle| \cdot A$$

sign not obvious

$\Rightarrow$ Extremal at $\boxed{\arg\langle\phi\rangle + \arg y = 0}$

7.3 The axion

- Write $\phi = \underset{\substack{\uparrow \\ |\langle\phi\rangle| \gg M_W}}{f_a} e^{i(\overset{\arg\langle\phi\rangle}{\overset{\downarrow}{\alpha}} + \underbrace{\frac{a}{f_a}}_{\text{dynamic field "axion"}})}$, neglecting radial excitations (heavy!)

Excitations of a $\Leftrightarrow$ chiral rotations. Using the axial anomaly, we can make the transformation

$$y f_a e^{i(\alpha + \frac{a}{f_a})} \bar{\Psi}_L \Psi_R + h.c. \longrightarrow y f_a e^{i\alpha} \bar{\Psi}_L \Psi_R + h.c.$$

$$+ \frac{a g^2}{16\pi^2 f_a} \mathrm{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

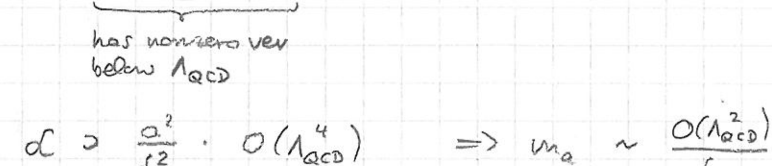
Low-E axion phenomenology depends on $a - A_\mu$ couplings since ϕ, Ψ are heavy.

If Ψ is charged under several gauge group factors, there is an $a F_{\mu\nu} \tilde{F}^{\mu\nu}$ coupling for each of them, in particular

$$\boxed{\frac{a e^2 \overset{\text{em charge of } \Psi}{q_\Psi^2}}{16\pi^2 f_a} \underbrace{F_{\mu\nu} \tilde{F}^{\mu\nu}}_{\text{em field strength tensor}}}$$

- Axion mass: [Note: $m_a = 0$ at the classical (Lagrangian) level

Bardeen Peccei Yanagida
Nucl Phys B 279 (1987) 401



$$\hookrightarrow \mathcal{L} \supset \frac{a^2}{f_a^2} \cdot O(\Lambda_{QCD}^4) \quad \Rightarrow m_a \sim \frac{O(\Lambda_{QCD}^2)}{f_a} \ll \Lambda_{QCD}$$

Calculation in chiral perturbation theory yields

Wantz, Shellard
0910.1066

$$\boxed{m_a = m_\pi \frac{f_\pi}{f_a} \frac{m_u m_d}{(m_u + m_d)^2} \times \text{model-dependent } O(1) \text{ correction factor}}$$

- Adding axions to the SM :

e.g. KSVZ (Kim Shifman Vainshtein Zakharov) model :

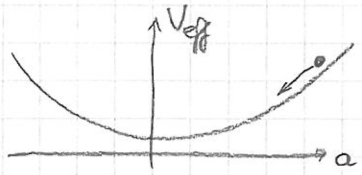
add scalar field ϕ with $\langle\phi\rangle \gg M_W$, and heavy quark Q with $m_Q \sim \langle\phi\rangle$; both charged under $U(1)_{PQ}$, all other fields uncharged

Q charged under $SU(3)$ and $U(1)_{em}$

↳ axion couples to gluons and photons

7.4 Production of axions in the early Universe: The misalignment mechanis

Axion potential:



assume random initial value

e.o.m: $\boxed{\ddot{a} + 3H\dot{a} + m_a^2 a = 0}$ (see chapter on inflation)

Early times: H very large $\rightarrow$ $\dot{a} \sim 0$

Later: Field "rolls down" the potential and oscillates
small m_a $\rightarrow$ shallow potential $\rightarrow$ slow oscillations,
but large amplitude if $\langle a \rangle_{initial}$ large

amplitude ~ # of particles

osc. frequency ~ energy of particles

=> production of many <u>non-relativistic</u> axions
(in spite of their low mass!)

=> $\boxed{\text{Cold DM candidate}}$

7.5 Detecting axions

Starting point: $\mathcal{L} \supset \frac{a}{f_a}\,\frac{e^2 q_\psi^2}{16\pi^2}\,F^{\mu\nu}\tilde{F}_{\mu\nu}$

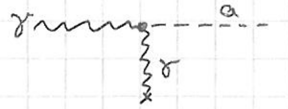
$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$\varepsilon^{0123} = -\varepsilon_{0123} = 1$]

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L} \supset \frac{a}{f_a}\,\frac{e^2 q_\psi^2}{32\pi^2}\left(-4\,\vec{E}\cdot\vec{B}\right)$$

$$= \boxed{-\frac{a}{f_a}\,\frac{\alpha q_\psi^2}{\pi}\,\vec{E}\cdot\vec{B}}$$



Photon can convert into axion in external $\vec{B}$-field ($\vec{E}$-field) if polarization along (orthogonal to) $\vec{B}$-field ($\vec{E}$-field) and propagation direction orthogonal to $\vec{E}$- or $\vec{B}$-field.

[show experiments and constraints]

end 18.07.2013

8. The baryon asymmetry of the Universe

8.1 Introduction

Observation: Only matter in the Universe, (almost) no antimatter

- Absence of annihilation lines (511 keV, 938 MeV, etc.)
- Low antimatter fraction in cosmic rays
  (→ mostly secondary production

Murayama
CERN lectures
25.05.2010

Koichi Funakubo:
"Baryon Asymmetry of the Universe"

Early Universe: Matter and antimatter thermally produced
Later: Annihilation via B (baryon number) and L (lepton number) conserving SM processes

⇒ There must have been <u>more matter than antimatter</u> when the annihilations happened

Asymmetry is tiny. CMB & BBN: $\boxed{\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} \sim 6 * 10^{-10}}$

baryon density

photon density

"Big Bang Nucleosynthesis"
production of elements
heavier than H at T ~ 100 keV

Where did the asymmetry come from?

↳ Initial conditions?
  → any initial asymmetry erased by inflation

↳ Particle physics processes!

