Quantum Field Theory
Exercise 6

November 24, 2015

-to be handed in by 03.12.2015 (12:00 h) to the "Theoretische Physik 6a" letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **Fierz Transformations (40 points)**
   a) (10 points) Normalize the 16 matrices Γ^A (scalar $\mathbb{1}$, pseudoscalar γ_5, vector γ^μ, axial vector $\gamma^\mu\gamma_5$ and tensor $\sigma^{\mu\nu}$) to the convention
   $$\mathrm{Tr}\left[\Gamma^A\Gamma^B\right] = 4\delta^{AB}.$$
   b) (10 points) We can define the so called *Fierz identity* as an equation
   $$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD}\,(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2),$$
   with unknown coefficients $C^{AB}{}_{CD}$. Show that
   $$C^{AB}{}_{CD} = \frac{1}{16}\mathrm{Tr}\left[\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right].$$
   c) (20 points) Work out explicitly the Fierz transformation laws for the products $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$ and $(\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4)$.

2. **Wick Theorem (20 points)**

   Using the Wick's theorem evaluate

   a) (10 points) $\langle 0|T(\phi^4(x)\phi^4(y))|0\rangle$
   b) (10 points) $\langle 0|T(\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y))|0\rangle$.

   Show results diagrammatically! Note that the contraction for the Dirac field is defined as $\psi(x)\bar{\psi}(y) = S_F(x-y)$, where $S_F(x-y)$ is the Feynman propagator. Keep also in mind that fermion fields anticommute!

3. ϕ^3 **Theory (40 points)**

   The Lagrangian of ϕ^3 theory reads
   $$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi^3.$$

a) (5 points)
What are the position space Feynman rules for this theory?

b) (15 points) Starting from $\langle 0|T\phi(x_1)\phi(x_2)...\phi(x_n)\mathrm{Exp}(-i\int d^4z\mathcal{H}_\mathrm{I})|0\rangle$, where $\mathcal{H}_\mathrm{I}$ is the interacting Hamilton density, determine all *connected* Feynman diagrams up to order λ^2 with up to $n = 3$ external legs.

c) (15 points) Using Feynman rules in position space, write the amplitude corresponding to the diagram shown in fig. 1. Apply the Fourier transform to the propagator terms. Finally, In order to solve the remaining integral over momentum space, apply the so called *Wick rotation* for transformation to Euclidean space (metric tensor=diag(1,1,1,1))

$$k^0 = ik_E^0, \qquad\qquad \mathbf{k} = \mathbf{k}_E,$$

where subscript E denotes components of Euclidean 4-monentum variable k_E. In addition, note that the surface "area" of a four-dimensional unit sphere is $2\pi^2$. Since the integral is not finite, perform a cut-off regularization, i.e. change the upper boundary of the radial integral from $+\infty$ to Λ (cut-off energy scale). The final result should contain the coupling λ, the mass of the scalar particle m and the aforementioned cut-off scale Λ.

d) (5 points) Draw the potential of the field ϕ. Can you determine the ground state energy? Discuss the validity of this theory.

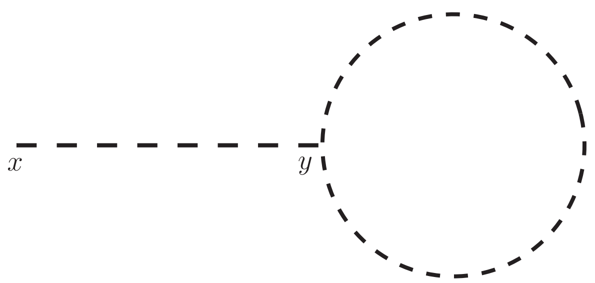

Figure 1: Tadpole diagram in ϕ^3 theory