Neutrino physics

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1. Neutrinos and their masses

1.1 The role of neutrinos in physics

The standard model fermions

Quarks $\begin{cases} u & c & t & q = +\frac{2}{3} \\ d & s & b & q = -\frac{1}{3} \end{cases}$

Leptons $\begin{cases} \nu_e & \nu_\mu & \nu_\tau & q = 0 \\ e & \mu & \tau & q = -1 \end{cases}$

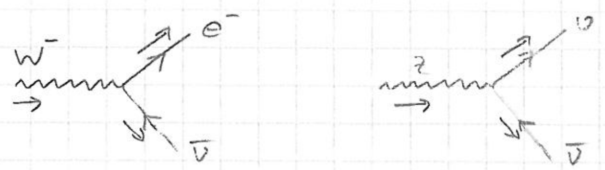
Role in physics

- Produced in nuclear reactions, e.g. $(A, Z) \to (A, Z+1) + e^- + \bar{\nu}_e$; $\pi^+ \to \mu^+ + \nu_\mu$, $p + p \to D + e^+ + \nu_e$
- May help elucidate the origin of flavor (why 3 families? why small mixing in quark sector, large in neutrino sector?)
- May be sensitive to GUT scale physics
- May help to explain the baryon asymmetry of the Universe
- can teach us about dark matter
- can teach us about sorces of high-E cosmic rays
- ...

Lagrangian:

$$\mathcal{L} \supset \sum_{\alpha = e,\mu,\tau} \Big[\bar{\nu}_{\alpha,L} i\not{\partial} \nu_{\alpha,L} + \frac{g}{\sqrt{2}} \left(W^{\mu +} \bar{\nu}_{\alpha,L} \gamma_\mu e_{\alpha,L} + h.c. \right)$$

$$+ \frac{g}{2\cos\theta_W} Z^\mu \bar{\nu}_{\alpha,L} \gamma_\mu \nu_{\alpha,L} \Big] + \text{mass term}$$



1.2 Dirac mass terms

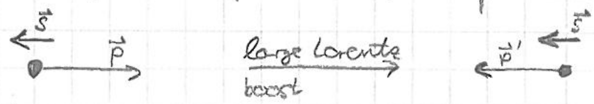
$\mathcal{L}$ involves only $\nu_L = \frac{1-\gamma^5}{2}\nu = \begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix}$

..... only upper components of 4-component spinor are physical
(LH ν + $\bar{\nu}$)

(Dirac) mass terms make ν_R physical:

$$\mathcal{L}_m \supset \sum_{\alpha,\beta} m_{\nu\alpha\beta}\, \bar{\nu}_{L\alpha} \nu_{R,\beta} + \text{h.c.}$$

Physical reason for 4 d.o.f.: Consider particle with LH helicity:



$\Rightarrow$ states with RH helicity must exist $\rightarrow$ 2 dof ; + antiparticles : 4 dof

<u>Problem:</u> Why is m_ν so much smaller than other fermion masse?

1.3 Majorana mass terms

Mass terms couple LH and RH fields.

The antiparticle of ν_L is a RH particle.

Could ν_R be identical to the antiparticle of ν_L ?

[Note: This can work only for neutrinos — for all other SM fermions, antiparticles carry opposite charge $\Rightarrow$ have to be different d.o.f.

More formally: charge conjugation:

$$\hat{C} : \quad \psi \longrightarrow \psi^c \equiv -\underbrace{i\gamma^2\gamma^0}_{\equiv C}\overline{\psi}^T = -i\gamma^2\psi^*$$

Effect on chirality:

$$\gamma^5\psi^c = +i\gamma^2\gamma^5\psi^* = -(\gamma^5\psi)^c$$

$\Rightarrow$ $\hat{C}$ flips chirality, transforms LH particle $\longleftrightarrow$ RH antiparticle

$\boxed{\text{Identify } \nu_R \equiv (\nu_L)^c}$

In 4-component spinor notation: $\nu_L = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$

$$(\nu_L)^c = -i\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \chi^* \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i\sigma^2\chi^* \end{pmatrix}$$

$$\Rightarrow \nu = \begin{pmatrix} \chi \\ i\sigma^2\chi^* \end{pmatrix} \quad ; \quad \nu^c = \nu$$

only 2 dof!

A new type of mass term:

$$\mathcal{L}_m \supset \sum_{\alpha,\beta}\frac{1}{2} m_{\alpha\beta}\,\overline{(\nu_{L\alpha})^c}\,\nu_{L,\beta} + \text{h.c.}$$

Problems: - How to obtain from $SU(2)$-invariant theory?
- Why is m so small?

1.4 Measuring neutrino mass

1.4.1 Kinematic measurement

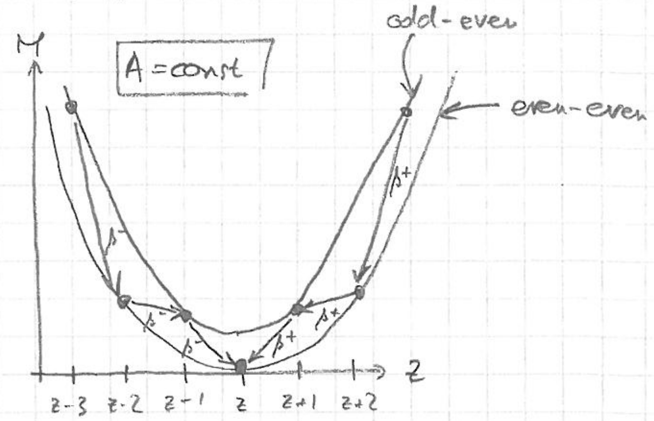
β-decay, e.g. $^3H \rightarrow {}^3He + e^- + \overline{\nu_e}$

