During the journey, Einstein explained his theory to me every day,
and when we arrived, I was convinced that he had understood it.
Chaim Weizmann, 1929

1. **Derivation of the Friedmann equation in flat spacetime**
Show that, for the energy-momentum tensor of a homogeneous, isotropic fluid, $T_{\mu\nu} = \mathrm{diag}\big[\rho(t), -p(t), -p(t), -p(t)\big]$, the (00) component of the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1}$$

yields, in a flat spacetime, the Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho(t)\,. \tag{2}$$

Make the ansatz

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) \tag{3}$$

for the metric in flat space, and use the definitions of the Ricci tensor,

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu} - \frac{\partial}{\partial x^\mu}\Gamma^\alpha{}_{\alpha\nu} + \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\alpha\mu}\,, \tag{4}$$

and of the Christoffel symbols,

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left(\frac{\partial}{\partial x^a} g_{bd} + \frac{\partial}{\partial x^b} g_{ad} - \frac{\partial}{\partial x^d} g_{ab}\right). \tag{5}$$

2. **Friedmann-Robertson-Walker metric in curved spacetime**
   (a) If we allow for curved spacetime, the FRW metric eq. (3) can be generalized to

$$ds^2 = dt^2 - R^2(t)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2) & \text{closed Universe} \\ d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\phi^2) & \text{flat Universe} \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\phi^2) & \text{open Universe} \end{cases}. \tag{6}$$

   The three cases correspond to generalized spherical coordinates for a 3-sphere, a flat Universe, and a 3-hyperboloid, respectively. Show that this metric can be written in the more compact form

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2\big[d\theta^2 + \sin^2\theta\, d\phi^2\big]\right), \tag{7}$$

   with $k = 1$ for a closed Universe, $k = 0$ for a flat Universe, and $k = -1$ for an open Universe.

   (b) In curved spacetime, the Friedmann equation takes the form

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho(t)\,. \tag{8}$$

   Consider a "curvature dominated" Universe, i.e. a Universe with $\rho \simeq 0$, but $k \neq 0$. Is such a Universe, open, closed, or flat? How does R scale with t in a curvature dominated Universe? What is the equation of state parameter w for a curvature dominated Universe?