Quantum Field Theory
Exercise 9

January 12, 2017

-to be handed in by 19.01.2017 (12:00 h) to the letterbox No. 37 ("relativistische QFT") in the foyer of Staudingerweg 7.

(1) The Scalar Electrodynamics Lagrangian from Symmetry Arguments (20 points)

Start from the free complex Klein-Gordon field Lagrangian

$$\mathcal{L}_{\text{free}} = (\partial^\mu \phi)(\partial_\mu \phi^*) - m\phi\phi^*. \tag{1}$$

Check that this Lagrangian is not invariant under $U(1)$ gauge transformations of the form $\phi(x) \to \phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x)$. By knowing the transformation property of the photon field $A^\mu \to A'^\mu = A^\mu + \frac{1}{e}\left(\partial^\mu \alpha(x)\right)$, supplement the starting Lagrangian with additional interaction terms ($\mathcal{L}_{\text{int}}$) which would help to restore the gauge invariance of the total Lagrangian $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$. Finally, express the Lagrangian in the form

$$\mathcal{L} = (\mathrm{D}^\mu \phi)(\mathrm{D}_\mu \phi)^* - m\phi\phi^* - \frac{1}{4}\mathrm{F}_{\mu\nu}\mathrm{F}^{\mu\nu}, \tag{2}$$

with a suitably defined covariant derivative D_μ.

(2) Electron-positron annihilation into charged scalar mesons, $e^+e^- \to J\bar{J}$ (30 points)

Consider electron-positron annihilation into a meson-antimeson pair, $e^+(k_2)e^-(k_1) \to J(p_2)\bar{J}(p_1)$. The process is represented by the Feynman diagram in fig. 1. Treat the electron as a massless Dirac particle and the meson as a massless Klein-Gordon particle. For the definition of the kinematic variables, refer to fig. 2.

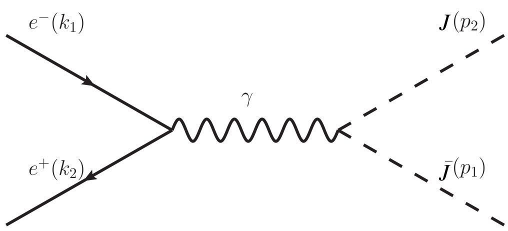

Figure 1: Electron-positron annihilation into a scalar meson-antimeson pair

(a) **(10 points)** The squared matrix element, obtained as an average over electron and positron spin configurations, can be expressed as

$$|\mathcal{M}|^2 = \frac{e^4 e_J^2}{s} L^{\mu\nu} J_{\mu\nu}, \tag{3}$$

with the meson charge e_J and the Mandelstam variable $s = (k_1 + k_2)^2$. Find the expressions for the meson tensor $J_{\mu\nu}$ and the lepton tensor $L_{\mu\nu}$ in terms of momenta k_1, k_2, p_1, p_2. The Feynman rule for the meson-antimeson-photon vertex is $ie_J e(p_2 - p_1)^\mu$.

(b) **(5 points)** Express the result for $L^{\mu\nu} J_{\mu\nu}$ in terms of the Mandelstam variable $s = (k_1 + k_2)^2$ and $t = (k_1 - p_1)^2$.

(c) **(10 points)** The general expression for the differential cross-section in the center-of-mass frame for a 2 → 2 process is

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{COM}}^2} \frac{|\mathbf{p}_{\text{fin}}|}{|\mathbf{p}_{\text{in}}|}, \tag{4}$$

where $|\mathbf{p}_{\text{in}}|$ and $|\mathbf{p}_{\text{fin}}|$ are the absolute values of the momentum in the center-of-mass system in the initial and final state, respectively.
Express it in terms of the energy squared in the center-of-mass frame, $s = (k_1 + k_2)^2$, and the angle θ between k_1 and p_2.

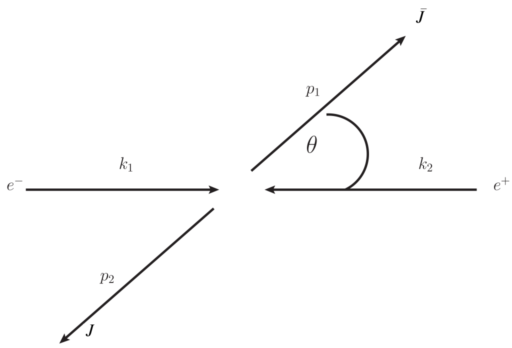

Figure 2: Parametrisation of the kinematic variables for electron-positron annihilation into a scalar meson-antimeson pair.

(3) Cross section of electron – muon scattering (50 points)

(a) **(5 points)** Use the Feynman rules to write down the amplitude for the process $e^- \, \mu^- \to e^- \, \mu^-$ shown in fig. 3

(b) **(40 points)** Show that the corresponding squared amplitude, averaged and summed over spins, is equal to

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{q^4} \left[(p_1 \cdot p_2')(p_1' \cdot p_2) + (p_1 \cdot p_2)(p_1' \cdot p_2') - m_\mu^2 (p_1 \cdot p_1') \right], \tag{5}$$

when neglecting the electron mass $m_e = 0$.

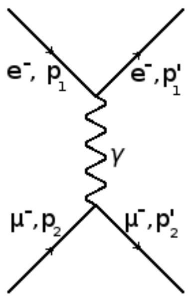

Figure 3: Feynman diagram for electron – muon scattering.

(c) **(5 points)** Assume to be in the center-of-mass system and calculate the differential cross section $\frac{d\sigma}{d\Omega}$ (again for vanishing electron mass, $m_e = 0$). What is the high energy limit of the differential cross section? Compare to the most general formula for the differential cross section for a $2 \to 2$ process (massive particles), given in exercise (2 c)). (refer to fig. 4 for definition of the variables)

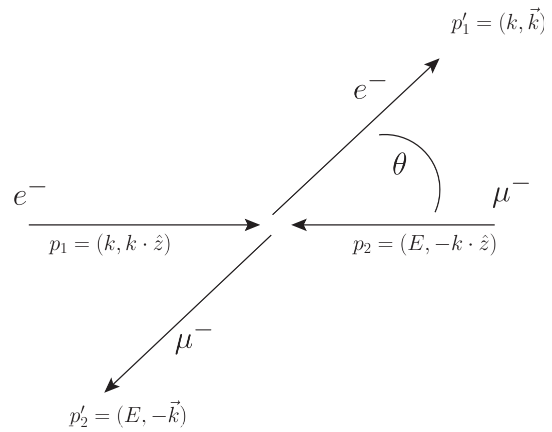

Figure 4: Parametrisation of the momenta of the in- and outgoing electron and muon in the center-of-mass system.