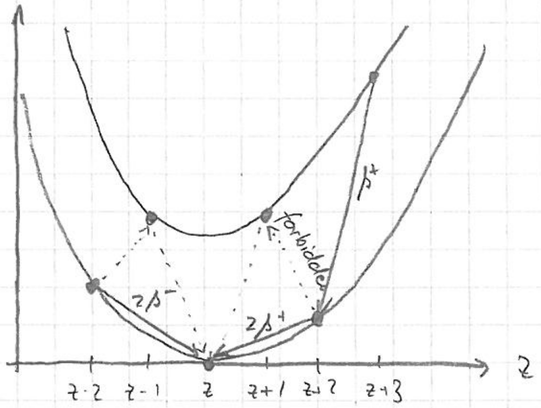
$$E_{e,max} = \underbrace{Q}_{m_H - m_{He}} - m_\nu$$

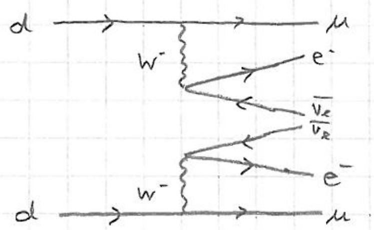
$\Rightarrow$ measure e^- spectrum precisely

1.4.2 Neutrinoless double beta decay

Nuclear masses:

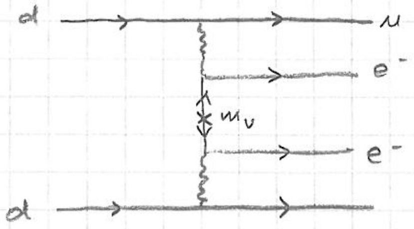






$$(A, Z) \rightarrow (A, Z+2) + 2e^- + 2\bar{\nu}_e$$

For Majorana neutrinos, also allowed



1.4.3 Cosmology

→ lecture by J. Jochum

neutrinos transport energy efficiently over large distances → wash out structure on small scales

1.5 The seesaw mechanism

__Goal:__ Explain why neutrino masses are so small

Augment SM with 3 RH neutrinos N_{Rj}, singlet under SM gauge group.
(Here: Consider just 1LH neutrino + 1 RH neutrino for simplicity)

$$\mathcal{L} \supset - \underbrace{m_D}_{\mathcal{O}(m_H)} \overline{\nu_L} N_R - \underbrace{\tfrac{1}{2} m_M}_{\text{can be very large}} \overline{(N_R)^c} N_R + \text{h.c.} \qquad (*)$$

see exercise

Write $n \equiv \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix}$

$$\hookrightarrow \mathcal{L} \supset -\tfrac{1}{2} \overline{n^c} M n + \text{h.c.}$$

$$= -\tfrac{1}{2} \begin{pmatrix} \overline{(\nu_L)^c} & \overline{N_R} \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & m_M \end{pmatrix} \begin{pmatrix} \nu_L \\ (N_R)^c \end{pmatrix} + \text{h.c.} \quad (**)$$

(see exercises for proof that (*) and (**) are equivalent.)

Diagonalize $M \longrightarrow$ eigenvalues $\sim -\frac{m_D^2}{m_M}, \ m_M$

eigenvectors $\nu_L' = \nu_L + \mathcal{O}\left(\frac{m_D}{m_M}\right) (N_R)^c$

$N_R' = N_R + \mathcal{O}\left(\frac{m_D}{m_M}\right) (\nu_L)^c$

Transform $\nu_L' \to i \nu_L'$ to absorb negative sign of eigenvalue into redefinition of phase of field

$$\hookrightarrow \mathcal{L} \supset -\tfrac{1}{2} \underbrace{\frac{m_D^2}{m_M}}_{\equiv m_\nu} \overline{(\nu_L')^c} \nu_L'$$

With $m_D \sim 100$ GeV, $m_M \sim 10^{14}$ GeV : $m_\nu \simeq \frac{m_D^2}{m_M} \sim 0.1$ eV

2. Neutrino mixing and oscillations

2.1 Three flavors of neutrinos

Remember:

$$\mathcal{L} \supset \sum_{\alpha=e,\mu,\tau} \Big[\overline{\nu_{\alpha,L}}\, i\partial\!\!\!/\, \nu_{\alpha,L} + \frac{g}{\sqrt{2}} \left(W^{\mu +}\, \overline{\nu_{\alpha L}}\, \gamma_\mu\, e_{\alpha L} + h.c. \right)$$

$$+ \frac{g}{2\cos\theta_W}\, Z^\mu\, \overline{\nu_{\alpha L}}\, \gamma_\mu\, \nu_{\alpha L} \Big]$$

$$+ \left(\sum_{\alpha,\beta} \frac{1}{2}\, m_{\alpha\beta}\, \overline{(\nu_{L\alpha})^c}\, \nu_{L\beta} + h.c. \right)$$

