Quantum Field Theory
Exercise 3

November 3, 2015

-to be handed in by 12.11.2015 (12:00 h) to the "Theoretische Physik 6a" letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **Lorentz Invariance (25 points)**

   Prove that the integral
   $$\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}} f(p_\mu p^\mu),$$
   where f is an arbitrary function of the four-momentum, is Lorentz invariant.

2. **Gordon Identity (25 points)**
   Derive the so-called *Gordon identity*
   $$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu}{2M} + \frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u(p),$$
   where $P = p' + p$, $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$.

3. **Projection Operators (20 points)**
   Show that $P_\pm = \pm\frac{\not{p}\pm m}{2m}$ represent a complete sets of projection operators, *i.e.* satisfy the conditions
   $$P_i P_j = \delta_{i,j}\,P_i, \qquad \sum_i P_i = 1.$$
   In addition, show that $P_\pm$ are the projection operators on positive and negative energy solutions (u and v spinors) for arbitrary particle momentum.

4. γ **Matrices (20 points)**
   Without using an explicit representation for the γ matrices show that :

   (a)(2.5 points) $\gamma_\mu\gamma^\mu = 4$;

   (b)(2.5 points) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$;

   (c)(5 points) $\mathrm{Tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\left[(a\cdot b)\;(c\cdot d) - (a\cdot c)\;(b\cdot d) + (a\cdot d)\;(b\cdot c)\right]$;

(d)(5 points) $\gamma_5 = \frac{i}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$;

(e)(5 points) $\gamma_5^2 = \mathbb{1}$.

5. **Identity with γ Matrices (10 points)**

Verify the following identity

$$\mathrm{Exp}\left[\gamma_5 \not{a}\right] = \cos\left(\sqrt{a_\mu a^\mu}\right) + \frac{1}{\sqrt{a_\mu a^\mu}}\gamma_5 \not{a} \sin\left(\sqrt{a_\mu a^\mu}\right),$$

where $a^2 > 0$.