Quantum Field Theory
Exercise 2

November 3, 2016

-to be handed in by 10.11.2016 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

Remember: In the lecture, for the creation and annihilation operators of the real scalar field the following commutation relations have been derived:

$$[a(\mathbf{k}), a(\mathbf{k}')] = 0, \tag{1}$$

$$\left[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = 0, \tag{2}$$

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = (2\pi)^3\,\delta^{(3)}\left(\mathbf{k}-\mathbf{k}'\right), \tag{3}$$

1. **Real Klein-Gordon Field (30 points)**

   Using the normal mode expansion of the real Klein-Gordon field

   $$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a(\mathbf{k})\,e^{-ik\cdot x} + a^\dagger(\mathbf{k})\,e^{ik\cdot x}\right], \tag{4}$$

   show that the momentum $\mathbf{P} = -\int d^3x\,\dot{\phi}\,\nabla\phi$ takes the form

   $$\mathbf{P} = \int \frac{d^3k}{(2\pi)^3}\,\mathbf{k}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\left[a(\mathbf{k}), a^\dagger(\mathbf{k})\right]\right). \tag{5}$$

2. **Complex Klein-Gordon Field (70 points)**

   The complex Klein-Gordon field is used to describe charged bosons with spin 0. Its Lagrangian is given by

   $$\mathcal{L} = (\partial_\mu\phi^\dagger)(\partial^\mu\phi) - \mu^2\phi^\dagger\phi, \tag{6}$$

   where the field ϕ has the following normal mode expansion

   $$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\left[a(\mathbf{k})\,e^{-ik\cdot x} + b^\dagger(\mathbf{k})\,e^{ik\cdot x}\right], \tag{7}$$

   and satifies the equal-time commutation relations

   $$[\phi(\mathbf{x}, t), \Pi_\phi(\mathbf{x}', t)] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{8}$$

   $$\left[\phi^\dagger(\mathbf{x}, t), \Pi_{\phi^\dagger}(\mathbf{x}', t)\right] = i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \tag{9}$$

(a) **(15 points)** Show that the Lagrangian in eq. (6) is equivalent to the Lagrangian of two independent real scalar fields with the same mass and satisfying the standard equal-time commutation relations.

Hint: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

(b) **(25 points)** Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$. Derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.

Hint: Recap the derivation of the expressions for the annihilation and creation operators in terms of ϕ *and* $\dot{\phi}$ *for a real scalar field (eqs. 1 - 3). You can then – without the full derivation – write down the corresponding expressions for* a, $a^\dagger$, b *and* $b^\dagger$ *when a complex scalar field is considered. For instance, by looking at eq.(4) and eq.(7) one can infer that the expression for* $a^\dagger$ *in the real Klein-Gordon theory corresponds to the expression for* $b^\dagger$ *in the complex one.*

(c) **(15 points)** Show that the Lagrangian in eq. (6) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ and express the conserved charge $Q = \int d^3\mathbf{x}\, J^0$ in terms of creation and annihilation operators.

(d) **(15 points)** Compute the commutators $[Q, \phi]$ and $[Q, \phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?