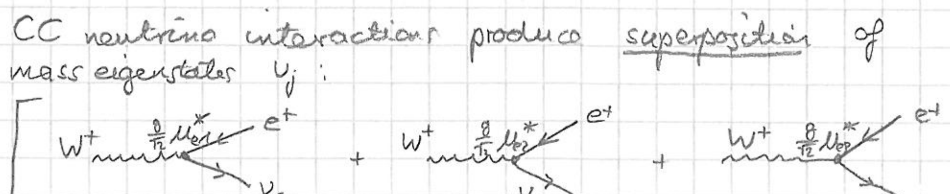
- $m_{\alpha\beta}$ in general off-diagonal
- Diagonalization: Write $\nu_{L\alpha} = U_{\alpha j}\, \nu_j$
  with $U^T m U = \mathrm{diag}(m_1, m_2, m_3)$
  For this to work, m must be symmetric, $m_{\alpha\beta} = m_{\beta\alpha}$
- Note. For Dirac mass-term $\sum_{\alpha\beta} m_{\alpha\beta}\, \overline{\nu_{L\alpha}}\, \nu_{R\beta}$,
  the transformation is $\nu_{L\alpha} = U_{\alpha j}\, \nu_j$; $\nu_{R\beta} = V_{\beta j}\, \nu_{Rj}$;
  $U^+ m V = \mathrm{diag}(m_1, m_2, m_3)$
  This works for arbitrary complex m.

- In the mass basis:

$$\mathcal{L} \supset \nu_{Lj}\, i\partial\!\!\!/\, \nu_{Lj} + \frac{g}{\sqrt{2}} \left(W^{\mu +}\, \overline{\nu_{Lj}}\, U^*_{\alpha j}\, \gamma_\mu\, e_{\alpha} + h.c. \right)$$

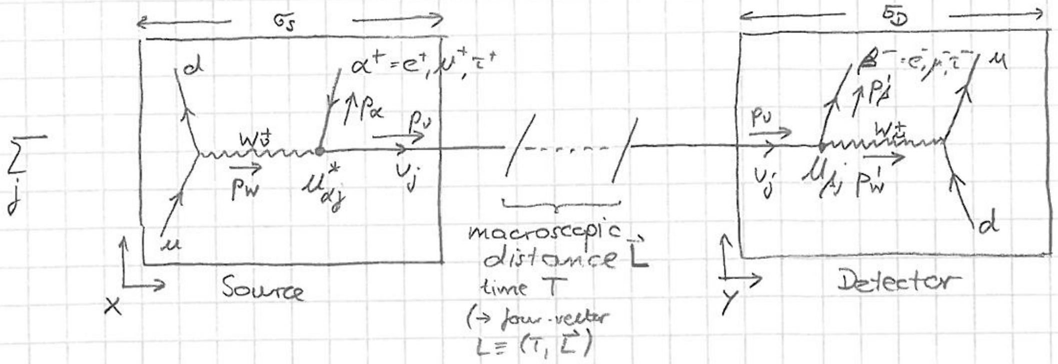
$$+ \frac{g}{2\cos\theta_W}\, Z^\mu\, \overline{\nu_{Lj}}\, \gamma_\mu\, \nu_{Lj} + \left(\frac{1}{2}\, m_j\, \overline{(\nu_{Lj})^c}\, \nu_{Lj} + h.c. \right)$$

$\Rightarrow$ CC neutrino interactions produce superposition of mass eigenstates ν_j:



2.2 Neutrino oscillations

Consider typical neutrino experiment



QFT

$$\mathcal{A} \propto \sum_j \int_{\text{Source}} d^4x \int_{\text{Detector}} d^4y \;\; \bar{u}_\beta(p_\beta')e^{ip_\beta' y}\gamma^\mu U_{\beta j}$$

$$\cdot \int \frac{d^4 p_\nu}{(2\pi)^4} \frac{\not{p}_\nu e^{-ip_\nu(y+L-x)}}{p_\nu^2 - m_j^2 + i\varepsilon} U^*_{\alpha j}\gamma^\nu v_\alpha(p_\alpha) e^{-ip_\alpha x}$$

$$\cdot \underbrace{\mathcal{M}_\nu^S \mathcal{M}_\mu^D}_{\text{hadronic matrix elements}} e^{ip_W' y} e^{-ip_W x}$$

$$= \sum_j U^*_{\alpha j} U_{\beta j} \mathcal{M}_\nu^S \mathcal{M}_\mu^D \int \frac{d^4 p_\nu}{(2\pi)^4}$$

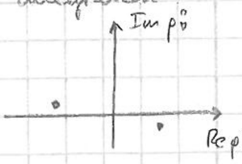
$$\cdot \hat{\delta}(p_\nu - p_\beta' - p_W')\,\hat{\delta}(p_\nu - p_\alpha - p_W)$$

approximate δ-functions, uncertainty $\sim (\text{size of } S, D)^{-1} \equiv \sigma_{S,D}^{-1}$

$$\cdot \bar{u}_\beta \gamma^\mu \frac{\not{p}_\nu e^{-ip_\nu L}}{p_\nu^2 - m_j^2 + i\varepsilon}\gamma^\nu v_\alpha$$

complex contour integration

$$\cong \sum_j U^*_{\alpha j} U_{\beta j} \mathcal{M}_\nu^S \mathcal{M}_\mu^D \int \frac{d^3 p_\nu}{(2\pi)^3}$$



$$\cdot \hat{\delta}(p_\nu - p_\beta' - p_W')\,\hat{\delta}(p_\nu - p_\alpha - p_W)$$

$$\cdot \bar{u}_\beta \gamma^\mu \frac{\not{p}_\nu}{2E_{\nu j}} e^{-i(E_\nu T - \vec{p}_\nu \vec{L})}\gamma^\nu v_\alpha$$

$$2E_{\nu j} = \sqrt{\vec{p}_\nu^{\,2} + m_j^2}$$

$(p_\nu^0 - \sqrt{\vec{p}_\nu^{\,2} + m_j^2} + i\varepsilon')$
$\cdot (p_\nu^0 + \sqrt{\vec{p}_\nu^{\,2} + m_j^2} - i\varepsilon')$
$= (p_\nu^0)^2 - (\sqrt{\vec{p}_\nu^{\,2} + m_j^2} - i\varepsilon')^2$
$= p_\nu^2 - m_j^2 + i\underbrace{\varepsilon}_{\equiv 2\sqrt{\vec{p}_\nu^{\,2} + m_j^2}\cdot\varepsilon'}$

