Quantum Field Theory
Exercise 4

November 17, 2016

-to be handed in by 24.11.2016 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

1. **Dirac Bilinears (60 points)**
   Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

   - $\Gamma_S = \mathbb{1}$ (scalar);
   - $\Gamma_P = \gamma_5$ (pseudoscalar);
   - $\Gamma_V^\mu = \gamma^\mu$ (vector);
   - $\Gamma_A^\mu = \gamma^\mu\gamma_5$ (axial vector);
   - $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (tensor).

   Without referring to any explicit representation for the Γ matrices,

   (a) **(5 points)** show that $\Gamma^2 = \pm\mathbb{1}$.

   (b) **(5 points)** show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$.

   (c) **(10 points)** check that the product of 2 different Γ's is proportional to some Γ different from Γ_S;

   (d) **(20 points)** using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check that the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma_5\psi$ and $\bar{\psi}'\sigma^{\mu\nu}\psi' = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi$.

   (e) **(20 points)** Calculate the following transformations:
   - $P\bar{\psi}\Gamma_T^{\mu\nu}\psi P$,
   - $C\bar{\psi}\Gamma_A^\mu\psi C$
   - $T\bar{\psi}\Gamma_V^\mu\psi T$

2. **The Quantized Dirac Field (40 points)**
   Express the following quantities in terms of creation and annihilation operators:

   (a) **(15 points)** momentum $\mathbf{P} = -i\int d^3x\, \psi^\dagger\nabla\psi$,

(b) **(15 points)** charge $Q = \int d^3x\, \psi^\dagger \psi$.

In addition, calculate:

(c) **(10 points)** $\big[\mathbf{P},\, a^\dagger(\mathbf{p}', s')\, a(\mathbf{p}', s')\big]$.