

# **Quantum Field Theory**

**Lecture Notes**

Joachim Kopp

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# Contents

<b>1</b>	<b>Introduction and Motivation</b>	<b>7</b>
1.1	Do you recognize the following equations? . . . . .	7
1.2	A Note on Notation . . . . .	7
<b>2</b>	<b>The Klein–Gordon Field</b>	<b>9</b>
2.1	Necessity of the Field Viewpoint . . . . .	9
2.2	Elements of Classical Field Theory . . . . .	10
2.2.1	The Euler–Lagrange Equations . . . . .	10
2.2.2	The Hamiltonian . . . . .	11
2.2.3	Noether’s Theorem . . . . .	12
2.3	Quantization of the Klein–Gordon Field . . . . .	13
2.3.1	Commutation Relations . . . . .	13
2.3.2	The Quantized Hamiltonian . . . . .	15
2.3.3	Time-Dependence of the Klein–Gordon Field Operator . . . . .	16
2.4	The Feynman Propagator for the Klein–Gordon Field . . . . .	17
2.4.1	Green’s Functions of the Klein–Gordon Operator . . . . .	17
2.4.2	The Feynman Propagator . . . . .	19
2.4.3	Relation to Correlation Functions . . . . .	19
<b>3</b>	<b>The Dirac Field</b>	<b>21</b>
3.1	The Dirac Equation and its Solutions . . . . .	21
3.1.1	The Equation and the Corresponding Lagrangian . . . . .	21
3.1.2	Solutions of the Dirac Equation . . . . .	22
3.1.3	Spin Sums . . . . .	24
3.2	Quantization of the Dirac Field . . . . .	25
3.2.1	How Not to Quantize the Dirac Field . . . . .	25
3.2.2	Quantizing the Dirac Field with Anticommutators . . . . .	26
3.2.3	Physical Significance of the Quantized Dirac Field . . . . .	28
3.3	The Feynman Propagator for the Dirac Field . . . . .	28
3.4	Symmetries of the Dirac Theory . . . . .	30
3.4.1	Lorentz Invariance . . . . .	30
3.4.2	Parity ( $P$ ) . . . . .	37
3.4.3	Time Reversal ( $T$ ) . . . . .	39
3.4.4	Charge Conjugation ( $C$ ) . . . . .	41
<b>4</b>	<b>Interacting Fields and Feynman Diagrams</b>	<b>43</b>

4.1	Time-Dependent Perturbation Theory for Correlation Functions . . . . .	43
4.1.1	$\phi^4$ Theory . . . . .	43
4.1.2	The Vacuum State of the Interacting Theory . . . . .	44
4.1.3	Correlation Functions . . . . .	44
4.1.4	Perturbation Theory . . . . .	45
4.2	Wick's Theorem . . . . .	50
4.3	Feynman Diagrams . . . . .	53
4.3.1	Basic Idea and Application to a Simple 4-Point Function . . . . .	54
4.3.2	An Example in $\phi^4$ Theory . . . . .	54
4.3.3	A More Advanced Example . . . . .	55
4.3.4	More Examples for Diagrams with Non-Trivial Symmetry Factors . . . . .	56
4.3.5	Position Space Feynman Rules . . . . .	57
4.3.6	Momentum Space Feynman Rules . . . . .	58
4.3.7	Disconnected Feynman Diagrams . . . . .	59
4.3.8	The Denominator of the Correlation Function . . . . .	61
4.4	The LSZ Reduction Formula . . . . .	61
4.5	Computing $S$ -Matrix Elements from Feynman Diagrams . . . . .	65
4.6	Feynman Rules for Fermions . . . . .	68
4.6.1	The Master Formula for Correlation Functions Involving Fermions . . . . .	68
4.6.2	Wick's Theorem for Fermions . . . . .	69
4.6.3	The LSZ Formula for Fermions . . . . .	70
4.6.4	Yukawa Theory . . . . .	71
4.6.5	The Yukawa Potential . . . . .	78
<b>5</b>	<b>Quantum Electrodynamics</b>	<b>81</b>
5.1	The QED Lagrangian from Symmetry Arguments . . . . .	81
5.2	The Feynman Rules for QED . . . . .	83
5.3	$e^+e^- \rightarrow \mu^+\mu^-$ . . . . .	86
5.3.1	Feynman Diagram and Squared Matrix Element . . . . .	86
5.3.2	Trace Technology . . . . .	87
5.3.3	The Squared Matrix Element for $e^+e^- \rightarrow \mu^+\mu^-$ (Part II) . . . . .	89
5.3.4	The Cross Section — General Results . . . . .	90
5.3.5	The Cross Section for $e^+e^- \rightarrow \mu^+\mu^-$ . . . . .	91
5.3.6	$e^+e^- \rightarrow \mu^+\mu^-$ : Summary . . . . .	92
5.4	More Technology for Evaluating QED Feynman Diagrams . . . . .	93
5.4.1	Scattering of Polarized Particles . . . . .	93
5.4.2	External Photons . . . . .	93
<b>6</b>	<b>Path Integrals</b>	<b>95</b>
6.1	Path Integrals in Quantum Mechanics . . . . .	95
6.2	The Path Integral for a Free Scalar Field . . . . .	97
6.3	The Feynman Propagator from the Path Integral . . . . .	100
6.4	Wick's Theorem from the Path Integral . . . . .	101

6.5	Interacting Field Theories in the Path Integral Formalism . . . . .	102
6.6	Quantization of the Photon Field . . . . .	103
6.7	Path Integrals for Fermions . . . . .	104
6.7.1	Grassmann Numbers . . . . .	105
6.7.2	Partition Function, Functional Derivative and Correlation Functions for Fermions . . . . .	107
6.8	The Quantum Equations of Motion: Schwinger–Dyson Equations . . . . .	109
6.9	The Ward–Takahashi Identity . . . . .	110
<b>7</b>	<b>Weyl and Majorana Fermions</b>	<b>115</b>
7.1	Spinor Indices . . . . .	116
7.1.1	Left-handed spinors . . . . .	116
7.1.2	Raising and lowering spinor indices . . . . .	117
7.1.3	Right-handed spinors . . . . .	118
7.1.4	Conjugate spinors . . . . .	119
7.1.5	Lorentz invariance of the Pauli matrices . . . . .	119
7.1.6	One more example: the vector current . . . . .	120
7.2	The QED Lagrangian in 2-Component Notation . . . . .	121
7.3	Majorana Fermions . . . . .	123
7.4	Application: Majorana Neutrinos and the Seesaw Mechanism . . . . .	124
7.4.1	Neutrino mass terms . . . . .	124
7.4.2	The seesaw mechanism . . . . .	125
7.4.3	Interlude: measuring neutrino masses . . . . .	127
7.5	Twistors . . . . .	131
7.5.1	Unifying spinors and momentum 4-vectors . . . . .	131
7.5.2	Twistor notation . . . . .	133
7.5.3	Examples . . . . .	135
<b>8</b>	<b>Radiative Corrections</b>	<b>137</b>
8.1	The Electron Vertex Function . . . . .	139
8.1.1	Preliminary Considerations . . . . .	139
8.1.2	Physical Interpretation . . . . .	141
8.1.3	Simplifying the Integrand . . . . .	143
8.1.4	The 4-Momentum Integral . . . . .	148
8.1.5	Lepton magnetic moments . . . . .	151
8.1.6	Renormalization: a First Glimpse . . . . .	154
8.1.7	Regularization of the Divergence . . . . .	156
8.1.8	Renormalization: The Counterterm . . . . .	159
8.1.9	Summary of our First 1-Loop Calculation . . . . .	160
8.2	Renormalization . . . . .	161
8.2.1	The renormalized QED Lagrangian . . . . .	161
8.2.2	1-Loop Corrections to the Fermion Propagator . . . . .	162
8.2.3	1-Loop Corrections to the Photon Propagator . . . . .	163

