Quantum Field Theory
Exercise 3

November 10, 2016

-to be handed in by 17.11.2016 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

1. **Relativistic particle in a homogeneous magnetic field (35 points)**
   The coupling of the Dirac equation to an electromagnetic field is accomplished by the substitution $\mathrm{i}\partial_\mu \to \mathrm{i}\partial_\mu - \mathrm{eA}_\mu$, where $A^\mu = (\Phi, \mathbf{A})$ is the electromagnetic 4-potential. The Dirac equation in presence of an electromagnetic field then reads:

$$\left((\mathrm{i}\not{\partial} - \mathrm{e}\not{A}) - \mathrm{m}\right)\psi = 0 \tag{1}$$

   (a) **(5 points)** Show that eq. 1 can be written as

$$\hat{E}\psi = \boldsymbol{\alpha}\left(\hat{\boldsymbol{p}} - e\mathbf{A}\right)\psi + e\Phi\psi + m\gamma^0\psi, \tag{2}$$

   with $\alpha^i = \gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and the usual quantum-mechanical replacements $\hat{p}^0 = \hat{p}_0 = \hat{E} = \mathrm{i}\partial_\mathrm{t}$ and $\hat{\boldsymbol{p}} = -i\nabla$. Note that here we use the *Dirac basis*, where $\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \beta$.

   Consider now a particle in a homogeneous magnetic field $\mathbf{B}$. Use the gauge $A^0 = A^2 = A^3 = 0$, $A^1 = By$.

   (b) **(5 points)** Show that eq. 2 now reads

$$(\hat{E} - m)\phi = \boldsymbol{\sigma}(\hat{\boldsymbol{p}} - eBy\,\hat{\boldsymbol{e}}_\mathbf{1})\chi \tag{3}$$

$$(\hat{E} + m)\chi = \boldsymbol{\sigma}(\hat{\boldsymbol{p}} - eBy\,\hat{\boldsymbol{e}}_\mathbf{1})\phi \tag{4}$$

   for the two components ϕ, χ of $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$.

   (c) **(15 points)** Derive the following equation for ϕ:

$$(\hat{E}^2 - m^2)\phi = \left(\hat{p}_y^2 + \hat{p}_z^2 + (\hat{p}_x - eB\hat{y})^2 + eB\sigma^3\right)\phi \tag{5}$$

$$= \left(\hat{p}_y^2 + \hat{p}_z^2 + (\hat{p}_x - eB\hat{y})^2 + 2eB\hat{s}_z\right)\phi \tag{6}$$

   Since the right hand sight of eq. 5 commutes with $\hat{p}_x$, $\hat{p}_z$ and $\hat{s}_z$, one can use the ansatz $\phi = \mathrm{e}^{\mathrm{i}(\mathrm{p_x x + p_z z})}\omega(\mathrm{p_y, y})\rho(\mathrm{s_z})$ for the stationary solution.

(d) **(10 points)** Derive a differential equation for $\omega(p_y, y)$. What does it remind you of? Deduce the corresponding energy levels for a relativistic particle in a homogeneous magnetic field.

2. **Gordon Identity (25 points)**
Derive the so-called *Gordon identity*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu}{2M} + \frac{i\sigma^{\mu\nu}q_\nu}{2M}\right]u(p),$$

where $P = p' + p$, $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

3. **Projection Operators (20 points)**
Show that $P_\pm = \pm\frac{\not{p}\pm m}{2m}$ represent a complete sets of projection operators, *i.e.* satisfy the conditions

$$P_i P_j = \delta_{i,j}\, P_i, \qquad \sum_i P_i = 1.$$

In addition, show that $P_\pm$ are the projection operators on positive and negative energy solutions (u and v spinors) for arbitrary particle momentum.

4. **γ Matrices (maximum 20 points)**: you are welcome to do as many as you like, but only the first to give 20 points will be marked.
Without using an explicit representation for the γ matrices show that:

(a) **(2.5 points)** $\gamma_\mu\gamma^\mu = 4$

(b) **(2.5 points)** $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$

(c) **(5 points)** $\mathrm{Tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\,[(a\cdot b)\,(c\cdot d) - (a\cdot c)\,(b\cdot d) + (a\cdot d)\,(b\cdot c)]$

(d) **(5 points)** $\gamma_5 = \frac{i}{4!}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

(e) **(5 points)** $\gamma_5^2 = \mathbb{1}$

(f) **(5 points)** $\{\gamma_5, \gamma^\mu\} = 0$

(g) **(5 points)** $\mathrm{Tr}[\gamma_5] = 0$

(h) **(5 points)** $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] = 0$

(i) **(5 points)** $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = 4i\,\varepsilon^{\mu\nu\rho\sigma}$

(j) **(5 points)** $\mathrm{Tr}[\gamma^{\mu_1}\cdots\gamma^{\mu_n}] = 0$ if n is odd

where $\not{a} \equiv \gamma^\mu a_\mu$, $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\varepsilon_{0123} = +1$