Quantum Field Theory
Exercise 4

November 11, 2015

-to be handed in by 19.11.2015 (12:00 h) to the "Theoretische Physik 6a" letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **Gamma Matrices II (25 points)**

   Without using an explicit representation for the Dirac matrices, show that:

   (a)(5 points) $\{\gamma_5, \gamma^\mu\} = 0$;

   (b)(5 points) $\mathrm{Tr}[\gamma_5] = 0$;

   (c)(5 points) $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0$;

   (d)(5 points) $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5] = 4i\,\varepsilon^{\mu\nu\rho\sigma}$;

   (e)(5 points) $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$ if n is odd;

   where $\not{a} \equiv \gamma^\mu a_\mu$, $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon_{0123} = +1$.

2. **Dirac Bilinears (60 points)**
   Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

   - $\Gamma_S = \mathbb{1}$ (scalar);
   - $\Gamma_P = \gamma_5$ (pseudoscalar);
   - $\Gamma_V^\mu = \gamma^\mu$ (vector);
   - $\Gamma_A^\mu = \gamma^\mu \gamma_5$ (axial vector);
   - $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (tensor).

   Without referring to any explicit representation for the Γ matrices,

   (a) (5 points) show that $\Gamma^2 = \pm\mathbb{1}$.

(b) (5 points) show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$.

(c) (10 points) check that the product of 2 different Γ's is proportional to some Γ different from Γ_S;

(d) (20 points) using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check that the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma_5\psi$ and $\bar{\psi}'\sigma^{\mu\nu}\psi' = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi$.

(e) (20 points) Show the following bilinear transformation properties :

- tensor $\Gamma_T^{\mu\nu}$ under parity P,
- axial vector Γ_A^μ under charge conjugation C,
- vector Γ_V^μ under time reversal T.

3. **Lorentz transformation identity (15 points)**
   Verify that for arbitrary proper Lorentz transformation S

$$S^{-1} = \gamma_0 S^\dagger \gamma_0.$$