8.2.4	On-Shell Renormalization . . . . .	164
8.2.5	Renormalizability . . . . .	165
8.2.6	Renormalization of the Electric Charge . . . . .	169
8.2.7	Renormalization group evolution and the Callan–Symanzik equation	173
8.2.8	The $\beta$ function . . . . .	174
8.3	Infrared Divergences . . . . .	176
8.3.1	IR behavior of virtual (1-loop) corrections . . . . .	176
8.3.2	IR behavior of real corrections . . . . .	178
<b>9</b>	<b>Non-Abelian Gauge Theories</b>	<b>183</b>
9.1	Gauge Interactions from Symmetry . . . . .	183
9.2	Non-Abelian Gauge Transformations . . . . .	186
9.3	Lie Algebras and Lie Groups . . . . .	190
9.3.1	Special Representations . . . . .	192
9.3.2	The Casimir Operator . . . . .	193
9.3.3	Product Representations . . . . .	194
9.3.4	The $SU(N)$ Groups . . . . .	195
9.4	Quantization of Non-Abelian Gauge Fields . . . . .	196
9.5	The Feynman Rules for Non-Abelian Gauge Theories . . . . .	200
9.5.1	The Gauge Boson Propagator . . . . .	200
9.5.2	Gauge Boson Self-Interactions . . . . .	200
9.5.3	Ghost Interactions . . . . .	201
9.5.4	Coupling to Fermions . . . . .	201
9.6	The Beta Function for Non-Abelian Gauge Theories . . . . .	202
9.7	Phenomenology of Quantum Chromodynamics (QCD) . . . . .	204
9.7.1	Deep-Inelastic Scattering . . . . .	205
9.7.2	Proton–Proton Interactions at the LHC . . . . .	207
9.7.3	Infrared divergences in QCD . . . . .	208
9.7.4	Multiple Splittings . . . . .	211
9.7.5	The DGLAP Equations . . . . .	212
<b>10</b>	<b>Spontaneous Symmetry Breaking</b>	<b>217</b>
10.1	The Abelian Higgs Mechanism . . . . .	217
10.2	Goldstone’s Theorem . . . . .	219
10.3	The Glashow–Salam–Weinberg Theory of Electroweak Interactions . . . . .	221
10.3.1	Higgs Mechanism . . . . .	221
10.3.2	Gauge Boson Masses . . . . .	221
10.3.3	Gauge Boson Self-Couplings . . . . .	224
10.3.4	Gauge Boson Coupling to Fermions . . . . .	225
10.3.5	Fermion Masses . . . . .	226
10.3.6	The Higgs Boson . . . . .	228
	<b>Bibliography</b>	<b>233</b>

# 1

## Introduction and Motivation

### 1.1 Do you recognize the following equations?

$$i\hbar\dot{\psi} = \hat{H}\psi \quad (1.1)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad (1.2)$$

$$\partial_t^2\psi - \nabla^2\psi + m^2\psi = 0 \quad (1.3)$$

$$i\not{\partial}\psi - m\psi = 0 \quad (1.4)$$

$$\partial_\mu \frac{\delta\mathcal{L}(\phi, \partial_\mu\phi)}{\delta(\partial_\mu\phi)} - \frac{\delta\mathcal{L}(\phi, \partial_\mu\phi)}{\delta\phi} = 0 \quad (1.5)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 \quad (1.6)$$

$$j^\mu = \frac{\delta\mathcal{L}(\phi, \partial_\mu\phi)}{\delta(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu \quad (1.7)$$

### 1.2 A Note on Notation

In this course, we work in *natural units*, where

$$\hbar = c = 1. \quad (1.8)$$

This implies that we set

$$1 = \hbar \cdot c = 6.58 \cdot 10^{-16} \text{ eV}^{-1} \text{ sec} \times 3 \cdot 10^8 \text{ m/sec} = 197 \cdot 10^{-9} \text{ eV} \cdot \text{m} \quad (1.9)$$

It follows for instance that in our units

$$1 \text{ m} = 5.066 \cdot 10^6 \text{ eV}^{-1}, \quad (1.10)$$

$$1 \text{ sec} = 1.52 \cdot 10^{15} \text{ eV}^{-1}. \quad (1.11)$$

Throughout the lecture, we use the “West Coast Metric”

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad (1.12)$$

which is also employed by Peskin and Schroeder [1], and by most particle physicists today. Note that the book by Srednicki, for instance, uses the “East Coast Metric”, with  $-1$  in the timelike component and  $+1$  in the spacelike components [2].



# 2

## The Klein–Gordon Field

### 2.1 Necessity of the Field Viewpoint

From relativistic quantum mechanics, we know how to deal with the dynamics of a single relativistic particle: the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi(x) = 0 \quad (2.1)$$

describes the time evolution of the wave function of a relativistic scalar particle in vacuum. (Here,  $x = (t, \mathbf{x})$  is the coordinate 4-vector and  $m$  is the mass of the particle.) The Klein–Gordon equation can be easily generalized to motions in an external field by making the replacements  $p^\mu \rightarrow p^\mu + ieA^\mu$ . The Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad (2.2)$$

achieves the same for a fermionic particle. However, these equations only describe the motion of a single particle. The relativistic equivalence of mass and energy tells us that kinetic energy can be converted into the production of new particle (e.g.  $e^+e^-$  pair production when an ultrarelativistic electron travels through matter), therefore it is clear that a complete description of a particles' dynamics must provide for creation and annihilation processes as well. Note that even in processes where too little energy is available to actually create an  $e^+e^-$  pair, the intermittent existence of such pairs for very short time intervals is allowed by the Heisenberg uncertainty relation  $\Delta E \cdot \Delta t \geq \hbar$  and will therefore happen.