Denis V. Perepelitsa:
"Sakharov conditions for baryogenesis"

Requirements (<u>Sakharov conditions</u>):

- $\boxed{\text{B violation}}$
- $\boxed{\text{C and CP violation}}$

↳ without C violation $\Gamma(X \to Y + B) = \Gamma(\bar{X} \to \bar{Y} + \bar{B})$

with C violation, but without CP violation:

$$\Gamma(X \to q_L q_L) = \Gamma(\bar{X} \to \bar{q}_R \bar{q}_R)$$

$$\Rightarrow \Gamma(X \to q_L q_L) - \Gamma(\bar{X} \to \bar{q}_L \bar{q}_L) =$$

$$\Gamma(\bar{X} \to \bar{q}_R \bar{q}_R) - \Gamma(X \to q_R q_R)$$

⇒ asymmetries in LH and RH sectors cancel

- **Departure from thermal equilibrium**

  ↳ in thermal equilibrium, all d.o.f. equally populated (chemical potentials ensure this)

Actually, the SM satisfies these:

- $\not{C}$, $\not{C}\not{P}$: weak interactions, CKM phase
- out of equilibrium : bubble formation during ewk phase transition → interactions at bubble boundaries out of equilibrium
- B violation ⟶ see next section

.... but the numbers don't work out.
(e.g. phase transition not 1st order at $m_h \cong 125$ GeV, CKM phase too small, ...)

8.2 Sphalerons: B+L violation in the SM

Consider pure $SU(2)_L$ theory:

$$\mathcal{L} = -\frac{1}{4} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} + i \overline{\Psi_L} \not{D} \Psi_L$$

Axial anomaly

$$\partial_\mu \overline{\Psi} \gamma^\mu \gamma^5 \Psi = -\frac{g^2}{16\pi^2} \underbrace{N_f}_{\text{\# of fermion families}} \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

[Proof: As in sec. 7.1 with $\alpha \to \alpha(x)$; vary w.r.t. $\alpha(x)$ in the end;
see Peskin/Schröder sec. 19.2; $\overline{\Psi} i \not{D} \Psi \to \overline{\Psi} \gamma^\mu \Psi (-\partial_\mu \alpha) \to + \alpha(x) \partial_\mu j^\mu$]

B current: $j^\mu_B \equiv \overline{\Psi_L} \gamma^\mu \Psi_L = \frac{1}{2} \overline{\Psi} \gamma^\mu \Psi - \frac{1}{2} \overline{\Psi} \gamma^\mu \gamma^5 \Psi$

$$\Rightarrow \partial_\mu j^\mu_B = \frac{g^2}{2 \cdot 8\pi^2} N_f \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Similarly $\partial_\mu \underbrace{j^\mu_L}_{\text{Lepton number current}} = \frac{g^2}{2\, 8\pi^2} N_f \operatorname{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}$

⇒ <u>B+L not conserved, B−L is conserved</u>

Connection to vacuum structure of non-Abelian gauge theories:

B+L violating processes ("sphaleron processes") change from one vacuum state to another

Consider non-infinitesmal gauge transformation

$$\Psi_L \rightarrow \mathcal{U}(x)\,\Psi_L \quad ; \quad \mathcal{U} = e^{i\alpha^a(x)\frac{\sigma^a}{2}} \in SU(2)$$

$$A_\mu \rightarrow \mathcal{U} A_\mu \mathcal{U}^{-1} - (\partial_\mu \mathcal{U})\,\mathcal{U}^{-1}$$

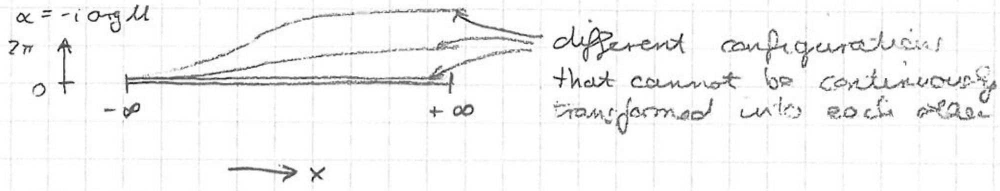
Vacuum state = "pure gauge":

$$A_\mu = -(\partial_\mu \mathcal{U})\,\mathcal{U}^{-1}$$

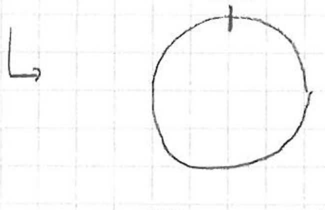
Chose boundary conditions

$$\mathcal{U} \xrightarrow{|x|\to\infty} 1 \quad ; \quad A_\mu \xrightarrow{|x|\to\infty} 0 \quad \text{mod } 2\pi$$

Simple analogy: $\mathcal{U}(1)$ in 1+1 dim



Since $\lim\limits_{|x|\to-\infty} A_\mu(x) = \lim\limits_{|x|\to+\infty} A_\mu(x)$, can "compactify" $\mathbb{R}$ to S^1 (a circle), with $-\infty \Leftrightarrow 0$; $+\infty \Leftrightarrow 2\pi \doteq 0$



Consider map $S^1 \rightarrow \mathcal{U}(1)$; $x \longmapsto e^{i\alpha(x)} \equiv \mathcal{U}$

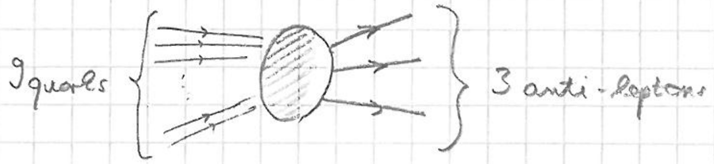
Classify vacuum gauge field configurations by "winding number" = number of times $\mathcal{U}$ travels around the circle corresponding to $\mathcal{U}(1)$ as x travels once around the circle corresponding to compactified space.

