Quantum Field Theory
Exercise 1

October 27, 2016

-to be handed in by 3.11.2016 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

Remember: In the case of n independent fields,

- the Hamiltonian of the system is given by $H = \int d^3x \left(\sum_{i=1}^{n} \pi_{\phi_i} \dot{\phi}_i - \mathcal{L} \right)$ where $\pi_{\phi_i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}$ is the conjugate momentum associated with i-th degree of freedom ϕ_i
- the Noether current of the system is given by $j^\mu = \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \Delta\phi_i - \mathcal{J}^\mu$

1. **Scalar Theory with $SO(2)$ Invariance (50 points)**

   Consider the following Lagrangian density for two real scalar fields $\phi_1(x)$, $\phi_2(x)$:

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi_1) (\partial_\mu \phi_1) + \frac{1}{2} (\partial^\mu \phi_2) (\partial_\mu \phi_2) - \frac{m^2}{2} \left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4!} \left(\phi_1^2 + \phi_2^2\right)^2 \tag{1}$$

   (a) **(20 points)** Identify the equations of motion and construct the Hamilton density $\mathcal{H}$.

   (b) **(15 points)** Show that the above Lagrangian is invariant under the transformations

$$\phi_1 \to \phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta, \tag{2}$$

$$\phi_2 \to \phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta. \tag{3}$$

   (c) **(15 points)** Calculate the Noether current j^μ and show explicitly, using equations of motion, that its divergence vanishes. Note that $\mathcal{J}^\mu$ equals zero in this particular case as the transformation in (b) does not generate any new term containing the total derivative.

2. **Real Scalar Field in 1+1 Dimensions (50 points)**
Consider the following Lagrangian for a real scalar field ϕ in $1+1$ (one spatial and one time) dimensions

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi\right)\left(\partial_\mu \phi\right) - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2, \tag{4}$$

where v is a constant.

(a) **(20 points)** Construct the corresponding Hamiltonian and find the condition on classical field configurations $\phi_0(x)$ which minimize the energy.

(b) **(10 points)** Find the equations of motion for the field ϕ.

(c) **(20 points)** The static solution (that is $\dot{\phi} = 0$) which interpolates between two vacuum states is called the kink solution. Prove that the kink solution

$$\phi_{\text{kink}}(x) = v \tanh\left(\sqrt{\frac{\lambda}{2}}\, v\, x\right) \tag{5}$$

is indeed a solution of the equations of motion.