Therefore a relativistic theory that can describe multiparticle processes, including in particular particle creation and annihilation, is highly desirable. Actually, we have encountered creation and annihilation processes before: when studying the quantum mechanical harmonic oscillator, the operators  $a$  and  $a^\dagger$  that transformed its wave function to a lower or higher energy state were called creation and annihilation operators, though

at the time this seemed like a bit of a misnomer. Quantum field theory (QFT) generalizes the concept of the harmonic oscillator to an extent that makes the terms “creation” and “annihilation” operator appropriate. It describes particles of a given species (e.g. electrons) as a *field*, i.e. a function  $\phi(x)$  that maps each spacetime point  $x$  to a scalar value (scalar fields, e.g. the Higgs boson), a Dirac spinor (fermion fields, e.g. the electron) or a Lorentz 4-vector (vector fields, e.g. the photon). We will show below that the equation of motion for each Fourier mode of this field has exactly the same form as the equation of motion of the harmonic oscillator. It can therefore be quantized in exactly the same way. The creation and annihilation operators then become  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^\dagger$ , e.g. they carry a momentum index  $\mathbf{p}$  corresponding to the Fourier mode they describe. The interpretation of a field is then the following: its ground state, where each momentum mode carries only the zero point energy, corresponds to the vacuum  $|0\rangle$ . If a momentum mode is in its first excited state  $a_{\mathbf{p}}^\dagger|0\rangle$ , this means that one particle with momentum  $\mathbf{p}$  exists. The operators  $a_{\mathbf{p}}^\dagger$  and  $a_{\mathbf{p}}$  then create and annihilate *particles*. The equations of motion of the theory describe the rules according to which such creation and annihilation processes occur.

In the following, we will first review a few concepts of classical (non-quantized) field theory (section 2.2) and then make the above statements more precise and more mathematical (section 2.3).

## 2.2 Elements of Classical Field Theory

### 2.2.1 The Euler–Lagrange Equations

In classical mechanics, the dynamics of a particle was described by its trajectory  $\mathbf{x}(t)$ , which was obtained as the solution of its equations of motion. These, in turn, were derived based on the principle that the action

$$S \equiv \int dt L(\mathbf{x}, \dot{\mathbf{x}}), \quad (2.3)$$

is stationary, i.e.

$$\delta S = 0, \quad (2.4)$$

where  $L(\mathbf{x}, \dot{\mathbf{x}})$  is the Lagrange functional and  $\delta S$  means the variation of  $S$  with respect to  $\mathbf{x}$  and  $\dot{\mathbf{x}}$ . In field theory, the Lagrange functional becomes the *Lagrange density* (or *Lagrangian* for short)  $\mathcal{L}(\phi, \partial_\mu \phi)$ , a functional of the field value and its spacetime derivatives. (It is called a density because we will see that it has units of  $[\text{length}]^{-3}[\text{time}]^{-1}$ , as opposed to the Lagrange functional in classical mechanics, which has units of  $[\text{time}]^{-1}$ .) The action is defined as

$$S \equiv \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (2.5)$$

The principal of stationary action,  $\delta S = 0$ , then implies

$$0 = \delta S = \int d^4x \left[ \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta(\partial_\mu \phi) + \frac{\delta \mathcal{L}}{\delta \phi} \delta \phi \right] \quad (2.6)$$

$$= \int d^4x \left[ -\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} + \frac{\delta \mathcal{L}}{\delta \phi} \right] \delta \phi, \quad (2.7)$$

where in the last step we have integrated by parts. Since eq. (2.7) is required to be satisfied for any variation  $\delta \phi$ , the term in square brackets must vanish. This leads to the *Euler-Lagrange equations*

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0. \quad (2.8)$$

As an example, consider a scalar field  $\phi$  (for instance the Higgs field), for which the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2. \quad (2.9)$$

The Euler–Lagrange equations then lead to the equation of motion

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0, \quad (2.10)$$

which is just the Klein–Gordon equation.

### 2.2.2 The Hamiltonian

In classical mechanics, the Lagrange functional  $L$  is related to the Hamilton functional  $H$  through a Legendre transform

$$H = \boldsymbol{\pi} \dot{\mathbf{x}} - L, \quad (2.11)$$

where the canonical momentum  $\boldsymbol{\pi}$  is defined as

$$\boldsymbol{\pi} \equiv \frac{\delta L}{\delta \dot{\mathbf{x}}}. \quad (2.12)$$

In analogy, we define the Hamiltonian density (or Hamiltonian for short) in field theory as

$$\mathcal{H} \equiv \pi(x) \dot{\phi}(x) - \mathcal{L}, \quad (2.13)$$

where  $\pi(x)$  in field theory is defined as

$$\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)}. \quad (2.14)$$

### 2.2.3 Noether's Theorem

A crucial concept in quantum field theory is symmetries, i.e. transformations of the system that leave the action invariant. *Noether's theorem* relates symmetries to conserved physical quantities. It applies to continuous symmetries, i.e. symmetry transformations that depend on a parameter  $\alpha$ , which can vary continuously from 0 to  $\infty$ , where  $\alpha = 0$  corresponds to the identity map. Examples include rotations or gauge transformations, which should be familiar from classical electrodynamics. It is for our purposes sufficient to consider infinitesimal transformations

$$\phi(x) \rightarrow \phi'(x) \equiv \phi(x) + \alpha \Delta\phi(x), \quad (2.15)$$

where  $\alpha$  is an infinitesimal parameter. We call this transformation a symmetry if it leaves the action  $S$  invariant. This requires in particular that the Lagrangian  $\mathcal{L}$  is left invariant up to possibly a 4-divergence  $\partial_\mu \mathcal{J}^\mu$ , which vanishes when integrated over  $d^4x$  by virtue of Gauss' theorem (We always assume boundary terms to be zero in such integrals.) A symmetry transformation thus satisfies

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu. \quad (2.16)$$

Thus,

$$\alpha \Delta\mathcal{L} \equiv \alpha \partial_\mu \mathcal{J}^\mu = \frac{\delta\mathcal{L}}{\delta\phi}(\alpha\Delta\phi) + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu(\alpha\Delta\phi) \quad (2.17)$$

$$= \alpha \partial_\mu \left( \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \Delta\phi \right) + \alpha \left[ \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \left( \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \right) \right] \Delta\phi. \quad (2.18)$$

In the second step, we have used the product rule of differentiation backwards. The term in square brackets vanishes due to the Euler–Lagrange equations. Therefore, we are left with the conclusion that the current

$$j^\mu \equiv \left( \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} \Delta\phi \right) - \mathcal{J}^\mu \quad (2.19)$$

is conserved:

$$\partial_\mu j^\mu = 0. \quad (2.20)$$

This implies in particular for the associated *charge* (which, except in the case of electromagnetic gauge transformations has nothing to do with electric charge)

$$Q \equiv \int d^3x j^0 \quad (2.21)$$

is constant:

$$\dot{Q} = \int d^3x \partial_0 j^0 = \int d^3x \partial_\mu j^\mu - \int d^3x \partial_k j^k = 0. \quad (2.22)$$

Here, as usual, the greek index  $\mu$  runs from 0 to 3, while the latin index  $k$  runs only from 1 to 3. The term containing  $\partial_\mu j^\mu$  vanishes by virtue of Noether's theorem (eq. (2.20)), the one containing  $\partial_k j^k$  vanishes thanks to Gauss' theorem.