We don't see B+L violation in perturbative calculations

↳ must be non-perturbative processes

Detailed investigation reveals:

Sphaleron processes:



$$\Rightarrow \Delta B = -3 \quad ; \quad \Delta L = -3$$

- energy barrier of order M_W
- accessible in early Universe at $T \gtrsim M_W$

8.3 Leptogenesis

Idea: - Generate lepton asymmetry in L and CP violating decays of heavy ($\gg M_W$) particles out of thermal equilibrium
- Sphaleron processes will convert ΔL to ΔB until $\Delta B \sim \Delta L$ ($\sim$ thermal equilibrium)

The heavy particles are RH neutrinos N_R, which could simultaneously explain why LH neutrinos are so light via the seesaw mechanism

$$SM + N_R: \quad \mathcal{L} \supset - \underset{\uparrow \text{"Dirac mass"} \sim M_W}{m_D} \overline{\nu_L} N_R - y_D h \overline{\nu_L} N_R$$

$$+ \frac{1}{2} \underset{\uparrow \text{"Majorana mass} \gg M_W}{m_M} \overline{(N_R)^c} N_R + h.c.$$

where $(N_R)^c = \underset{\uparrow \text{particle-antiparticle conjugation}}{\hat{C}} N_R \equiv -i\gamma^2 N_R^*$.

Physical meaning: $\hat{C}$ transforms RH particle to LH antiparticle ($\gamma^5 \hat{C} \psi = +i\gamma^2\gamma^5 \psi^* = -\hat{C}(\gamma^5\psi)^*$)

No reason why LH antiparticle $\neq$ LH particle if particle is uncharged (like N_R)

Mass terms couple LH fields to RH fields
↳ $\overline{(N_R)^c} N_R$ is a valid mass term

Integrating out N_R: Write $n \equiv \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix}$

$$\hookrightarrow + \mathcal{L} \supset -\frac{1}{2} \overline{\left(\nu_L \;\; (N_R)^c \right)^c} \begin{pmatrix} 0 & m_D \\ m_D^{(T)} & m_M \end{pmatrix} \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix} + h.c.$$

$$= -\frac{1}{2} \overline{(\nu_L)^c} m_D (N_R)^c - \frac{1}{2} \overline{((N_R)^c)^c} m_M (N_R)^c$$

$$-\frac{1}{2} \overline{((N_R)^c)^c} m_D^{(T)} \nu_L + h.c.$$

$$\overline{(\nu_L)^c} m_D (N_R)^c = \overline{(-i\gamma^2 \nu_L^*)} m_D (-i\gamma^2 N_R^*)$$

$$= \nu_L^T \gamma^{2\dagger} i \gamma^0 m_D - i\gamma^2 N_R^*$$

$$= \nu_L^T \gamma^0\gamma^2\gamma^0\gamma^0\gamma^2 m_D N_R^*$$

$$\overset{\gamma^2\cdot\gamma^2=-1}{=\ -}\ \nu_{L\alpha}\ \gamma^0_{\alpha\beta}\ N^*_{R\beta}$$

spinor indices

fermion anticomm.

$$= +N^*_{R\beta}\ \gamma^0_{\beta\alpha}\ \nu_{L\alpha}\ m_D$$

use $\gamma^0_{\alpha\beta} = \gamma^0_{\beta\alpha}$

$$= \overline{N_R}\ \nu_L$$

$$\Rightarrow \mathcal{L} \supset -\tfrac{1}{2} m_D \overline{N_R}\nu_L - \tfrac{1}{2}\overline{(N_R)^c}\, m_M N_R - \tfrac{1}{2} m_D \overline{N_R}\nu_L + h.c.$$

□

Diagonalize mass matrix → eigenvalues ~ $m_M, -\frac{m_D^2}{m_M}$
(fix sign of second eigenvalue by rephasing fields)

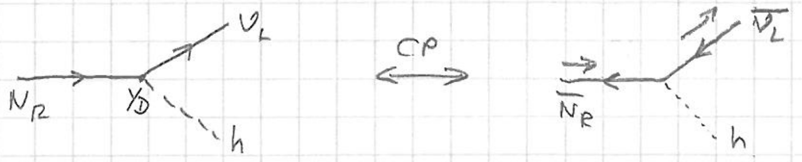
↳ $\mathcal{L} \supset -\frac{1}{2}\frac{m_D^2}{m_M}\overline{(\nu_L')^c}\,\nu_L'$

with $\nu_L' \simeq \nu_L + O\left(\frac{m_D}{m_M}\right)(N_R)^c$

↳ with $m_D \sim 100$ GeV, $m_M \sim 10^{14}$ GeV, we obtain

$$m_\nu \equiv \frac{m_D^2}{m_M} \sim 0.1\,\text{eV}$$

Heavy neutrino decay :



$$i\mathcal{M} = i y_D \, \bar{u}_\nu \underbrace{P_R}_{=\frac{1+\gamma^5}{2}} u_N \qquad\qquad i\overline{\mathcal{M}} = i y_D^* \, \bar{v}_\nu \, P_L \, v_N$$

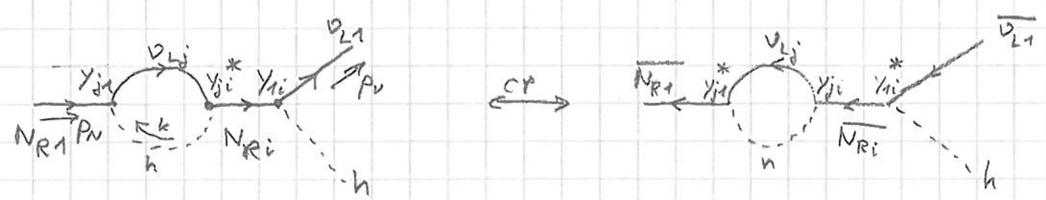
$$|\mathcal{M}|^2 = |y_D|^2 \, \mathrm{tr}\left[(\not{p}_N + m_R) \, P_L \, \not{p}_\nu \, P_R\right]$$

$$= \frac{1}{2}|y_D|^2 \, \mathrm{tr} \, \not{p}_N \not{p}_\nu$$

$$= 2|y_D|^2 \, p_N \cdot p_\nu \qquad\qquad = |\overline{\mathcal{M}}|^2$$

$$\Rightarrow \quad A_{CP} \equiv \frac{|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2}{|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2} = 0$$

↳ no $\cancel{CP}$ at tree level, but there are loop corrections, and there is more than one neutrino flavor.