Consider phase $\phi_j(\vec{p}_\nu) \equiv E_{\nu j}T - \vec{p}_\nu \vec{L}$

$$\phi(\vec{p}_\nu) \cong \vec{p}_\nu\left(T\frac{\vec{p}_\nu}{|\vec{p}_\nu|} - \vec{L}\right) + \frac{m_j^2}{2\vec{p}_\nu}T$$

QM

Produced neutrino state:

$$|\nu_\alpha\rangle = U^*_{\alpha j}\, |\nu_j\rangle$$

Detected neutrino state

$$\langle \nu_\beta | = U_{\beta k} \langle \nu_k |$$

Time and space evolution

$$|\nu_\alpha\rangle \to |\nu_\alpha(T, L)\rangle = \sum_j U^*_{\alpha j} e^{-iE_jT + ip_jL}|\nu_j\rangle$$

$$\hookrightarrow \mathcal{A} = \langle \nu_\beta | \nu_\alpha(t)\rangle$$

$$= \sum_{j,k} U^*_{\alpha j} U_{\beta k} e^{-iE_jT + ip_jL}\langle \nu_k | \nu_j\rangle$$

$$= \sum_j U^*_{\alpha j} U_{\beta k} e^{-iE_jT + ip_jL}$$

<u>Note:</u> States with different energies and momenta $(E_1, p_1), (E_2, p_2), (E_3, p_3)$ can interfere only if E, p-uncertainties in source, detector are larger than $|E_j - E_k|, |p_j - p_k|$

$\hat\delta$ functions have width $\sim \frac{1}{\sigma_{S,D}} < \frac{1}{Å} \simeq \mathrm{keV}$

↳ terms that vary slowly over such a momentum interval can be pulled out of the integral

$$\Rightarrow \mathcal{A} \propto \left(\bar u_\beta \gamma^\mu \frac{\overset{\sim\, p_e + p_W}{\not p_{\nu 0}}}{2 p_{\nu 0}} \gamma^\nu \ell_e \mathcal{M}^S_\nu \mathcal{M}^D_\mu + O\left(\frac{m_j^2}{p_{\nu 0}^2}\right) \right)$$

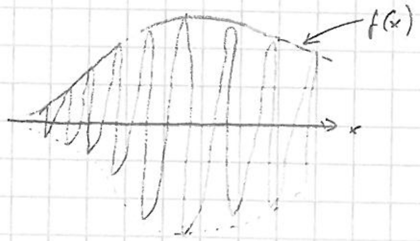
$$\cdot \sum_j U^*_{\alpha j} U_{\beta j} \int \frac{d^3 p_\nu}{(2\pi)^3}\, e^{-i(E_{\nu j} T - \vec p_\nu \vec L)}$$

$$\cdot\, \hat\delta(p_\nu - p'_\beta - p'_W)\, \hat\delta(p_\nu - p_e - p_W)$$

Method of stationary phase:

$$\int dx\, \overset{\text{function that varies significantly over interval } \Delta x}{f(x)}\, e^{i\phi(x)}$$

can be large only if $\phi(x)$ has $O(1)$ variations over intervals $\gtrsim \Delta x$ (otherwise, just oscillations average integral to zero)



Here, this means only region with $\vec p_\nu \parallel \vec L$ relevant $\Longrightarrow$ Reduce int. to 1D

Moreover, enforces $T \simeq \vec L \frac{\vec p_{\nu 0}}{\underbrace{E_{\nu j 0}}_{= \sqrt{p_{\nu 0}^2 + m_j^2}}} + O(\sigma_{S,D})$

Let us choose $T \equiv |\vec L|$

↳ $\mathcal{A} \propto \sum_j U^*_{\alpha j} U_{\beta j} \int dp_\nu\, e^{-i \frac{m_j^2 L}{2 p_\nu}}$

$$\cdot\, \hat\delta(p_\nu - p'_\beta - p'_W)\, \hat\delta(p_\nu - p_e - p_W)$$

$$\simeq \sum_j U^*_{\alpha j} U_{\beta j}\, e^{-i \frac{m_j^2 L}{2 p_{\nu 0}}}$$

$E_{\nu_j} = \sqrt{p_\nu^2 + m_j^2}$

$\simeq p_\nu + \frac{m_j^2}{2 p_\nu}$

$\frac{m_j^2 L}{2 p_\nu} - \frac{m_j^2 L}{2(p_\nu + \sigma_{S,D}^{-1})}$

$= \frac{m_j^2 L}{2 p_\nu} \frac{\sigma_{S,D}^{-1}}{p_\nu}$

$\lesssim \frac{(1\mathrm{eV})^2 \cdot \mathrm{km}}{2 \cdot \mathrm{MeV}} \frac{\mathrm{keV}}{\mathrm{MeV}} \ll 1$