As an example, consider infinitesimal spacetime shifts

$$x^\mu \rightarrow x^\mu - \alpha a^\mu, \quad (2.23)$$

for some constant 4-vector  $a^\mu$ , applied to the free scalar field from eq. (2.9). The corresponding field transformation is

$$\phi(x) \rightarrow \phi(x + \alpha a) = \phi(x) + \alpha [\partial_\mu \phi(x)] a^\mu. \quad (2.24)$$

Plugging this into the Lagrangian eq. (2.9), we obtain (remembering that  $\alpha$  is infinitesimal)

$$\mathcal{L} \rightarrow \mathcal{L} + \alpha (\partial_\mu \partial_\nu \phi) (\partial^\mu \phi) a^\nu + \alpha m^2 \phi (\partial_\nu \phi) a^\nu = \mathcal{L} + \alpha a^\nu \partial_\nu \mathcal{L}. \quad (2.25)$$

This implies

$$\mathcal{J}^\nu = a^\nu \mathcal{L} \quad (2.26)$$

and

$$j^\mu = \left[ (\partial^\mu \phi) [\partial_\nu \phi(x)] - \mathcal{L} \delta^\mu_\nu \right] a^\nu. \quad (2.27)$$

Consider in particular time-like shifts, i.e.  $a^\nu = (1, 0, 0, 0)$ . Then the term in square brackets becomes

$$j'^\mu \equiv (\partial^\mu \phi) [\partial_0 \phi(x)] - \mathcal{L} \delta^\mu_0. \quad (2.28)$$

The 0-component of this expression is just the Legendre transform of the Lagrangian, i.e. the Hamiltonian density. The  $d^3x$  integral over the Hamiltonian density (or Hamiltonian for short) is just the energy. Thus, invariance under time-like shifts implies energy conservation. Had we chosen  $a^\nu$  to correspond to space-like shifts, we would in analogy have found momentum conservation.

## 2.3 Quantization of the Klein–Gordon Field

### 2.3.1 Commutation Relations

Let us now proceed to quantized field theory. We use as an example the real scalar field  $\phi(x)$ , whose Lagrangian eq. (2.9) we repeat here:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2. \quad (2.29)$$

When going from classical mechanics to quantum mechanics, the procedure is to promote the position  $\mathbf{q}$  and the canonical momentum  $\boldsymbol{\pi} \equiv (\delta L)/(\delta \dot{\mathbf{q}})$  to operators that satisfy canonical commutation relations

$$[q^j, \pi^k] = i\delta^{jk} \quad (2.30)$$

$$[q^j, q^k] = [\pi^j, \pi^k] = 0. \quad (2.31)$$

We now do this analogously for the field  $\phi(x)$  by promoting  $\phi(x)$  and

$$\pi(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\phi}(x)} \quad (2.32)$$

to operators and postulating that they satisfy

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= i\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\ [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] &= [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \end{aligned} \quad (2.33)$$

Note that the time coordinate of the spacetime points  $x$  and  $y$  is the same here.<sup>1</sup> Since we normally deal with particles of known energy (for instance the particles in the LHC beams), without being too much interested in their position, it is convenient to go to Fourier space and write

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} \phi(\mathbf{p}, t), \quad (2.34)$$

$$\pi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\mathbf{x}} \pi(\mathbf{p}, t). \quad (2.35)$$

Note that  $\phi(-\mathbf{p}, t) = \phi(\mathbf{p}, t)$  because  $\phi(\mathbf{x}, t)$  was assumed to be real. The Klein–Gordon equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0, \quad (2.36)$$

then becomes

$$\left[ \frac{\partial^2}{\partial t^2} + |\mathbf{p}|^2 + m^2 \right] \phi(\mathbf{p}, t) = 0. \quad (2.37)$$

This is precisely the equation of motion for a harmonic oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (2.38)$$

---

<sup>1</sup>Allowing for different time coordinates would not make sense. For points with a spacelike separation, causality dictates that all commutators must be zero because measurements at points that are outside each other's light cone cannot influence each other. Measurements at points within the light cone, however, can affect each other. Thus, the commutator of fields at different  $t$  is more complicated.

We know how to quantize the harmonic oscillator: we introduce *creation and annihilation operators*  $a_{\mathbf{p}}^\dagger(t)$  and  $a_{\mathbf{p}}(t)$ , defined such that

$$\phi(\mathbf{p}, t) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}(t) + a_{-\mathbf{p}}^\dagger(t)) \quad (2.39)$$

$$\pi(\mathbf{p}, t) = -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}}(a_{\mathbf{p}}(t) - a_{-\mathbf{p}}^\dagger(t)). \quad (2.40)$$

Note that, by convention, we assign the creation operator an index  $-\mathbf{p}$ . It will become clear later why this choice makes sense. Note also that we work in the Heisenberg picture here where the field operators are time dependent. We will see below in sec. 2.4 what the explicit form of this time dependence is. From the commutation relations eq. (2.33), we can then derive

$$[a_{\mathbf{p}}(t), a_{\mathbf{p}'}^\dagger(t)] = \frac{1}{2} \left[ \sqrt{\omega_{\mathbf{p}}} \phi(\mathbf{p}, t) + \frac{i}{\sqrt{\omega_{\mathbf{p}}}} \pi(\mathbf{p}, t), \sqrt{\omega_{\mathbf{p}'}} \phi(-\mathbf{p}', t) - \frac{i}{\sqrt{\omega_{\mathbf{p}'}}} \pi(-\mathbf{p}', t) \right] \quad (2.41)$$

$$= -\frac{i}{2} \left( \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}'}}} [\phi(\mathbf{p}, t), \pi(-\mathbf{p}', t)] + \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} [\phi(-\mathbf{p}', t), \pi(\mathbf{p}, t)] \right) \quad (2.42)$$

$$= -\frac{i}{2} \int d^3x e^{-i\mathbf{p}\mathbf{x}} \int d^3x' e^{i\mathbf{p}'\mathbf{x}'} \left( \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}'}}} [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] + \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} [\phi(\mathbf{x}', t), \pi(\mathbf{x}, t)] \right) \quad (2.43)$$

$$= \frac{1}{2} \int d^3x e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{x}} \left( \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}'}}} + \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \right) \quad (2.44)$$

$$= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \quad (2.45)$$

### 2.3.2 The Quantized Hamiltonian

Let us next consider the Hamilton operator  $H = \int d^3x \mathcal{H}$ . According to eq. (2.13) it reads

$$H = \int d^3x \left( \frac{1}{2} [\pi(\mathbf{x}, t)]^2 + \frac{1}{2} [\nabla \phi(\mathbf{x}, t)]^2 + \frac{1}{2} m^2 [\phi(\mathbf{x}, t)]^2 \right) \quad (2.46)$$

$$\begin{aligned} &= \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\mathbf{x}} \left[ -\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4} (a_{\mathbf{p}}(t) - a_{-\mathbf{p}}^\dagger(t)) (a_{\mathbf{p}'}(t) - a_{-\mathbf{p}'}^\dagger(t)) \right. \\ &\quad \left. + \frac{-\mathbf{p} \cdot \mathbf{p}' + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (a_{\mathbf{p}}(t) + a_{-\mathbf{p}}^\dagger(t)) (a_{\mathbf{p}'}(t) + a_{-\mathbf{p}'}^\dagger(t)) \right] \end{aligned} \quad (2.47)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^\dagger(t) a_{\mathbf{p}}(t) + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]). \quad (2.48)$$