$$i\mathcal{M}_{loop} \hat{=} \sum_{ij} \int \frac{d^4k}{(2\pi)^4} \, \bar{u}_{\nu 1} \, i y_{1i} \, P_R \, \frac{i(\not{p}_N + m_{Ni})}{p_N^2 - m_{Ni}^2} \, i y_{ji}^* \, P_L \, \frac{i(\not{p}_N + \not{k})}{(p_N + k)^2} \, i y_{j1} P_R u_{N1} \cdot \frac{i}{k^2}$$

$$= \underset{\substack{\uparrow \\ i^6}}{-} \sum_{i,j} y_{1i} y_{ji}^* y_{j1} \, \bar{u}_{\nu 1} \, P_R \, \frac{\not{p}_N}{p_N^2 - m_{Ni}^2} \left[\int \frac{d^4k}{(2\pi)^4} \, \frac{\not{p}_N + \not{k}}{(p_N + k)^2} \cdot \frac{1}{k^2}\right] u_{N1}$$

<u>Idea:</u> $|\mathcal{M}_{tree} + \mathcal{M}_{loop}|^2 \neq |\overline{\mathcal{M}_{tree}} + \overline{\mathcal{M}_{loop}}|^2$

$\left[\left(\text{use } |\mathcal{M}_{loop}|^2 = |\overline{\mathcal{M}_{loop}}|^2 \text{ and } |\mathcal{M}_{tree}|^2 = |\overline{\mathcal{M}_{tree}}|^2\right)\right]$

$\Leftrightarrow \quad 2 \, \mathrm{Re} \, \mathcal{M}_{tree} \mathcal{M}_{loop}^* \neq 2 \, \mathrm{Re} \, \overline{\mathcal{M}_{tree}} \, \overline{\mathcal{M}_{loop}}^*$

$\Leftrightarrow \quad \cancel{\arg \mathcal{M}_{tree}} - \arg \mathcal{M}_{loop} \neq \cancel{\arg \overline{\mathcal{M}_{tree}}} - \arg \overline{\mathcal{M}_{loop}}$

$\Leftrightarrow$ [Phase from couplings] · [phase from loop integral]

$\neq$ [Phase from couplings]* · [Phase from loop int.

Loop integrals yield complex phases if particles in the loop can go on-shell simultaneously

$$\int \frac{d^4k}{(2\pi)^4} \frac{\not{p}_N + \not{k}}{(p_N+k)^2} \frac{1}{k^2} = \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\not{p}_N + \not{k}}{\left[x(p_N^2+2kp_N+k^2) + (1-x)k^2\right]^2}$$

$$\overset{\ell \equiv k+p_N x}{=} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\not{\ell} + (1-x)\not{p}_N}{\left[\ell^2 + x(1-x)p_N^2\right]^2}$$

$$= \int_0^1 dx \, (1-x)\not{p}_N \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2 + x(1-x)p_N^2\right]^2}$$

$$= \int_0^1 dx \, (1-x)\not{p}_N \frac{i}{(4\pi)^2} \Big(\underbrace{\frac{2}{\varepsilon} - \gamma + \log 4\pi}_{\text{removed by renormalizati}}$$

$$- \log \big[\underbrace{-x(1-x)p_N^2}_{<0}\big]$$

$\Rightarrow$ $\log\left[-x(1-x)p_N^2\right]$ imaginary!

[Note: If particles in loop were heavy, we would have obtained $\log\left[-x(1-x)p_N^2 + x m_\nu^2 + (1-x) m_h^2\right] \in \mathbb{R}$]

$$\Rightarrow \mathcal{M}_{tree} \mathcal{M}_{loop}^* \propto y_{11} \cdot \sum_{i,j} y_{1i}^* y_{ji} y_{j1}^* \frac{1}{m_{N1}^2 - m_{Ni}^2} \cdot \underbrace{i}_{\text{from loop}}$$

$$\overline{\mathcal{M}_{tree}} \, \overline{\mathcal{M}_{loop}}^* \propto y_{11}^* \sum_{i,j} y_{1i} y_{ji}^* y_{j1} \frac{1}{m_{N1}^2 - m_{Ni}^2} \cdot i$$

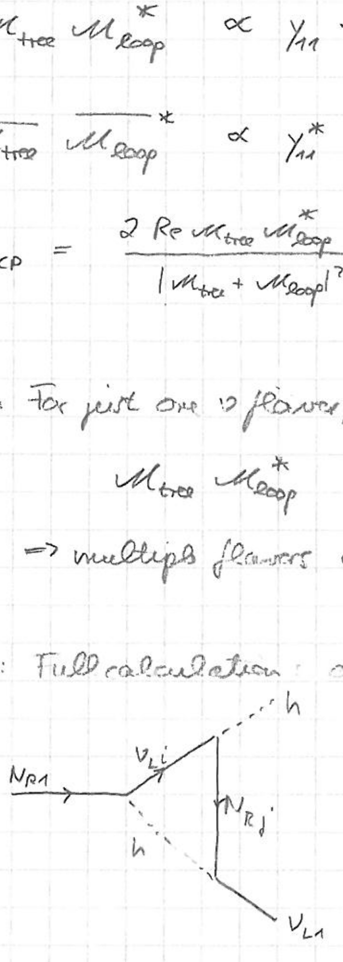
$$\Rightarrow A_{CP} = \frac{2 \operatorname{Re} \mathcal{M}_{tree} \mathcal{M}_{loop}^* - 2 \operatorname{Re} \overline{\mathcal{M}_{tree}} \, \overline{\mathcal{M}_{loop}}^*}{|\mathcal{M}_{tree} + \mathcal{M}_{loop}|^2 + |\overline{\mathcal{M}_{tree}} + \overline{\mathcal{M}_{loop}}|^2} \neq 0$$

[**Note**: For just one ν flavor, we would have obtained

$$\mathcal{M}_{tree} \mathcal{M}_{loop}^* \propto |y_{11}|^4 \cdot i = \overline{\mathcal{M}_{tree}} \, \overline{\mathcal{M}_{loop}}^*$$