Probability:

$$P_{\alpha\to\beta} \equiv \frac{1}{\mathcal{N}} |\mathcal{A}|^2$$

$$\propto \frac{1}{\mathcal{N}} \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \int \underbrace{dp_\nu\, d\tilde{p}_\nu}_{\simeq dE_\nu d\tilde{E}_\nu}$$

$$\cdot\, e^{-i(E_\nu T - p_\nu L) + i(\tilde{E}_\nu T - \tilde{p}_\nu L)}$$

$$\cdot\, \hat{\delta}(p_\nu - p_\beta' - p_N')\, \hat{\delta}(p_\nu - p_\alpha - p_N)$$

$$\cdot\, \hat{\delta}(\tilde{p}_\nu - p_\beta' - p_N')\, \hat{\delta}(\tilde{p}_\nu - p_\alpha - p_N)$$

Assume continuous neutrino flux ("stationary source"), and assume we're not interested in <u>when</u> each ν is detected

$$\hookrightarrow \overline{P_{\alpha\beta}} = \frac{1}{\mathcal{N}'} \int dT\, |\mathcal{A}|^2$$

$$= \frac{1}{\mathcal{N}'} \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*$$

$$\cdot \int dE_\nu\, e^{i(\sqrt{E_\nu^2 - m_j^2} - \sqrt{E_\nu^2 - m_k^2})L}$$

$$\cdot\, \hat{\delta}(\sqrt{E_\nu^2 - m_j^2} - p_\beta' - p_N')\, \hat{\delta}(\sqrt{E_\nu^2 - m_j^2} - p_\alpha - p_N)$$

$$\cdot\, \hat{\delta}(\sqrt{E_\nu^2 - m_k^2} - p_\beta' - p_N')\, \hat{\delta}(\sqrt{E_\nu^2 - m_k^2} - p_\alpha - p_N)$$

Use $\left(\sqrt{E_\nu^2 - m_j^2} - \sqrt{E_\nu^2 - m_k^2}\right)L \simeq -\frac{\overbrace{m_j^2 - m_k^2}^{\equiv \Delta m_{jk}^2}}{2E_\nu} L$

varies slowly over interval $[E_\nu, E_\nu + \sigma_{S,D}]$

$\to$ pull out of integral as well.

$$\Rightarrow \overline{P_{\alpha\beta}} = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*$$

$$\cdot \exp\left[-i \frac{\Delta m_{jk}^2}{2E_\nu} L\right]$$

$$P_{\alpha\to\beta} = |\mathcal{A}|^2 = \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j} U_{\beta k}^* e^{-i(E_j - E_k)T + i(p_j - p_k)L}$$

Assume continuous neutrino flux ("stationary source"), and assume we're not interested in <u>when</u> each neutrino is detected

$$\hookrightarrow \overline{P_{\alpha\to\beta}} = \underbrace{\frac{1}{\mathcal{N}}}_{\text{normalization}} \int dT\, |\mathcal{A}|^2$$

$$= \frac{1}{\mathcal{N}} \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j} U_{\beta k}^* \cdot 2\pi\, \delta(E_j - E_k)$$

$$\cdot \exp\left[i\left(\sqrt{E^2 - m_j^2} - \sqrt{E^2 - m_k^2}\right)L\right]$$

$$= \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j} U_{\beta k}^*$$

$$\cdot \exp\left[-i \frac{\Delta m_{jk}^2}{2E} L\right]$$

Check normalization: $\sum_\beta \overline{P_{\alpha\beta}} = \sum_{j,k} U_{\alpha j}^* U_{\alpha k}\, \delta_{jk}\, e^{-\frac{i \Delta m_{jk}^2 L}{2E_\nu}}$

$= 1$

Consider 2-flavor case :

$$\mathcal{U} = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

$$\Rightarrow P_{e\to\mu} = |\mathcal{U}_{e1}|^2 |\mathcal{U}_{\mu 1}|^2 + |\mathcal{U}_{e2}|^2 |\mathcal{U}_{\mu 2}|^2 + \mathcal{U}_{e1}\mathcal{U}_{\mu 1}\mathcal{U}_{e2}\mathcal{U}_{\mu 2}\left(e^{-i\frac{\Delta m^2}{2E_\nu}L} + e^{+i\frac{\Delta m^2}{2E_\nu}L}\right)$$

$$= \cos^2\Theta \sin^2\Theta \cdot 2 \quad - \cos^2\Theta \sin^2\Theta \cdot 2\cos\frac{\Delta m^2 L}{2E_\nu}$$

$$= 2\cdot\frac{1}{4}\sin^2 2\Theta - 2\cdot\frac{1}{4}\sin^2 2\Theta \cos\frac{\Delta m^2 L}{2E_\nu}$$

$$= \sin^2 2\Theta \ \sin^2\frac{\Delta m^2 L}{4E}$$

2.3 3-flavor neutrino oscillations

3-flavor mixing matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \mathcal{U} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Parameters:

General 3×3 matrix:	9 real params	9 phases
Unitary: $\sum_j \lvert \mathcal{U}_{\alpha j} \rvert^2 = 1$	-3	
$\sum_j \mathcal{U}_{\alpha j} \mathcal{U}^*_{\beta j} \overset{\alpha \neq \beta}{=} 0$	-3	-3
Field redefinitions $\nu_j \to e^{i\phi_j} \nu_j$, $\nu_\alpha \to e^{i\phi_\alpha} \nu_\alpha$		-5 (a common rescaling of all ν_α, ν_j does not alter $\mathcal{U}$)