In the last line, we have used that  $\omega_{-\mathbf{p}} = \omega_{\mathbf{p}}$ . As for the harmonic oscillator in quantum mechanics, the operator  $a_{\mathbf{p}}^\dagger(t) a_{\mathbf{p}}(t)$  is the particle number operator. When applied to a

quantum state  $|\psi\rangle$ , it gives the number of particles with momentum  $\mathbf{p}$  in that state:

$$a_{\mathbf{p}}^\dagger(t)a_{\mathbf{p}}(t)|\psi\rangle = n_{\mathbf{p}}|\psi\rangle \quad (2.49)$$

The term  $[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]$  is an infinite constant according to the commutation relation eq. (2.45) and corresponds to the zero point energies of all the individual  $\mathbf{p}$ -modes. However, experiments measure only energy differences, therefore a constant (albeit infinite) contribution to  $H$  can be dropped without changing the physics. This is what we will do from now on.<sup>2</sup>

We now discuss the eigenstates of the Hamiltonian eq. (2.48). As before, we call the vacuum state  $|0\rangle$ . It is normalized according to

$$\langle 0|0\rangle = 1. \quad (2.50)$$

A state containing exactly one particle of momentum  $\mathbf{p}$  will be written as  $|\mathbf{p}\rangle \equiv ca_{\mathbf{p}}^\dagger|0\rangle$ . Here,  $a_{\mathbf{p}}$  (without the argument  $t$ ) denotes  $a_{\mathbf{p}}(0)$ . Remember that we are working in the Heisenberg picture, so the states are time-independent and all the time-dependence is included in the operators. The normalization constant  $c$  is chosen such that

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad (2.51)$$

where  $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ . This normalization condition has the advantage of being Lorentz invariant, as can be easily shown by applying a Lorentz transformation and using the properties of the  $\delta$ -function (Exercise!). It implies

$$|\mathbf{p}\rangle \equiv \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^\dagger|0\rangle. \quad (2.52)$$

This is easily seen by directly computing

$$\langle \mathbf{p}|\mathbf{q}\rangle = 2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}\langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^\dagger|0\rangle \quad (2.53)$$

$$= 2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}\left(\langle 0|a_{\mathbf{q}}^\dagger a_{\mathbf{p}}|0\rangle + \langle 0|[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]|0\rangle\right) \quad (2.54)$$

$$= 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}). \quad (2.55)$$

### 2.3.3 Time-Dependence of the Klein–Gordon Field Operator

So far, we have not made the time dependence of the creation and annihilation operators  $a_{\mathbf{p}}^\dagger$  and  $a_{\mathbf{p}}$  explicit. To do so, we use the Heisenberg equation of motion, which tells us that

$$\frac{d}{dt}a_{\mathbf{p}}(t) = i[H, a_{\mathbf{p}}(t)]. \quad (2.56)$$

---

<sup>2</sup>There is one situation where the absolute energy scale is relevant, and that is cosmology. The expansion rate of the Universe depends on the total energy it contains, which includes vacuum energy (i.e. energy that is there even if no particles exist so that all fields are in their ground states). Such vacuum energy is what dominated the energy density of the Universe during the early phase of inflation, and is dominating again nowadays, where it is driving the accelerated expansion of the Universe. In this context, it is dubbed *dark energy*, and we have no idea what it is or what determines its magnitude.



It is easy to show that

$$[H, a_{\mathbf{p}}(t)] = \int \frac{d^3 p'}{(2\pi)^3} E_{\mathbf{p}'} [a_{\mathbf{p}'}^\dagger(t) a_{\mathbf{p}'}(t), a_{\mathbf{p}}(t)] \quad (2.57)$$

$$= -E_{\mathbf{p}} a_{\mathbf{p}}(t), \quad (2.58)$$

which implies

$$a_{\mathbf{p}}(t) = e^{-iE_{\mathbf{p}}t} a_{\mathbf{p}}. \quad (2.59)$$

We again use the notation  $a_{\mathbf{p}} \equiv a_{\mathbf{p}}(0)$ . In a similar way we can show that

$$a_{\mathbf{p}}^\dagger(t) = e^{iE_{\mathbf{p}}t} a_{\mathbf{p}}^\dagger. \quad (2.60)$$

Consequently, making the  $t$ -dependence explicit, the Klein–Gordon field becomes, according to eqs. (2.34) and (2.39),

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}). \quad (2.61)$$

As usual, the symbols  $p$  and  $x$  denote 4-vectors, i.e.  $px = E_{\mathbf{p}}t - \mathbf{p}\mathbf{x}$ . In the term containing  $a_{\mathbf{p}}^\dagger(t)$ , we have substituted  $\mathbf{p} \rightarrow -\mathbf{p}$ .

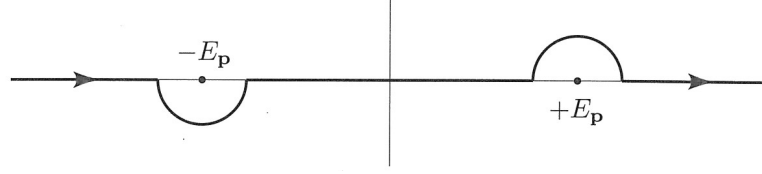
Note that the field described by eq. (2.61) contains a term with positive frequency (proportional to  $e^{-ipx}$ ) and a term with negative frequency (proportional to  $e^{+ipx}$ ). The positive frequency term comes with an operator that destroys a positive energy state, and the negative frequency term comes with an operator that creates a positive energy state. So the Hilbert space contains only positive energy states, and thus has a straightforward physical interpretation—in contrast to the wave function solutions to the Klein–Gordon equation in relativistic quantum mechanics.

## 2.4 The Feynman Propagator for the Klein–Gordon Field

### 2.4.1 Green's Functions of the Klein–Gordon Operator

So far, we have discussed the theory of a free (non-interacting) real scalar field. This is admittedly a bit boring, and we would eventually like to discuss interactions among particles. Adding interactions will make the Lagrangian more complicated. For instance to introduce electromagnetic couplings in the canonical way, we would replace  $\partial_\mu \rightarrow \partial_\mu + ieA_\mu$ , where  $e$  is the electromagnetic coupling constant and  $A_\mu$  is the electromagnetic 4-potential. A convenient way of dealing with these extra terms is the Green's function method. A Green's function  $D(x - y)$  of the Klein–Gordon operator is defined by the requirement

$$(\partial_x^2 + m^2)D(x - y) = -i\delta^{(4)}(x - y). \quad (2.62)$$



**Figure 2.1:** The integration contour for the  $p^0$  integral in the Feynman propagator, shown in the complex  $p^0$  plane. Figure taken from [1].