$\Rightarrow$ multiple flavors essential

Note: Full calculation: add'l diagrams:



Kusenko 0906.2968

9. Sterile neutrinos as warm dark matter

9.1 Active and sterile neutrino mixing

$$\mathcal{L} \supset \sum_{\alpha=e,\mu,\tau} \overline{\nu}_{\alpha L}\, i\not{\partial}\, \nu_{\alpha L} + \sum_{\alpha=e,\mu,\tau} \Big[\frac{g}{\sqrt{2}} \Big(W^{\mu+}\, \overline{\nu}_{\alpha L}\, \gamma_\mu\, \overset{\text{SM charged leptons}}{e_{\alpha L}} + h.c. \Big)$$

$$+ \frac{g}{2\cos\theta_W}\, Z^\mu\, \overline{\nu_{\alpha L}}\, \gamma_\mu\, \nu_{\alpha L} \Big]$$

$$- \sum_{\substack{\alpha=e,\mu,\tau \\ j=1\ldots n}} Y_{\alpha j}\, \underbrace{\overline{L}_\alpha}_{=\begin{pmatrix}\nu_{\alpha L}\\ e_{\alpha L}\end{pmatrix}}\, \underbrace{\tilde{H}}_{= i\sigma^2 H^* = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}G^+(x)\\ \langle v\rangle + h(x) + iG^0(x)\end{pmatrix}^*}\, \overset{\text{SM singlet fermions ("sterile neutrinos")}}{N_{jR}} + h.c.$$

$$- \frac{1}{2} \sum_{j,k=1\ldots n} M_{jk}\, \underbrace{\overline{(N_{jR})^c}}_{\substack{\text{charge conjugation:} \\ \hat{C}:\ \psi \longmapsto \psi^c \equiv i\gamma^2\psi^*}}\, N_{jR} + h.c. \qquad + \sum_{j=1\ldots n} \overline{N}_{\alpha R}\, i\not{\partial}\, N$$

<u>Sterile neutrinos</u> N (singlet under $SU(3)_c \times SU(2)_L \times U(1)_Y$) appear in most neutrino mass models; here, we have n of them.
[Exception: Models with triplet Higgs field]

Mass term can also be written as

$$\mathcal{L}_{mass} \supset -\frac{1}{2} \begin{pmatrix} \overline{(\nu_L)^c} & \overline{N_R} \end{pmatrix} \begin{pmatrix} 0 & Y \overset{\text{Higgs vev}}{\frac{\langle v\rangle}{\sqrt{2}}} \\ Y^T \frac{\langle v\rangle}{2} & M \end{pmatrix} \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix}$$

Diagonalizing this matrix gives, for $Y_{\alpha j} \sim O(1)$; $M \gg \langle v\rangle$:

n eigenvalues of order M

3 eigenvalues of order $\frac{\langle v\rangle^2}{M}$ $\Longrightarrow$ 3 very light neutrinos

The <u>seesaw mechanism</u>

Mixings between $(N_{jR})^c$ and $\nu_{\alpha L}$ are of order $\frac{\langle v\rangle}{M}$.

But: not all eigenvalues of M must bee $\gg \langle v \rangle$
(we don't know what physics determines M)

↳ it is possible that some sterile neutrinos are lighter (e.g. keV)

<u>Lagrangian in the mass basis</u> $\quad \nu_i \equiv \mathcal{U}^*_{\alpha i} \, \nu_\alpha$ $\quad$ $(i = 1 \dots 3+n$

leptonic mixing matrix

$\nu_\alpha = (\nu_e, \nu_\mu, \nu_\tau, N^c_{1R}, \dots N_{nR}$

$$\mathcal{L} \supset \sum_{i=1\dots 3+n} \frac{g}{\sqrt{2}} \left(W^{\mu +} \, \bar{\nu}_{iL} \, \mathcal{U}^*_{\alpha i} \, \gamma_\mu \, e_{\alpha L} + \text{h.c.} \right)$$

$$+ \sum_{\substack{i,k = 1 \dots 3+n \\ \alpha = e, \mu, \tau}} \frac{g}{2 \cos \Theta_W} \, Z^\mu \, \bar{\nu}_{iL} \, \mathcal{U}^*_{\alpha i} \, \gamma_\mu \, \mathcal{U}_{\alpha j} \, \nu_j$$

+ diagonal mass terms $\quad$ + kinetic terms

In a simple two flavor framework (which we will use from now on):

active neutrino $\rightarrow \quad \nu_a = \cos\Theta \; \nu_1 + \sin\Theta \; \nu_2$

sterile neutrino $\rightarrow \quad \nu_s = -\sin\Theta \; \nu_1 + \cos\Theta \; \nu_2$

9.2 Neutrino oscillations

Assume a neutrino of flavor $\alpha = e, \mu, \tau$ is produced

$$|\nu_\alpha\rangle = \mathcal{U}^*_{\alpha i} \, |\nu_i\rangle \qquad \text{(sum over } i \text{ implied, } i = 1 \dots 3+n)$$

Detected neutrino state

$$\langle \nu_\beta | = \mathcal{U}_{\beta k} \, \langle \nu_k |$$

Time and space evolution:

energy/momentum of i-th mass eigenstate

$$|\nu_\alpha\rangle \longrightarrow |\nu_\alpha(t,x)\rangle = \Sigma_i \, \mathcal{U}^*_{\alpha i} \, e^{-iE_i t + i p_i x} \, |\nu_i\rangle$$

⇒ <u>Oscillation amplitude:</u>

$$\mathcal{A} \equiv \langle \nu_\beta | \nu_\alpha(t,x) \rangle$$

$$= \sum_{i,k} \mathcal{U}^*_{\alpha i} \mathcal{U}_{\beta k} \, e^{-iE_i t + ip_i x} \langle \nu_k | \nu_j \rangle$$

$$= \sum_{i} \mathcal{U}^*_{\alpha i} \mathcal{U}_{\beta i} \, e^{-iE_i t + ip_i x}$$

<u>Note:</u> States with different energies and momenta (E_i, p_i) can interfere only if E, p - Heisenberg uncertainties are larger than $|E_i - E_k|$; $|p_i - p_k|$

