Quantum Field Theory
Exercise 11

January 26, 2017

-to be handed in by 02.02.2017 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

(1) Path integrals in Quantum Mechanics (45 points)

In the lecture you derived the amplitude for the propagation of a particle from q at time 0 to q' at time t to be

$$\langle q'(t)|e^{-iHt}|q(0)\rangle = \int \prod_{k=1}^{n} dq_k \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t}, \tag{1}$$

$$\tag{2}$$

where the potential term is now set to zero for simplicity. Useful formulae are

$$\dot{q}_j = \frac{q_{j+1}-q_j}{\delta t}, \qquad \delta t = \frac{t}{n+1}. \tag{3}$$

(a) **(20 points)** Consider the general p_j integral. Complete the square in order to make it gaussian. When solving the integral treat it as a real integral, in particular use $\int_{-\infty}^{\infty} e^{-cx^2}dx = (\frac{\pi}{c})^{\frac{1}{2}}$ where c can contain imaginary unit. Using the result of the integration prove

$$\int \prod_{j=0}^{n} \frac{dp_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\frac{p_j^2}{2m}\delta t} = \left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \exp\left[\frac{im}{2\delta t}\sum_{j=0}^{n}(q_{j+1}-q_j)^2\right]. \tag{4}$$

(b) **(25 points)** Now the total amplitude equals

$$\left(\frac{m}{2\pi i\delta t}\right)^{\frac{n+1}{2}} \int \prod_{k=1}^{n} dq_k \exp\left[\frac{im}{2\delta t}\sum_{j=0}^{n}(q_{j+1}-q_j)^2\right]. \tag{5}$$

Try to solve also the integrals over dq_k. First integrate over q_1, then q_2, etc. and try to look for a pattern. Again, as in part (1 a), complete the square, treat the integrals as real and apply the formula for gaussian integrals.

Hint : The result is

$$\int \prod_{k=1}^{n} dq_k \exp\left[\frac{im}{2\delta t}\sum_{j=0}^{n}(q_{j+1}-q_j)^2\right] = \left[\frac{2i\pi\delta t}{m}\right]^{\frac{n}{2}} \sqrt{\frac{n!}{(n+1)!}} \exp\left[\frac{im(q_{n+1}-q_0)^2}{2(n+1)\delta t}\right].$$

Since $q_0 = q$ and $q_{n+1} = q'$ it is now easy to express this result, together with a prefactor from (5) (and using eqs. (3)) as

$$\langle q'(t)|e^{-iHt}|q(0)\rangle = \sqrt{\frac{m}{2\pi i t}} \exp\left[\frac{im(q'-q)^2}{2t}\right],$$

which depends only on the initial and final position, the time and the mass of the particle.

(2) Path Integral for a Free Complex Scalar Field (35 points)
Obtain the following form of a path integral $Z_0(J, J^*)$ for a free complex scalar field

$$Z_0(J, J^*) = \text{Exp}\left[-\int d^4x \int d^4x' J^*(x')\Delta(x-x')J(x)\right], \tag{6}$$

where $\Delta(x - x')$ is the Feynman propagator in position space.

Hint: Note that in the case of a complex scalar field one has to introduce two source terms of the form $J^*(x)\phi(x) + J(x)\phi^*(x)$.

(3) Polarized $e^+e^- \to \mu^+\mu^-$ Cross Section (20 points)

Starting from the matrix element for $e^+e^- \to \mu^+\mu^-$ process

$$i\mathcal{M}(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')) = \frac{ie^2}{(p+p')^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right) \tag{7}$$

calculate the differential cross section

$$\frac{d\sigma}{d\Omega}(e^+_R e^-_L \to \mu^+_L \mu^-_R), \tag{8}$$

where the indices L and R label left and right handed particles, respectively. Work in the massless limit ($m_e, m_\mu \to 0$). Express the differential cross section in terms of the incoming electron energy E and the angle between the incoming electron and the outgoing muon.