(In general, we should write  $D(x, y)$ , but translational invariance implies that the dependence can only be on  $x - y$ .) A solution of the equation

$$(\partial_x^2 + m^2)\phi(x) = j(x), \quad (2.63)$$

where  $j(x)$  denotes the extra terms, can then be obtained as

$$\phi(x) = \int d^4y iD(x - y)j(y). \quad (2.64)$$

To find  $D(x - y)$ , we go to Fourier space and write

$$D(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{D}(p) \quad (2.65)$$

Equation (2.62) then becomes

$$(\partial_x^2 + m^2) \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \tilde{D}(p) = -i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \quad (2.66)$$

$$\Leftrightarrow \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} (-p^2 + m^2) \tilde{D}(p) = -i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}. \quad (2.67)$$

We arrive at

$$D(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2}. \quad (2.68)$$

Note that in going from eq. (2.67) to eq. (2.68), we have divided by  $p^2 - m^2$ , which is only well-defined if  $p^2 \neq m^2$ . This, of course, need not be the case—in particular,  $p^2 = E_{\mathbf{p}}^2 - \mathbf{p}^2 = m^2$  is just the relativistic mass-shell condition, satisfied for “real” particles. We therefore have to regularize eq. (2.68). We do so by shifting the poles away from the real axis by an infinitesimal amount  $\epsilon$ . There are four ways of doing this, and all of them lead to valid Green’s functions.

### 2.4.2 The Feynman Propagator

We will see below that the prescription we need in QFT is the one where the pole at  $p^0 = -\sqrt{\mathbf{p}^2 + m^2}$  is shifted upwards (positive imaginary part), while the one at  $p^0 = \sqrt{\mathbf{p}^2 + m^2}$  is shifted downwards (negative imaginary part), see fig. 2.1. This is achieved by writing

$$D_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}. \quad (2.69)$$

To see this, rewrite the expression as

$$D_F(x - y) = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{2p^0} \left[ \frac{1}{p^0 + \sqrt{\mathbf{p}^2 + m^2} - i\epsilon'} + \frac{1}{p^0 - \sqrt{\mathbf{p}^2 + m^2} + i\epsilon'} \right]. \quad (2.70)$$

Here,  $\epsilon' \equiv \epsilon/(2\sqrt{\mathbf{p}^2 + m^2})$  is also infinitesimal. The quantity  $D_F(x - y)$  is called the *Feynman propagator* (hence the index  $F$ ). To carry out the complex contour integral over  $p^0$  in eq. (2.70) explicitly, we close the integration contour in the complex plane by a half-circle at infinity. For  $x^0 > y^0$ , the contour should be closed in the lower half plane, for  $x^0 < y^0$ , it should be closed in the upper half plane for the exponential to go to zero on the half circle. Using the residual theorem, we then obtain

$$D_F(x - y) = \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} \left[ e^{-iE_{\mathbf{p}}(x^0 - y^0) + i\mathbf{p}(\mathbf{x} - \mathbf{y})} \theta(x^0 - y^0) + e^{iE_{\mathbf{p}}(x^0 - y^0) + i\mathbf{p}(\mathbf{x} - \mathbf{y})} \theta(y^0 - x^0) \right] \quad (2.71)$$

$$= \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} \left[ e^{-ip(x-y)} \theta(x^0 - y^0) + e^{ip(x-y)} \theta(y^0 - x^0) \right]. \quad (2.72)$$

In the second term on the second line, we have substituted  $\mathbf{p} \rightarrow -\mathbf{p}$ .

### 2.4.3 Relation to Correlation Functions

To find a physical interpretation for the Feynman propagator  $D_F(x - y)$ , we will relate it to the *2-point correlation function*

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle, \quad (2.73)$$

which can be interpreted as the amplitude for a particle to propagate from  $y$  to  $x$ . Indeed, since  $a_{\mathbf{p}}|0\rangle = 0$ , only the  $a_{\mathbf{p}}^\dagger$  contributions are picked out of  $\phi(y)$ , i.e.  $\phi(y)|0\rangle$  can be interpreted as creating a particle at  $y$ . Similarly, only the annihilation operators contribute to  $\langle 0 | \phi(x)$ , so  $\phi(x)$  here annihilates the particle at  $x$ . For the overall correlation function to be nonzero, the particle must propagate from  $y$  to  $x$ . We can compute the 2-point correlation function for the free Klein–Gordon field:

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \left\langle 0 \left| \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \int \frac{d^3 p'}{(2\pi)^3 \sqrt{2E_{\mathbf{p}'}}} e^{-ipx + ip'y} a_{\mathbf{p}} a_{\mathbf{p}'}^\dagger \right| 0 \right\rangle \quad (2.74)$$

$$= \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} e^{-ip(x-y)}. \quad (2.75)$$

Comparing to eq. (2.72), we can write

$$D_F(x-y) = \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle. \quad (2.76)$$

In other words, the Feynman propagator always describes the propagation of a particle in the *positive* time direction. This motivates the particular choice for the shift of the poles in eq. (2.69). We also see now why  $D_F(x-y)$  deserves the name “propagator”.

A more compact way of writing  $D_F(x-y)$  is obtained by introducing the *time ordering symbol*  $T$ , which tells us to order any following field operators according to the zero component of their argument in descending order:

$$D_F(x-y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle. \quad (2.77)$$

# 3

## The Dirac Field

### 3.1 The Dirac Equation and its Solutions

#### 3.1.1 The Equation and the Corresponding Lagrangian

Now that we have some understanding of how to quantize a scalar field, let us repeat the same for fermions. Our starting point is the Dirac equation (written in covariant form)

$$(i\cancel{\partial} - m)\psi(x) = 0, \quad (3.1)$$

where we use the notation  $\cancel{\partial} \equiv \gamma^\mu \partial_\mu$ , and  $\gamma^\mu$  are the Dirac matrices, which satisfy the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (3.2)$$

Here,  $g^{\mu\nu} = (1, -1, -1, -1)$  is the Minkowski metric and  $\{\cdot, \cdot\}$  is the anticommutator, i.e.

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu. \quad (3.3)$$

One can get quite far without specifying a specific representation for the Dirac matrices  $\gamma^\mu$ , but it is much easier to have one in mind. In the following, we will always use the *chiral representation*, which reads

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (3.4)$$

Here,  $\sigma^i$  are the Pauli matrices. Often, we will also encounter the 5-th gamma matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (3.5)$$

$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma. \quad (3.6)$$

In the chiral representation, it reads

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_{2 \times 2} & 0 \\ 0 & \mathbb{1}_{2 \times 2} \end{pmatrix}. \quad (3.7)$$

The matrix  $\gamma^5$  has the property that it anticommutes with the other  $\gamma$  matrices, as can be easily seen by using eq. (3.6) and the anticommutator (3.2).

