Quantum Field Theory
Exercise 9

January 7, 2016

-to be handed in by 14.01.2016 (12:00 h) to the "Theoretische Physik 6a" letterbox (No. 37) in the foyer of Staudingerweg 7.

1. **Cross section of electron – muon scattering (50 points)**

a) (5 points) Use the Feynman rules to write down the amplitude for the process $e^- \, \mu^- \to e^- \, \mu^-$

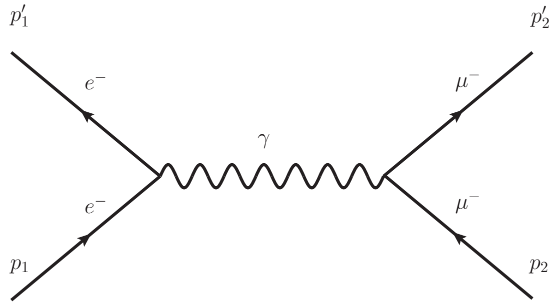

Figure 1: Feynman diagram for electron – muon scattering.

b)(40 points) Show that the corresponding squared amplitude, averaged and summed over the spins, is equal to

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\right] , \qquad (1)$$

when neglecting the electron mass $m_e = 0$.

c) (5 points) Assume to be in the centre of mass (com) system and calculate the differential cross section $\frac{d\sigma}{d\Omega}$ (again for vanishing electron mass, $m_e = 0$). What is the high energy limit of the differential cross section? The most general formula for the differential cross section for $2 \to 2$ process (massive particles) is (see exercise 8, problem 2 d))

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{COM}}^2} \frac{|\mathbf{p}_{\text{fin}}|}{|\mathbf{p}_{\text{in}}|}, \tag{2}$$

where $|\mathbf{p}_{\text{in}}|$ and $|\mathbf{p}_{\text{fin}}|$ are absolute values of momentum in com system in initial and final state, respectively.

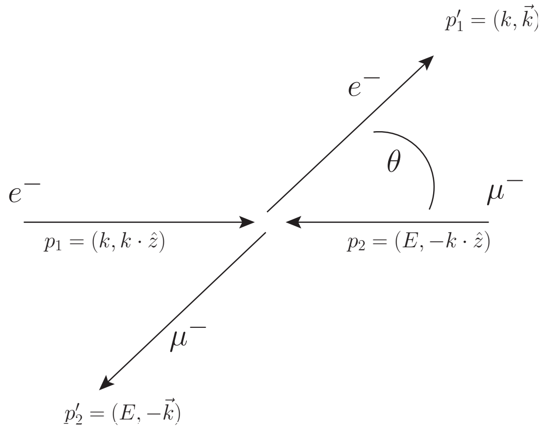

Figure 2: Parametrisation of the momenta of the in- and outgoing electron and muon in the com system.

2. **Path integrals in Quantum Mechanics (50 points)**

In the lecture you derived the amplitude for the propagation of a particle from q at time 0 to q' at time t to be

$$\begin{aligned} \langle q'(t)|e^{-iHt}|q(0)\rangle &= \int \prod_{k=1}^{n} dq_k \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t}, \qquad (3) \\ & \qquad (4) \end{aligned}$$

where the potential term is now set to zero for simplicity. Useful formulae are

$$\dot{q}_j = \frac{q_{j+1} - q_j}{\delta t}, \qquad \delta t = \frac{t}{n+1}. \tag{5}$$

(a)(20 points) Consider the general p_j integral. Complete the square in order to make it gaussian. When solving the integral treat it as a real integral, in particular use $\int_{-\infty}^{\infty} e^{-cx^2} dx = (\frac{\pi}{c})^{\frac{1}{2}}$ where c can contain imaginary unit. Using the result of the integration prove

$$\int \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t} = \left(\frac{m}{2\pi i \delta t}\right)^{\frac{n+1}{2}} \exp\left[\frac{im}{2\delta t} \sum_{j=0}^{n} (q_{j+1} - q_j)^2\right]. \tag{6}$$

(b)(30 points) Now the total amplitude equals

$$\left(\frac{m}{2\pi i \delta t}\right)^{\frac{n+1}{2}} \int \prod_{k=1}^{n} dq_k \exp\left[\frac{im}{2\delta t} \sum_{j=0}^{n} (q_{j+1} - q_j)^2\right]. \tag{7}$$

Try to solve also the integrals over dq_k. First integrate over q_1, then q_2, etc. and try to look for a pattern. Again, as in a) part, complete the square,treat the integrals as real and apply the formula for gaussian integrals. **Help** : The result is

$$\int \prod_{k=1}^{n} dq_k \exp\left[\frac{im}{2\delta t}\sum_{j=0}^{n}(q_{j+1}-q_j)^2\right] = \left[\frac{2i\pi\delta t}{m}\right]^{\frac{n}{2}} \sqrt{\frac{n!}{(n+1)!}} \exp\left[\frac{im(q_{n+1}-q_0)^2}{2(n+1)\delta t}\right]. \tag{8}$$

Since $q_0 = q$ and $q_{n+1} = q'$ it is now easy to express this result, together with a prefactor from (7) (and using eqs. (5)) as

$$\langle q'(t)|e^{-iHt}|q(0)\rangle = \sqrt{\frac{m}{2\pi it}} \exp\left[\frac{im(q'-q)^2}{2t}\right], \tag{9}$$

which depends only on the initial and final position, the time and the mass of the particle.