$\Rightarrow$ 3 real parameters ("mixing angles"), 1 phase

Convenient parameterization: $\theta_{12}, \theta_{13}, \theta_{23}, \delta$

$c_{ij} \equiv \cos\theta_{ij}$
$s_{ij} \equiv \sin\theta_{ij}$

$$\mathcal{U} = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} e^{-i\delta} \\ & 1 & \\ -s_{13} e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

In addition, $P_{\alpha \to \beta}$ depends on two independent "mass squared differences"

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2 \quad \text{and} \quad \Delta m_{31}^2 \equiv m_3^2 - m_1^2$$

2.4 Neutrino oscillation experiments

Neutrino sources

↳ see presentation

Neutrino detectors

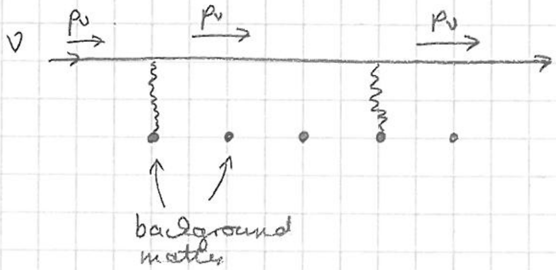
↳ see presentation

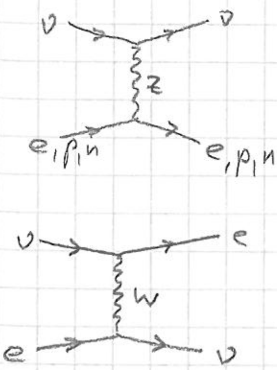
Neutrino oscillation experiments

↳ see presentation

2.5 Neutrino oscillations in matter

Coherent forward scattering:

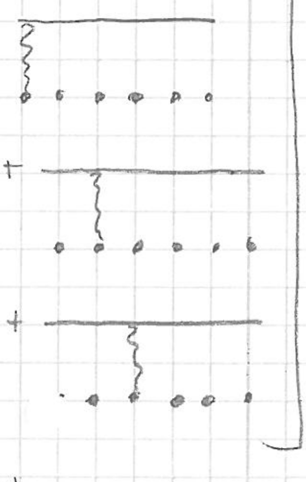




Exchange of force carrier (W, Z) with background matter, leaving energies and momenta unchanged.

Analogy: Photon travelling through glass

$\mathcal{M} =$



$+ \dots$

Crucial observation:
Interactions with different particles add up <u>coherently</u>

$\hookrightarrow \ |\mathcal{M}|^2 \propto N^2 \cdot G_F^2$

(N: number of matter particles)

(cf. incoherent scattering: $|\mathcal{M}|^2 \propto N \cdot G_F^2$)

$\Rightarrow$ enhancement by N

Coherent forward scattering creates a <u>potential</u>:

Consider  :

$$\mathcal{H}_{cfs} = \frac{G_F}{\sqrt{2}} \left[\bar{e}\gamma^\mu(1-\gamma^5)\,\nu_e\right]\left[\bar{\nu}_e\gamma^\mu(1-\gamma^5)\right]$$

$$= \frac{G_F}{\sqrt{2}} \left[\bar{e}\gamma^\mu(1-\gamma^5)\,e\right]\left[\bar{\nu}_e\gamma^\mu(1-\gamma^5)\right]$$

Fierz transformation

Treat electrons as fixed background field
=> take expectation value $\langle \cdot \rangle_e$

$$\langle \mathcal{H}_{cfs} \rangle_e = \frac{G_F}{\sqrt{2}} \left\langle \bar{e} \gamma^\mu (1-\gamma^5) e \right\rangle_e \cdot \left[\bar{\nu}_e \gamma^\mu (1-\gamma^5) \nu_e \right]$$

Rember:

$u = \begin{pmatrix} \sqrt{\sigma p}\, \xi \\ \sqrt{\bar{\sigma} p}\, \xi \end{pmatrix}$

$\mathrm{tr}\, \sigma^\mu \bar{\sigma}^\nu = 2 g^{\mu\nu}$

- $\langle \bar{e} \gamma^0 e \rangle \sim \langle e^+ e \rangle = n_e$ ← e^- number density
- $\langle \bar{e} \vec{\gamma} e \rangle \sim \langle \vec{v}_e \rangle = 0$ in matter at rest
- $\langle \bar{e} \gamma^0 \gamma^5 e \rangle \sim \left\langle \frac{\vec{\sigma} \cdot \vec{p}_e}{E} \right\rangle$ = avg. helicity $= 0$ for unpolarized matter / matter at rest
- $\langle \bar{e} \vec{\gamma} \gamma^5 e \rangle \sim \langle \vec{\sigma} \rangle$ = average spin $= 0$ for unpolarized matter

Use
$$e = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{\text{spins } s} \left(a^s_{\vec{p}}\, u^s(p)\, e^{-ipx} + b^{s\dagger}_{\vec{p}}\, v^s(p)\, e^{ipx} \right)$$

$E_{\vec{p}} = \sqrt{\vec{p}^2 + m_e^2}$; $a^s_{\vec{p}}$: particle annihilation operator; $b^{s\dagger}_{\vec{p}}$: antiparticle creation operator

$$|e^-(\vec{p}, s)\rangle = \sqrt{2E_{\vec{p}}}\; a^{s\dagger}_{\vec{p}} |0\rangle$$

Assume BG e^- follow p-distribution $f(\vec{p})$; $\int \frac{d^3p}{2E_{\vec{p}}} f(\vec{p}) = n_e$ ← e^- number density