Note that, by taken the Hermitian transpose of the Dirac equation and multiplying by  $\gamma^0$  from the right, we can immediately derive an equation for  $\bar{\psi} \equiv \psi^\dagger \gamma^0$ . (We will see shortly why it is useful to consider  $\bar{\psi}$  instead of simply  $\psi^\dagger$ .)

$$-i[\partial_\mu \psi^\dagger(x)]\gamma^{\mu\dagger}\gamma^0 - m\bar{\psi} = 0. \quad (3.8)$$

This can be simplified using the identity

$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu, \quad (3.9)$$

which holds for the chiral representation (but not all representations) of the Dirac matrices, as can be easily checked by direct computation. The complex conjugate Dirac equation then becomes

$$-i[\partial_\mu \bar{\psi}(x)]\gamma^\mu - m\bar{\psi} = 0. \quad (3.10)$$

The Lagrangian from which the Dirac equation and its complex conjugate are obtained is

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi. \quad (3.11)$$

To check this, simply apply the Euler–Lagrange equations eq. (2.8), taking into account that  $\psi(x)$  and  $\bar{\psi}(x)$  are independent fields. (We could equivalently treat the real and imaginary parts of  $\psi(x)$  as the independent degrees of freedom, but this would lead to much more cumbersome equations.)

### 3.1.2 Solutions of the Dirac Equation

In relativistic quantum mechanics, the free solutions of the Dirac equation are given by

$$\psi_{p,s,+}(x) = u^s(p)e^{-ipx}, \quad (\text{positive energy solution}) \quad (3.12)$$

$$\psi_{p,s,-}(x) = v^s(p)e^{ipx}, \quad (\text{negative energy solution}) \quad (3.13)$$

where  $p$  is an on-shell 4-momentum (i.e.  $p^0 = E_{\mathbf{p}}$ ) and the index  $s$  denotes the spin orientation. The physical meaning of the negative energy solutions is a highly nontrivial topic in relativistic quantum mechanics. Here, we will neglect it for the moment since field theory will provide an elegant solution once we have proceeded to the quantized

Dirac field. The spinors  $u^s(p)$  and  $v^s(p)$  must satisfy the relations (momentum space Dirac equation)

$$\begin{aligned} (\not{p} - m)u^s(p) &= 0, \\ (\not{p} + m)v^s(p) &= 0, \\ \bar{u}^s(p)(\not{p} - m) &= 0, \\ \bar{v}^s(p)(\not{p} + m) &= 0 \end{aligned} \tag{3.14}$$

for  $\psi_{p,s,+}(x)$  and  $\psi_{p,s,-}(x)$  to satisfy the Dirac equation eq. (3.1).

Explicitly, we have

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{and} \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}. \tag{3.15}$$

We introduced here even more notation, namely we define the 4-vectors  $\sigma \equiv (\mathbb{1}_{2 \times 2}, \boldsymbol{\sigma})$  and  $\bar{\sigma} \equiv (\mathbb{1}_{2 \times 2}, -\boldsymbol{\sigma})$ . The 2-component vectors  $\xi^1$  and  $\xi^2$  distinguishing the two spin orientations are simply two orthonormal basis vectors of  $\mathbb{R}^2$ , for instance  $\xi^1 = (1, 0)$  and  $\xi^2 = (0, 1)$ . We can see that  $u^s(p)$  and  $v^s(p)$  are indeed solutions of eq. (3.14) in the following way:

$$\begin{aligned} \not{p}u^s(p) &= \begin{pmatrix} 0 & p \cdot \sigma \\ p \cdot \bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{(p \cdot \sigma)^2} p \cdot \bar{\sigma} \xi^s \\ \sqrt{(p \cdot \bar{\sigma})^2} p \cdot \sigma \xi^s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{(E - \mathbf{p} \cdot \boldsymbol{\sigma})^2} (E + \mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s \\ \sqrt{(E + \mathbf{p} \cdot \boldsymbol{\sigma})^2} (E - \mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{(E^2 - 2E(\mathbf{p} \cdot \boldsymbol{\sigma}) + \mathbf{p}^2)} (E + \mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s \\ \sqrt{(E^2 + 2E(\mathbf{p} \cdot \boldsymbol{\sigma}) + \mathbf{p}^2)} (E - \mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{E^3 - 2E^2(\mathbf{p} \cdot \boldsymbol{\sigma}) + E\mathbf{p}^2 + E^2(\mathbf{p} \cdot \boldsymbol{\sigma}) - 2E\mathbf{p}^2 + \mathbf{p}^2(\mathbf{p} \cdot \boldsymbol{\sigma})} \xi^s \\ \sqrt{E^3 + 2E^2(\mathbf{p} \cdot \boldsymbol{\sigma}) + E\mathbf{p}^2 - E^2(\mathbf{p} \cdot \boldsymbol{\sigma}) - 2E\mathbf{p}^2 - \mathbf{p}^2(\mathbf{p} \cdot \boldsymbol{\sigma})} \xi^s \end{pmatrix} \\ &= \begin{pmatrix} m\sqrt{E - \mathbf{p} \cdot \boldsymbol{\sigma}} \xi^s \\ m\sqrt{E + \mathbf{p} \cdot \boldsymbol{\sigma}} \xi^s \end{pmatrix} \\ &= \begin{pmatrix} m\sqrt{p \cdot \sigma} \xi^s \\ m\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \\ &= mu^s(p). \end{aligned} \tag{3.16}$$

Here, we have used multiple times the relation  $(\mathbf{p} \cdot \boldsymbol{\sigma})^2 = p^i p^j \sigma^i \sigma^j = \frac{1}{2} p^i p^j \{\sigma^i, \sigma^j\} = \mathbf{p}^2$ . A similar derivation shows that also  $(\not{p} + m)v^s(p) = 0$ .

The spinors  $u^s(p)$  and  $v^s(p)$  are normalized according to the Lorentz-invariant condition

$$\bar{u}^s(p) u^r(p) = -\bar{v}^s(p) v^r(p) = 2m\delta^{rs}, \tag{3.17}$$

where  $\bar{u}^s \equiv (u^s)^\dagger \gamma^0$ . Equation (3.17) can be easily checked by explicit calculation. It is also easy to check that normalizing  $u^\dagger u$  rather than  $\bar{u}u$  would not yield a Lorentz-invariant normalization condition. In fact,

$$u^{s\dagger}(p) u^r(p) = v^{s\dagger}(p) v^r(p) = 2E_{\mathbf{p}} \delta^{rs}. \quad (3.18)$$

Therefore, in QFT, we almost always work with  $\bar{\psi}$  instead of  $\psi^\dagger$ . Note that the  $u$  and  $v$  spinors are orthogonal:

$$\bar{u}^s(p) v^r(p) = -\bar{v}^s(p) u^r(p) = 0. \quad (3.19)$$

A similar relation does *not* hold for  $u^{s\dagger}(p) v^r(p)$  and  $v^{s\dagger}(p) u^r(p)$  — these products are in general nonzero. A relation that is, however, useful sometimes is

$$u^{s\dagger}(\mathbf{p}) v^r(-\mathbf{p}) = v^{s\dagger}(\mathbf{p}) u^r(-\mathbf{p}) = 0. \quad (3.20)$$

(We write the argument of the  $u$  and  $v$  spinors as 3-vectors here to emphasize the sign of the 3-momentum. It is of course implied that the 0-component of the momentum 4-vector is set by the relativistic energy–momentum relation.)