<u>Oscillation probability</u>

$$P_{\alpha\to\beta} = |\mathcal{A}|^2 = \sum_{i,k} \mathcal{U}^*_{\alpha i} \mathcal{U}_{\beta i} \mathcal{U}_{\alpha k} \mathcal{U}^*_{\beta k} \, e^{-i(E_i - E_k)t + i(p_i - p_k)x}$$

In the early Universe, we don't care about where exactly a particular neutrino is produced or interacts
⇒ average over x

$$\hookrightarrow \overline{P_{\alpha\to\beta}} \equiv \frac{1}{\mathcal{N}} \int dx \, P_{\alpha\to\beta}$$

$\uparrow$ normalization

$$= \frac{1}{\mathcal{N}} \sum_{i,k} \mathcal{U}^+_{\alpha i} \mathcal{U}_{\beta i} \mathcal{U}_{\alpha k} \mathcal{U}^*_{\beta k} \, e^{-i(E_i - E_k)t} \, \delta(p_i - p_k)$$

$$= \sum_{i,k} \mathcal{U}^*_{\alpha i} \mathcal{U}_{\beta i} \mathcal{U}_{\alpha k} \mathcal{U}^*_{\beta k} \exp\left[-i\left(\sqrt{p^2 + m_i^2} - \sqrt{p^2 + m_k^2}\right)t\right]$$

$$\simeq \sum_{i,k} \mathcal{U}^+_{\alpha i} \mathcal{U}_{\beta i} \mathcal{U}_{\alpha k} \mathcal{U}^*_{\beta k} \exp\left[-i \frac{\Delta m^2_{ik}}{2p} t\right]$$

$$\equiv m_i^2 - m_k^2$$

In the 2-flavor approximation: $\mathcal{U} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \equiv \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$

$$\hookrightarrow P_{a\to s} = \mathcal{U}^2_{a1} \mathcal{U}^2_{s1} + \mathcal{U}^2_{a2} \mathcal{U}^2_{s2} + \mathcal{U}_{a1} \mathcal{U}_{a2} \mathcal{U}_{s1} \mathcal{U}_{s2} \Big(e^{-i\frac{\Delta m^2}{2p}t} + e^{+i\frac{\Delta m^2}{2p}t}$$

$$= 2c^2 s^2 - c^2 s^2 \cdot 2\cos\frac{\Delta m^2 t}{2p}$$

$$= \sin^2 2\Theta \; \frac{1-\cos\frac{\Delta m^2 t}{2p}}{2}$$

$$= \sin^2 2\Theta \cdot \sin^2 \frac{\Delta m^2 t}{4p}$$

Time average: $\langle P_{a\to s} \rangle = \frac{1}{2}\sin^2 2\Theta$

Dodelson - Widrow
hep-ph/9303287

Kainulainen
Phys. Lett. B 244 (1990) 191

9.3 Sterile neutrino production through oscillations:

The Dodelson-Widrow mechanism

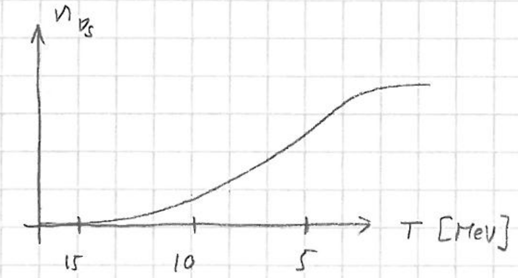
Active neutrino from the hot plasma ($T \gg$ MeV) oscillates → acquires a ν_s component of average magnitude $\frac{1}{2}\sin^2 2\Theta$.

After an average collision time t_{coll} ($\gg \frac{p}{\Delta m^2}$) a hard scattering process ("measurement") interrupts the coherent evolution of the mixed $\nu_a - \nu_s$ state.

[Scattering corresponds to a flavor measurement because only the ν_a component can scatter, not ν_s

With probability $\frac{1}{2}\sin^2 2\Theta$, the neutrino becomes a pure ν_s, otherwise it becomes a pure ν_a and starts oscillating again. (Oscillations of ν_s back into ν_a occur, but are negligible)

As this happens many times, the ν_s abundance increases over time until neutrinos decouple at $T \sim$ MeV (⇒ no more scattering)



Note: $\left.\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\frac{E_0}{a(t)} \\ &= -\frac{E_0}{a^2}\dot{a} \\ &= -E\cdot H \end{aligned}\right]$

$$\left(\frac{\partial}{\partial t} - H E \frac{\partial}{\partial E}\right) f_{\nu_s}(E,t) = \left[\frac{1}{2}\sin^2(2\Theta) \;\; \Gamma(E,t)\right] f_{\nu_a}(E,t)$$

Hubble parameter

ν_s distribution function

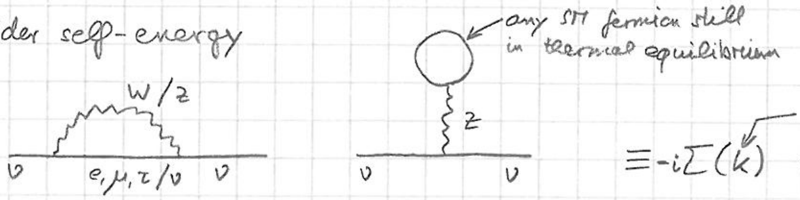
active ν distribution

9.4 The effective mixing angle

At nonzero ρ, T neutrinos have a dispersion different from $E = p$ $\Rightarrow$ "effective potential" V_{eff}

$$E = p + V_{eff}$$

Consider self-energy



ν momentum

$$\equiv -i\Sigma(k) \equiv$$

$\Rightarrow$ Neutrino propagator

$$= \frac{i}{\not k - m_0} + \frac{i}{\not k - m_\nu}\left[-i\Sigma(k)\right]\frac{i}{\not k - m_\nu} + \frac{i}{\not k - m_\nu}\left[-i\Sigma(k)\right]\frac{i}{\not k - m_\nu}\left[-i\Sigma(k)\right]\frac{i}{\not k - m_\nu} + \dots$$