$\left\{ a^s_{\vec{p}}, a^{s'\dagger}_{\vec{p}'} \right\} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \delta_{ss'}$

$$\Rightarrow \langle \bar{e} \gamma^\mu (1-\gamma^5) e \rangle_e = \frac{1}{2} \sum_s \int \frac{d^3p}{2E_{\vec{p}}} f(\vec{p}) \langle e^-(\vec{p}, s) | \bar{e} \gamma^\mu (1-\gamma^5) e | e^-(\vec{p}, s) \rangle$$

$\frac{1}{2}\sum_s$: spin average

$$= \frac{1}{2} \sum_s \int \frac{d^3p}{2E_{\vec{p}}} f(\vec{p})\; \bar{u}^s(p) \gamma^\mu (1-\gamma^5) u^s(p)$$

$$= \frac{1}{2} \int \frac{d^3p}{2E_{\vec{p}}} f(\vec{p})\; \mathrm{tr}\left((\not{p} + m_e) \gamma^\mu (1-\gamma^5) \right)$$

$$= \frac{1}{2} \int \frac{d^3p}{2E_{\vec{p}}} f(\vec{p})\; \underbrace{\mathrm{tr}\, \not{p} \gamma^\mu}_{= 4p^\mu}$$

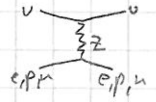
$$= \begin{cases} n_e & \text{for } \mu = 0 \\ 0 & \text{for } \mu = 1, 2, 3 \end{cases} \quad \text{assuming } f(\vec{p}) = f(|\vec{p}|)$$

$\mathrm{tr}\, \gamma^\mu = 0$
$\mathrm{tr}\, \gamma^\mu \gamma^5 = 0$
$\mathrm{tr}\, \gamma^\mu \gamma^\nu \gamma^5 = 0$

$$\Rightarrow \langle \mathcal{H}_{eff} \rangle = \frac{G_F}{\sqrt{2}}\, n_e\, \overline{\nu_e}\gamma^0(1-\gamma^5)\nu_e$$

[cf. similar structure: em potential A_μ
↳ $\mathcal{L} \supset \bar{e}\gamma^\mu e A_\mu$]

$$= \underbrace{\sqrt{2}\, G_F\, n_e}_{\equiv V_{CC}}\ \overline{\nu_{eL}}\gamma^0\nu_{eL}$$

Similarly, potential from  :

$$V_{NC} = -\frac{1}{2}\cdot\sqrt{2}\, G_F\, n_e$$

affects all neutrino flavors

Note: $\boxed{\text{Potentials flip sign under CP}}$ (deeper reason: $\overline{\nu_{eL}}\gamma^0\nu_{eL} = -\overline{(\nu_{eL})^c}\gamma^0(\nu_{eL})^c$)

In flavor basis $\psi \equiv \begin{pmatrix} a\,|\nu_e\rangle \\ b\,|\nu_\mu\rangle \end{pmatrix}$; $a^2+b^2=1$

$$P = \left|\psi^+\, U \begin{pmatrix} e^{ip_1L} & \\ & e^{ip_2L} \end{pmatrix} U^+\psi\right|^2$$

$$= \left|\psi^+ e^{i\hat{p}L}\psi\right|^2$$

$$= \left|\psi^+ e^{i\sqrt{\underbrace{\hat{H}_{kin}}_{\hat{H}_{kin}=\hat{H}-\hat{V}}^2-\hat{M}^2}}\psi\right|^2$$

$$\simeq \left|\psi^+ \exp\left[i\left(\hat{H} + \frac{\hat{M}^2}{2\hat{H}} - \hat{V}\right)\right]\psi\right|^2$$

↳ in flavor basis: $\begin{pmatrix} V_{CC}-V_{NC} & \\ & -V_{NC} \end{pmatrix}$

Consider phase $\phi = \hat{p}\cdot L$

$$= \sqrt{(\hat{H}-\hat{V})^2 - \hat{M}^2}$$

M and V small →

$$\simeq \hat{H} - \frac{\hat{M}^2}{2\hat{H}} - \hat{V}$$

Need to determine states of definite momentum

↳ Diagonalize $\hat{H} - \frac{\hat{M}^2}{2\hat{H}} - \hat{V} = E\cdot\mathbb{1} - U\begin{pmatrix} \frac{m_1^2}{2E} & \\ & \frac{m_2^2}{2E} \end{pmatrix}U^+ - \begin{pmatrix} V_{CC}-V_{NC} & \\ & -V_{NC} \end{pmatrix}$

$$= \left(E - \frac{m_1^2}{4E} - \frac{m_2^2}{4E} - V_{NC}\right) - \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -\frac{\Delta m^2}{4E} & \\ & \frac{\Delta m^2}{4E} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \underbrace{\sin\theta}_{\equiv s} & \underbrace{\cos\theta}_{\equiv c} \end{pmatrix} - \begin{pmatrix} V_{CC} & \\ & 0 \end{pmatrix}$$

see exercise, mention only outline of calculation and result

$$= \left(E - \frac{m_1^2}{4E} - \frac{m_2^2}{4E} - V_{NC}\right) - \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} -c & s \\ s & c \end{pmatrix}\frac{\Delta m^2}{4E} - \begin{pmatrix} V_{cc} & \\ & 0 \end{pmatrix}$$

$$= \quad '' \quad - \begin{pmatrix} -c^2+s^2 & 2sc \\ 2sc & -s^2+c^2 \end{pmatrix} \underbrace{\frac{\Delta m^2}{4E}}_{\equiv \Delta/2} - \quad ''$$

