1. **Neutrino oscillations in matter**

(a) Diagonalize the 2-flavor neutrino Hamiltonian in matter,

$$\mathcal{H}_{\text{eff}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2}{4E} & 0 \\ 0 & \frac{\Delta m^2}{4E} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + \begin{pmatrix} \sqrt{2}G_F n_e & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

to show that the effective mass squared difference and mixing angle in matter are given by

$$\Delta m^2_{\text{eff}} = \sqrt{\Big(\sqrt{2}G_F n_e - \frac{\Delta m^2}{2E}\cos 2\theta\Big)^2 + \Big(\frac{\Delta m^2}{2E}\Big)^2} \tag{2}$$

$$\sin 2\theta_{\text{eff}} = \frac{\sin 2\theta\, \frac{\Delta m^2}{2E}}{\sqrt{\big(\sqrt{2}G_F n_e - \frac{\Delta m^2}{2E}\cos 2\theta\big)^2 + \big(\frac{\Delta m^2}{2E}\big)^2}} \tag{3}$$

(b) Consider now neutrino oscillations in spatially *inhomogeneous* matter of density $n_e = n_e(x)$. This is relevant for instance for solar neutrinos propagating out of the core of the Sun. Neutrino evolution is described by the Schrödinger-like equation

$$-i\frac{d}{dx}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_{\text{eff}} & \sin\theta_{\text{eff}} \\ -\sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} -\frac{\Delta m^2_{\text{eff}}}{4E} & 0 \\ 0 & \frac{\Delta m^2_{\text{eff}}}{4E} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta_{\text{eff}} & -\sin\theta_{\text{eff}} \\ \sin\theta_{\text{eff}} & \cos\theta_{\text{eff}} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{4}$$

Rewrite this equation in the basis of *matter eigenstates*, i.e. states of definite energy and momentum in matter: $(\nu_A, \nu_B) = U^\dagger_{\text{eff}}(x)\,(\nu_e, \nu_\mu)$, where U_{eff} is the effective mixing matrix in matter. You should find

$$-i\frac{d}{dx}\begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix} = \begin{pmatrix} p_A & i\frac{d\theta}{dx} \\ -i\frac{d\theta}{dx} & p_B \end{pmatrix} \begin{pmatrix} \nu_A \\ \nu_B \end{pmatrix}, \tag{5}$$

where p_A, p_B are the momentum eigenvalues in matter.

(c) For slowly varying matter density, $d\theta/dx \ll |p_A - p_B|$, the off-diagonal terms on the right hand side of eq. (5) are neglgible. Solve the equation to show that the survival probability of solar neutrinos $P(\nu_e \to \nu_e)$ is given by

$$P(\nu_e \to \nu_e) = \frac{1}{2}\Bigg(1 + \cos 2\theta_i \cos 2\theta_f + \sin 2\theta_i \sin 2\theta_f \cos\Bigg[\int_{x_i}^{x_f} dx\, \frac{\Delta m^2_{eff}(x)}{2E}\Bigg]\Bigg). \tag{6}$$

Here θ_i and θ_f are the effective mixing angles corresponding to the center of the Sun and its exterior, respectively. The integral in the last term runs along the neutrino trajectory from its production point x_i to its detection point x_f.

(d) What is the maximum conversion probability in the case of small vacuum mixing angle $\theta \ll 1$? Consider the case that the matter density at the center of the Sun lies well above the MSW resonance, whereas the density outside the Sun is far below.

2. **The solar neutrino problem and its resolution** For a long time, solar neutrino experiments observed a deficit of ν_e compared to theoretical predictions (which did not include oscillations). It was not until 2002 that the SNO (Sudbury Neutrino Observatory) proved that the missing ν_e were in fact converted into ν_μ and ν_τ.

   (a) SNO used a heavy water (D_2O) target and was sensitive to reactios

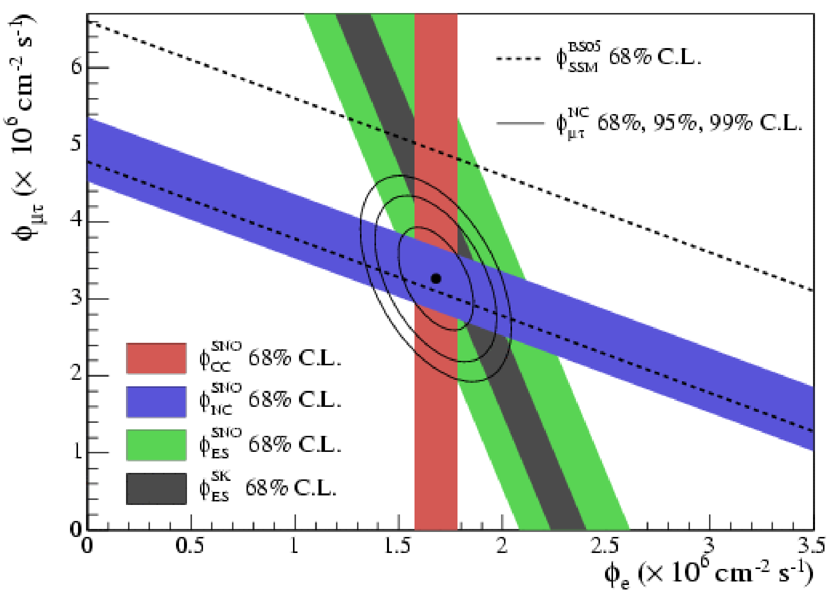
$$\nu + D \to p + n + \nu\,, \tag{7}$$
$$\nu + e \to \nu + e\,, \tag{8}$$

   and

$$\nu_e + (A, Z) \to e^- + (A, Z-1)\,. \tag{9}$$
$$\tag{10}$$

   Explain why charged current interactions cannot be used to show that solar ν_e convert into other neutrino flavors.

   (b) The following plot shows the famous SNO result.



   The green, blue and red bands correspond to constraints from reactions (7), (8) and (9), respectively. The horizontal axis shows the flux of ν_e, the vertical one the flux of $\nu_\mu + \nu_\tau$. Explain (qualitatively) the slope of the colored bans.

   (c) Why is the solar neutrino flux expected to vary throughout the year?

3. **Neutrino anomalies** Think about possible explanations (within the Standard Model and beyond) of the LSND and MiniBooNE results, as well as the reactor and gallium anomalies. Consider experimental backgrounds, systematic uncertainties, and new physics scenarios beyond the simple $3+n$ sterile neutrino scenarios from the lecture.

   If you have formed a hypothesis for how one or several of the anomalous results could be explained, discuss how your hypothesis could be tested.