Quantum Field Theory
Exercise 8

December 10, 2015

-to be handed in by 17.12.2015 (12:00 h) to the Theoretische Physik 6a letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **The Scalar Electrodynamics Lagrangian from Symmetry Arguments (30 points)**

   Start from the free complex Klein-Gordon field Lagrangian

   $$\mathcal{L}_{\text{free}} = (\partial^\mu \phi)(\partial_\mu \phi^*) - m\phi\phi^*.$$

   Check that this Lagrangian is not invariant under U(1) gauge transformations of the form $\phi(x) \to \phi'(x) = e^{-i\alpha(x)}\phi(x)$. By knowing the transformation property of the photon field $A^\mu \to A'^\mu = A^\mu + \frac{1}{e}\left(\partial^\mu \alpha(x)\right)$, supplement the starting Lagrangian with additional interaction terms ($\mathcal{L}_{\text{int}}$) which would help to restore the gauge invariance of the total Lagrangian $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$. Finally, express the Lagrangian in the form

   $$\mathcal{L} = (\mathrm{D}^\mu \phi)(\mathrm{D}_\mu \phi)^* - \mathrm{m}\phi\phi^* - \frac{1}{4}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu},$$

   with $D_\mu = \partial_\mu + ieA_\mu$.

2. **Electron-positron annihilation into scalar quark-antiquark pair $e^+e^- \to q\bar{q}$ (70 points)**

   Consider electron-positron annihilation into quark-antiquark pair $e^+(k_2)e^-(k_1) \to q(p_2)\bar{q}(p_1)$. The process is represented by the Feynman diagram in fig. 1. Treat the electron as a massless Dirac particle and the quark as a massless Klein-Gordon particle.

   (a)(25 points) The squared matrix element obtained as an average over electron and positron spin configurations can be expressed as

   $$|M|^2 = \frac{e^4 e_q^2}{s} L^{\mu\nu} Q_{\mu\nu},$$

   with the quark charge $e_q e$ and the Mandelstam variable $s = (k_1 + k_2)^2$. Find the expressions for the quark tensor $Q_{\mu\nu}$ and the lepton tensor $L_{\mu\nu}$ in terms of momenta

k_1, k_2, p_1, p_2. The Feynman rule for quark-antiquark-photon vertex is $ie_q e(p_2 - p_1)^\mu$.

(b)(15 points) Calculate $L^{\mu\nu}Q_{\mu\nu}$ in terms of Mandelstam variable s and the angle between the initial electron momenta and the final anti-quark momenta in the center-of-mass frame.

(c)(10 points) Express the result for $L^{\mu\nu}Q_{\mu\nu}$ in terms of the Mandelstam variable s and $t = (k_1 - p_1)^2$.

(d)(20 points) The general expression for the differential cross-section in the center-of-mass frame for $2 \to 2$ process is

$$d\sigma = \frac{1}{2E_\mathcal{A} 2E_\mathcal{B} |\mathbf{v}_\mathcal{A} - \mathbf{v}_\mathcal{B}|} \frac{d^3\mathbf{p}_1}{(2\pi)^3} \frac{1}{2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3} \frac{1}{2E_2} |\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to p_1, p_2)|^2$$
$$(2\pi)^4 \delta^{(4)}(p_\mathcal{A} + p_\mathcal{B} - p_1 - p_2),$$

where the indices $\mathcal{A}$ and $\mathcal{B}$ correspond to the colliding initial state particles, and the indices 1 and 2 are labelling terms corresponding to final state particles. Integrate over the quark phase space $d^3\mathbf{p}_2$ and the antiquark momentum $d|\mathbf{p}_1|$ and obtain the expression for the differential cross section $\frac{d\sigma}{d\Omega}$, where Ω represents the angular part of the antiquark phase space integral. Express it in terms of the energy in the center-of-mass frame squared $s = (k_1 + k_2)^2$ and the angle θ between k_1 and p_1.

The differential cross-section for this process is $\propto \sin^2\theta$, whereas the differential cross-section for the proces in which quarks are considered to be fermions $\propto (1 + \cos^2\theta)$, latter being in the perfect agreement with the experimental results.

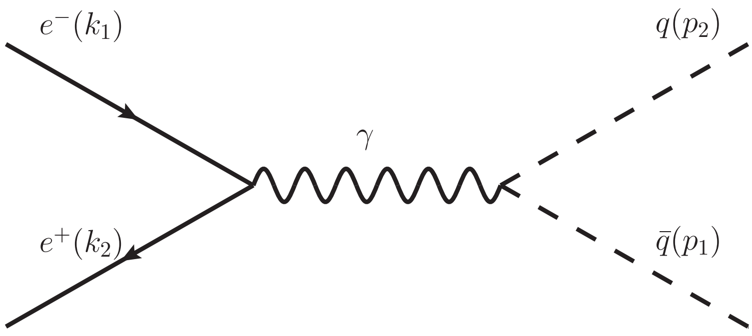

Figure 1: Electron-positron annihilation into a scalar quark-antiquark pair