Quantum Field Theory
Exercise 2

October 29, 2015

-to be handed in by 5.11.201**5** (12:00 h) to the "Theoretische Physik 6a" letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **Real Klein-Gordon Field (40 points)**

   Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\mathbf{x},t) \quad = \quad \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \left[a(\mathbf{k})\, e^{-ik\cdot x} + a^\dagger(\mathbf{k})\, e^{ik\cdot x}\right], \tag{1}$$

   and the equal-time commutation relations

$$[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] \quad = \quad 0, \tag{2}$$

$$\left[\dot{\phi}(\mathbf{x},t), \dot{\phi}(\mathbf{x}',t)\right] \quad = \quad 0, \tag{3}$$

$$\left[\phi(\mathbf{x},t), \dot{\phi}(\mathbf{x}',t)\right] \quad = \quad i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{4}$$

   show that

   (a)(20 points) the creation and annihilation operators satisfy the following commutation relations

$$[a(\mathbf{k}), a(\mathbf{k}')] \quad = \quad 0, \tag{5}$$

$$\left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] \quad = \quad 0, \tag{6}$$

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] \quad = \quad (2\pi)^3\,\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right), \tag{7}$$

   (b)(20 points) the momentum $\mathbf{P} = -\int d^3x\, \dot{\phi}\, \nabla\phi$ takes the form

$$\mathbf{P} = \int \frac{d^3k}{(2\pi)^3}\, \mathbf{k} \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\left[a(\mathbf{k}), a^\dagger(\mathbf{k})\right]\right). \tag{8}$$

2. **Complex Klein-Gordon Field (60 points)**
   The complex Klein-Gordon field is used to describe charged bosons with spin 0. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - \mu^2\phi^\dagger\phi, \tag{9}$$

where the field ϕ has the following normal mode expansion

$$\phi(\mathbf{x},t) \;=\; \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a(\mathbf{k})\,e^{-ik\cdot x}+b^\dagger(\mathbf{k})\,e^{ik\cdot x}\right], \tag{10}$$

and satifies the equal-time commutation relations

$$[\phi(\mathbf{x},t),\Pi_\phi(\mathbf{x}',t)] \;=\; i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{11}$$
$$\left[\phi^\dagger(\mathbf{x},t),\Pi_{\phi^\dagger}(\mathbf{x}',t)\right] \;=\; i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{12}$$

In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

(a)(15 points) Show that the Lagrangian in eq. (9) is equivalent to the Lagrangian of two independent real scalar fields with the same mass and satisfying the standard equal-time commutation relations. *Hint*: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}\,(\phi_1 + i\phi_2)$.

(b)(15 points) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$. Derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.
Hint: Assuming you have derived expressions for annihilation and creation operators in terms of ϕ and $\dot\phi$ for a real scalar field (problem 1), you can without the full derivation write down the corresponding expressions for a, $a^\dagger$, b and $b^\dagger$ when complex scalar field is considered. For instance, by looking at eq.(1) and eq.(10) one can infer that the expression for $a^\dagger$ in real Klein-Gordon theory corresponds to the expression for $b^\dagger$ in the complex one.

(c)(15 points) Show that the Lagrangian in eq. (9) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ and express the conserved charge $Q = \int d^3\mathbf{x}\, J^0$ in terms of creation and annihilation operators.

(d)(15 points) Compute the commutators $[Q,\phi]$ and $[Q,\phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?