geometric series

$$= \frac{i}{\not k - m_\nu}\,\frac{1}{1 - \frac{\Sigma(k)}{\not k - m_\nu}}$$

$$= \frac{i}{\not k - m_\nu - \Sigma(k)}$$

Particle states in QFT are defined as states for which the propagator is singular ("on-shell"), i.e. here

$$\boxed{\not k - m_\nu - \Sigma(k) = 0}$$

In vacuum, $\Sigma(k)$ is removed by renormalization

$$\hookrightarrow \not k - m_\nu = 0 \qquad \Big| \cdot (\not k + m_\nu)$$

$$\Leftrightarrow k^2 - m_\nu^2 = 0$$

$$\Leftrightarrow E_k^2 - \vec{k}^2 - m_\nu^2 = 0 \qquad \checkmark$$

Enqvist Kainulainen Maalampi;
Nucl. Phys. B 349
(1991) 754

At $T > 0$ this is no longer possible

↳ Need to compute $\Sigma(k)$ using finite-T propagators for charged leptons (and gauge bosons if $T \gtrsim m_W$)

$$S(p) = i(\not p + m)\left[\frac{1}{p^2 - m^2} + i\,\Gamma_f(p)\right]$$ for fermions (mass m)

$$D_{\mu\nu}(p) = i\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right)\left[\frac{1}{p^2 - M^2} - i\,\Gamma_b(p)\right]$$ for gauge bosons (mass M)

$$\Gamma_f = 2\pi\,\delta(p^2 - m^2)\left[\Theta(x)\,n_f^+(x) + \Theta(-x)\,n_f^-(x)\right]$$

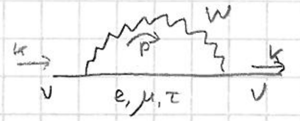
$$\Gamma_b = 2\pi\,\delta(p^2 - m^2)\,n_b(x)$$

$$n_f^\pm(x) = \frac{1}{e^{(|x| \mp \mu)/T} + 1} \quad ; \quad x = p \cdot u$$

$$n_b(x) = \frac{1}{e^{|x|/T} - 1}$$

u = 4velocity of the thermal bath = (1, 0, 0, 0) for plasma at rest

Then, for instance:



$$= \left(\frac{g}{\sqrt{2}}\right)^2 \int \frac{d^4p}{(2\pi)^4}\, \gamma_\mu \frac{1-\gamma^5}{2}\, S(k-p)\, \gamma_\nu \frac{1-\gamma^5}{2}\, D^{\mu\nu}(p)$$

A lengthy calculation (see eg. Enqvist, Kainulainen, Maalampi, Nucl. Phys. B 349 (1991) 754) gives for instance for ν_e:

$$\Sigma^{\nu_e} = \sqrt{2}\, G_F\, \underbrace{n_\gamma}_{\text{photon number density}} \Big(\overbrace{L_e}^{\text{from tadpole}} \overbrace{- 4\left(\frac{7\,\zeta(4)}{2\,\zeta(3}\right)^2 \frac{T^2}{M_W^2}}^{\text{from bubble}} \Big)\, \gamma^0 \frac{1-\gamma^5}{2} \equiv \tilde{\Sigma}\gamma^0 \frac{1-\gamma^5}{2}$$

$$L_e \equiv \frac{n_{e^-} - n_{e^+}}{n_\gamma}$$

$\sim 10^{-10}$ in standard cosmology

Riemann ζ function

$\Rightarrow$ dispersion relation ($m_\nu = 0$) $\Rightarrow$ can omit $\frac{1-\gamma^5}{2}$ in Σ^{ν_e} since RH ν fully decoupled, i.e. unphysical, never produced in this approximation

$$\not k - \gamma^0 \tilde{\Sigma} = 0 \qquad \Big| \cdot \not k - \gamma^0 \tilde{\Sigma}$$

$$k^2 - \underbrace{\tilde{\Sigma}^2}_{\text{small}} + 2k^0 \tilde{\Sigma} = 0$$

$$(k^0)^2 - \vec{k}^2 + 2k^0 \tilde{\Sigma} = 0$$

Write $k^0 = |\vec{k}| + V_{eff}$, use $V_{eff} \ll k^0, |\vec{k}|$

↳ $$\cancel{|\vec{k}|^2} + 2|\vec{k}| V_{eff} + 2|\vec{k}|\tilde{\Sigma} + \underbrace{2 V_{eff}\tilde{\Sigma}}_{\substack{\text{small}^2 \\ \to \text{neglect}}} = 0$$

$$\Leftrightarrow \quad \boxed{V_{eff} = -\tilde{\Sigma}}$$

$$\boxed{V_{eff} = -\sqrt{2}\, G_F\, n_\gamma \left(L_e - 39.7 \frac{T^2}{M_W^2} \right)}$$

Effect on neutrino mixing:

↳ Compute modified neutrino energy eigenstates in presence of a potential

$$\hat{H}_{\substack{\text{in flavor} \\ \text{basis}}} \quad \mathcal{U} \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} \mathcal{U}^+ + \begin{pmatrix} V_{eff} & \\ & 0 \end{pmatrix}$$

Goal: Determine matrix $\mathcal{U}_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}$ that diagonalizes $\hat{H}$.

Since flavor-independent terms ($\sim \mathbb{1}$) do not matter for this, it is sufficient to consider

$$\tilde{\hat{H}} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2}{4p} & \\ & \frac{\Delta m^2}{4p} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} V_{eff} & \\ & 0 \end{pmatrix}$$

Homework: Diagonalize

Result: $$\sin^2 2\theta_m = \frac{\sin^2 2\theta}{\left(\frac{2 p V_{eff}}{\Delta m^2} - \cos 2\theta\right)^2 + \sin^2 2\theta}$$

↳ If the resonance condition $\cos 2\theta = \frac{2 p V_{eff}}{\Delta m^2}$ is fulfilled, mixing becomes maximal, independent of θ.