$$= \quad '' \quad - \begin{pmatrix} -\frac{\Delta}{2}\cos 2\Theta + V_{cc} & \frac{\Delta}{2}\sin 2\Theta \\ \frac{\Delta}{2}\sin 2\Theta & \frac{\Delta}{2}\cos 2\Theta \end{pmatrix}$$

$$\Rightarrow \left(V_{cc} - \frac{\Delta}{2}\cos 2\Theta - \lambda\right)\left(\frac{\Delta}{2}\cos 2\Theta - \lambda\right) - \frac{\Delta^2}{4}\sin^2 2\Theta = 0$$

$$\Leftrightarrow \lambda^2 - \lambda \cdot V_{cc} - \frac{\Delta^2}{4}\cos^2 2\Theta + V_{cc}\frac{\Delta}{2}\cos 2\Theta - \frac{\Delta^2}{4}\sin^2 2\Theta = 0$$

$$\lambda = \frac{V_{cc} \pm \sqrt{V_{cc}^2 + \Delta^2 - 2V_{cc}\Delta\cos 2\Theta}}{2}$$

$$= \frac{V_{cc} \pm \sqrt{(V_{cc} - \Delta\cos 2\Theta)^2 + \Delta^2\sin^2 2\Theta}}{2}$$

Eigenvectors:
$$\begin{matrix}(I)\\(II)\end{matrix}\begin{pmatrix} -\frac{\Delta}{2}\cos 2\Theta + V_{cc} & \frac{\Delta}{2}\sin 2\Theta \\ \frac{\Delta}{2}\sin 2\Theta & \frac{\Delta}{2}\cos 2\Theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\Big(V_{cc} \pm \underbrace{\sqrt{\quad}}_{\equiv G}\Big)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\cos 2\Theta\,(I) - \sin 2\Theta\,(II): \quad -\frac{\Delta}{2}a + V_{cc}\cos 2\Theta\, a = \frac{1}{2}\left(V_{cc} \pm G\right)(\cos 2\Theta\, a - \sin 2\Theta\, b)$$

$$\Rightarrow \sin 2\Theta \cdot b \cdot \frac{1}{2}(V_{cc} \pm G)$$

$$= \left[\cos 2\Theta \cdot \frac{1}{2}(V_{cc} \pm G) + \frac{\Delta}{2} - V_{cc}\cos 2\Theta\right] a$$

$$= \left[\frac{\Delta}{2} + \frac{1}{2}\cos 2\Theta\left(-V_{cc} \pm G\right)\right] a$$

$$\Rightarrow \tan\Theta_{eff} = -\frac{b}{a}\Big|_{''-''} = -\frac{\frac{\Delta}{2} + \frac{1}{2}\cos 2\Theta(-V_{cc} - G)}{\sin 2\Theta \cdot \frac{1}{2}(V_{cc} - G)}$$

$$\sin 2\Theta_{eff} = \frac{2\tan\Theta_{eff}}{1 + \tan^2\Theta_{eff}}$$

$$= -2\,\frac{\Delta + \cos 2\Theta(-V_{cc} - G)}{\sin 2\Theta(V_{cc} - G)\left[1 + \left(\frac{\Delta + \cos 2\Theta(-V_{cc} - G)}{\sin 2\Theta(V_{cc} - G)}\right)^2\right]}$$

$$= -2 \frac{(\Delta - \cos 2\theta (V_{cc} + G)) \sin 2\theta (V_{cc} - G)}{\sin^2 2\theta (V_{cc} - G)^2 + (\Delta - \cos 2\theta (V_{cc} + G))^2}$$

$$= -2 \frac{\sin 2\theta [\Delta V_{cc} - \Delta G - \cos 2\theta (V_{cc}^2 - G^2)]}{\sin^2 2\theta (V_{cc}^2 - 2V_{cc}G + G^2) + \Delta^2 + \cos^2 2\theta (V_{cc}^2 + G^2 + 2V_{cc}G) - 2\Delta\cos 2\theta (V_{cc}}$$

$$= -2 \sin 2\theta \frac{\Delta [V_{cc} - G - \cos 2\theta (-\Delta + 2V_{cc}\cos 2\theta)]}{V_{cc}^2 + G^2 + 2V_{cc} G (\cos^2 2\theta - \sin^2 2\theta) + \Delta^2 - 2\Delta \cos 2\theta (V_{cc} + G)}$$

$$= -2 \sin 2\theta \cdot \Delta \frac{V_{cc}(\sin^2 2\theta - \cos^2 2\theta) - G + \Delta \cos 2\theta}{2G^2 + 2V_{cc} G (\cos^2 2\theta - \sin^2 2\theta) - 2G\Delta \cos 2\theta}$$

$$= \frac{\Delta \cdot \sin 2\theta}{G}$$

Result: $$\frac{\Delta m^2_{eff}}{2E} = \sqrt{\left(\sqrt{2} G_F n_e - \frac{\Delta m^2}{2E} \cos 2\theta\right)^2 + \left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta}$$

$$\sin^2 2\theta_{eff} = \frac{\sin 2\theta \cdot \frac{\Delta m^2}{2E}}{\sqrt{\left(\sqrt{2} G_F n_e - \frac{\Delta m^2}{2E} \cos 2\theta\right)^2 + \left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta}}$$

At the MSW (Mikheyev – Smirnov – Wolfenstein) resonance,

$$\frac{\Delta m^2}{2E} \cos 2\theta = \sqrt{2} G_F n_e$$

the effective mixing θ_{eff} is maximal.

3. Neutrino anomalies

$\hookrightarrow$ see presentation