Quantum Field Theory
Exercise 1

October 20, 2015

-to be handed in by 29.10.2012 (12:00 h) to the WA-THEP letterbox (No. 22) in the foyer of Staudingerweg 7.

1. **Complex Scalar Field (20 points)**

   Consider the following Lagrangian density for a complex scalar field ϕ

   $$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) + \mu^2 \phi^* \phi - \frac{\lambda}{2} (\phi^* \phi)^2. \tag{1}$$

   (a)(10 points) Treating ϕ and ϕ^* as independent, show that the corresponding Hamiltonian is given by

   $$H = \int d^3x \left(\dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi - \mu^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 \right). \tag{2}$$

   Use that, in the case of n independent fields,the Hamiltonian of the system is given by $H = \int d^3x \left(\sum_{i=1}^{n} \pi_{\phi_i} \dot{\phi}_i - \mathcal{L} \right)$ where $\pi_{\phi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$ is the conjugate momentum associated with i-th degree of freedom ϕ_i.

   (b)(10 points) Show that the kinetic term in eq. (1) can be effectively taken as

   $$\mathcal{L}_{\text{Kin}} = -\phi^* \Box \, \phi. \tag{3}$$

2. **Scalar Theory with $SO(2)$ Invariance (35 points)**

   Consider the following Lagrangian density for two real scalar fields $\phi_1(x)$, $\phi_2(x)$:

   $$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi_1)(\partial_\mu \phi_1) + \frac{1}{2} (\partial^\mu \phi_2)(\partial_\mu \phi_2) - \frac{m^2}{2} \left(\phi_1^2 + \phi_2^2 \right) - \frac{\lambda}{4!} \left(\phi_1^2 + \phi_2^2 \right)^2. \tag{4}$$

(a)(10 points)
Identify the equations of motion and construct the Hamilton density $\mathcal{H}$.

(b)(10 points) Show that the above Lagrangian is invariant under the transformations

$$\phi_1 \to \phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta, \tag{5}$$

$$\phi_2 \to \phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta. \tag{6}$$

(c)(15 points) Calculate the Noether current j^μ and show explicitly, using equations of motion, that its divergence vanishes. Noether current for n independent fields is given as $j^\mu = \sum\limits_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta\phi_i - \mathcal{J}^\mu$ where $\mathcal{J}^\mu$ (see the script) equals zero in this particular case as transformation in b) does not generate any new term containing the total derivative.

3. **Real Scalar Field in 1+1 Dimensions (45 points)**
Consider the following Lagrangian for a real scalar field ϕ in $1+1$ (one spatial and one time) dimensions

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{\lambda}{4} \left(\phi^2 - v^2\right)^2, \tag{7}$$

where v is a constant.

(a)(20 points) Construct the corresponding Hamiltonian and find the condition on classical field configurations $\phi_0(x)$ which minimizes the energy.

(b)(10 points) Find the equations of motion for the field ϕ.

(c)(15 points) The static solution $\left(\dot{\phi} = 0\right)$ which interpolates between 2 vacuum states

$$\phi_0(x) = v \tanh\left(\sqrt{\frac{\lambda}{2}}\, v\, x\right), \tag{8}$$

is called the kink solution. Prove that the kink is indeed a solution of the equations of motion.