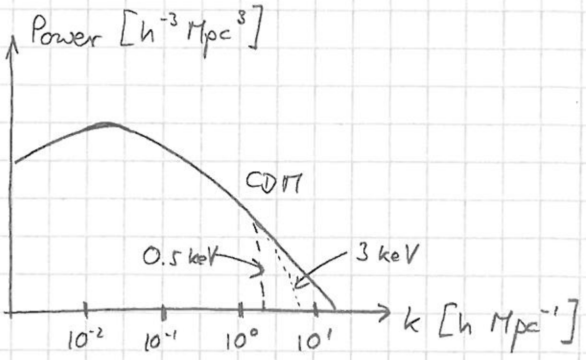
⟹ Needs to be taken into account when evaluating the sterile neutrino abundance

If $L_e \gg 10^{-10}$, resonant production more efficient

↳ Shi-Fuller mechanism. But: whence $L_e \gg 10^{-10}$?

9.5 Sterile neutrinos as warm dark matter (WDM)

Lower DM mass ⇒ efficient at transporting energy over larger scales ⇒ washout of structure up to some maximum scale



1109.6291

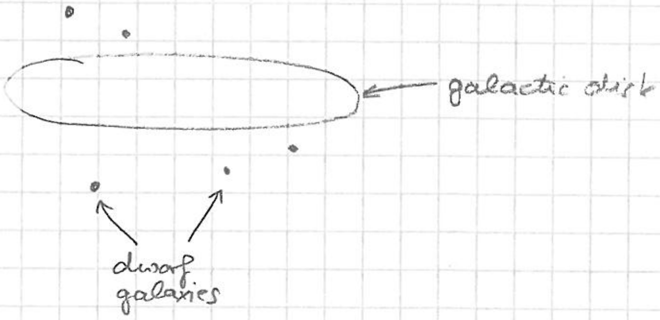
This may help to solve the following problems:

Kusenko 0906.2968

James Bullock
"Solving Too Big To Fail"
(Harvard talk)

- Missing satellites

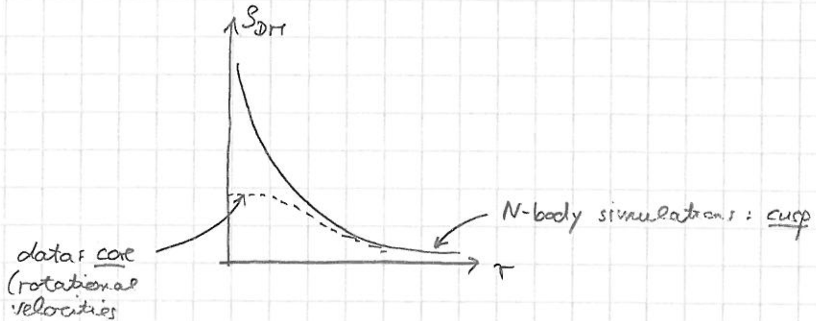
  Satellite galaxies (or dwarf galaxies) are small companions of a galaxy, presumed to form in overdense regions of the main halo.



However: The observed number of satellites is smaller than predicted by N-body simulations
⇒ small scale structure appears to be suppressed ⇒ WDM?

- Cusp-vs.-core problem

  DM density profiles in dwarf galaxie



WDM suppresses high Fourier modes ⇒ more cored profiles

- <u>Too Big To Fail - Problem</u> (TBTF)

  Simulations predict several very large DM subhalos in a Milky Way-type galaxy

  But: None observed

  Do those subhalos fail to form stars → standard star formation models say "no", they are "too big to fail"?

  WDM: Reduces subhalo counts

- ... + several other inconsistencies

None of these provides convincing evidence (e.g. may be due to baryon effects typically neglected in N-body simulations)

Also: may be difficult to solve all of them simultaneously
e.g. solving TBTF requires parameters that predict too <u>few</u> satellites

Kusenko, 0906.2968

<u>Side remark:</u> <u>Pulsar kicks</u>

Pulsar = fast rotating, strongly magnetized neutron star from supernova explosion

Observation: Pulsars travel at velocities ~ 250 $\frac{km}{s}$
up to 1600 km/s

Whence these "kicks"?

SN simulations do not predict them.

<u>Hypothesis</u>: Anisotropic neutrino emission → pulsar recoils
(Note: ν's carry 99% of the SN energy release)

$\overset{(-)}{\nu}_e$ are produced anisotropically in strong B-field by

$$n + e^+ \longrightarrow p + \bar{\nu}_e$$

$$p + e^- \longrightarrow n + \nu_e$$

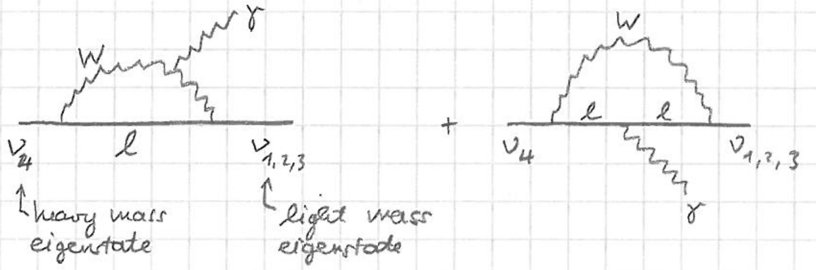
(polarized $e^\pm$ —— ν spin tends to point in the same direction; ν's are left-handed → anisotropic momentum distribution)

<u>But</u>: Active ν's cannot leave SN core immediately (too dense)
→ diffusion → isotropization

Sterile ν <u>can</u> leave without scattering ⇒ strong "kick"

9.6 Detection of sterile neutrino DM: x-ray lines

ν_s can decay through



[Note: Neutrinos exist as pure mass eigenstates in the Universe — any coherent superposition will have become incoherent long ago since wave packets corresponding to different mass eigenstates have different group velocities

Kusenko 0906.2968

$$\Gamma = \frac{9}{256\pi^4}\,\alpha_{em}\,G_F^2\,\sin^2\Theta\;m_s^5$$

$$\sim \frac{1}{1.8\cdot 10^{21}\,s}\,\sin^2\Theta\left(\frac{m_s}{keV}\right)^5$$

[cf. age of the Universe $\sim 4\cdot 10^{17}\,s$

Signature: Monochromatic x-ray flux from regions of high DM density (galaxy clusters, galactic center, dwarf galaxies, ...)