### 3.1.3 Spin Sums

We will often deal with systems in which the polarization of the incoming particles is random, and the polarization of the outgoing particles is not measured. In these cases, we need to sum over polarizations, which leads to *spin sums* of the form

$$\sum_{s=1,2} u^s(p) \bar{u}^s(p) \quad (3.21)$$

Plugging in eq. (3.15) for  $u^s(p)$  and using  $\sum_{s=1,2} \xi^s \xi^{s\dagger} = \mathbb{1}_{2 \times 2}$ , we obtain

$$\sum_{s=1,2} u^s(p) \bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} (\xi^{s\dagger} \sqrt{p \cdot \bar{\sigma}}, \xi^{s\dagger} \sqrt{p \cdot \sigma}) \quad (3.22)$$

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \end{pmatrix} \quad (3.23)$$

$$= \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} \quad (3.24)$$

$$= \not{p} + m. \quad (3.25)$$

In the third equality, we have used

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = (p^0 - p^i \sigma^i)(p^0 + p^i \sigma^i) \quad (3.26)$$

$$= (p^0)^2 - (p^i)^2 \quad (3.27)$$

$$= m^2. \quad (3.28)$$

In analogy to eq. (3.25), we can also show that

$$\sum_{s=1,2} v^s(p) \bar{v}^s(p) = \not{p} - m. \quad (3.29)$$



## 3.2 Quantization of the Dirac Field

### 3.2.1 How Not to Quantize the Dirac Field

The naïve way of quantizing the Dirac field would be to proceed in analogy to section 2.3 and postulate the commutation relations

$$\begin{aligned} [\psi_a(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] &= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ab}, \\ [\psi_a(\mathbf{x}, t), \psi_b(\mathbf{y}, t)] &= [\psi_a^\dagger(\mathbf{x}, t), \psi_b^\dagger(\mathbf{y}, t)] = 0 \end{aligned} \quad (3.30)$$

where  $a, b = 1 \cdots 4$  are spinor indices. Note that, again, we take the time coordinate of the two fields to be the same. Let us try to quantize the Dirac field based on these commutation relations and see what goes wrong.

We write the field  $\psi(x)$  as a superposition of all solutions of the free Dirac equation, with operator-valued coefficients:

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} e^{i\mathbf{p}\mathbf{x}} (a_{\mathbf{p}}^s(t) u^s(\mathbf{p}) + b_{-\mathbf{p}}^s(t) v^s(-\mathbf{p})). \quad (3.31)$$

Note that, unlike for the real Klein–Gordon field, the two terms in parentheses are not related to each other for the Dirac field, which is complex. If the creation and annihilation operators  $a_{\mathbf{p}}^s(t)$  and  $b_{\mathbf{p}}^s(t)$  satisfy the commutation relations

$$[a_{\mathbf{p}}^r(t), a_{\mathbf{q}}^{s\dagger}(t)] = [b_{\mathbf{p}}^r(t), b_{\mathbf{q}}^{s\dagger}(t)] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs}, \quad (3.32)$$

(and all other commutators being zero) we can show that  $\psi$  and  $\psi^\dagger$  satisfy the postulated relations eq. (3.30):

$$\begin{aligned} [\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}} \sqrt{2E_{\mathbf{q}}}} e^{i\mathbf{p}\mathbf{x} - i\mathbf{q}\mathbf{y}} \sum_{r,s} \\ &\quad \times \left( [a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}] u^r(\mathbf{p}) \bar{u}^s(\mathbf{q}) + [b_{-\mathbf{p}}^r, b_{-\mathbf{q}}^{s\dagger}] v^r(-\mathbf{p}) \bar{v}^s(-\mathbf{q}) \right) \gamma^0 \end{aligned} \quad (3.33)$$

$$= \int \frac{d^3 p}{(2\pi)^3 2E_{\mathbf{p}}} e^{i\mathbf{p}(\mathbf{x} - \mathbf{y})} (\gamma^0 E_{\mathbf{p}} - \gamma^i \mathbf{p} + m + \gamma^0 E_{\mathbf{p}} + \gamma^i \mathbf{p} - m) \gamma^0 \quad (3.34)$$

$$= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \mathbb{1}_{4 \times 4}. \quad (3.35)$$

Note that, for brevity of notation, we do not explicitly write out the time dependence of the creation and annihilation operators here and in the following.

Let us look at the quantized Hamilton operator, which is as usual obtained from a Legendre transform of the Lagrangian:

$$H = \int d^3 x \left[ \frac{\delta \mathcal{L}}{\delta(\partial_0 \psi)} \partial_0 \psi - \mathcal{L} \right] \quad (3.36)$$

$$= \int d^3x [i\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}(i\boldsymbol{\nabla} - m)\psi] \quad (3.37)$$

$$= \int d^3x \bar{\psi} [-i(\boldsymbol{\gamma} \cdot \boldsymbol{\nabla}) + m] \psi \quad (3.38)$$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} \sum_{s,r} \left[ \exp^{-i\mathbf{p}\cdot\mathbf{x}} (a_{\mathbf{p}}^{s\dagger} \bar{u}^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} \bar{v}^s(-\mathbf{p})) \right. \\ \left. \cdot [-i(\boldsymbol{\gamma} \cdot \boldsymbol{\nabla}) + m] \exp^{i\mathbf{q}\cdot\mathbf{x}} (a_{\mathbf{q}}^r u^r(\mathbf{q}) + b_{-\mathbf{q}}^r v^r(-\mathbf{q})) \right] \quad (3.39)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \sum_{s,r} \left[ a_{\mathbf{p}}^{s\dagger} \bar{u}^s(\mathbf{p}) (\boldsymbol{\gamma} \cdot \mathbf{p} + m) a_{\mathbf{p}}^r u^r(\mathbf{p}) \right. \\ + a_{\mathbf{p}}^{s\dagger} \bar{u}^s(\mathbf{p}) (\boldsymbol{\gamma} \cdot \mathbf{p} + m) b_{-\mathbf{p}}^r v^r(-\mathbf{p}) \\ + b_{-\mathbf{p}}^{s\dagger} \bar{v}^s(-\mathbf{p}) (\boldsymbol{\gamma} \cdot \mathbf{p} + m) a_{\mathbf{p}}^r u^r(\mathbf{p}) \\ \left. + b_{-\mathbf{p}}^{s\dagger} \bar{v}^s(-\mathbf{p}) (\boldsymbol{\gamma} \cdot \mathbf{p} + m) b_{-\mathbf{p}}^r v^r(\mathbf{p}) \right] \quad (3.40)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s [E_{\mathbf{p}} a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s]. \quad (3.41)$$

To understand the sign of the derivative term in the third line, note that  $\not{\partial} = \gamma^0\partial_0 + \gamma^i\partial_i$ . No raising or lowering of indices is necessary, hence there is *no* extra minus sign from the metric. In the fourth equality, we have plugged in eq. (3.31) and its complex conjugate. In the sixth equality, we have used  $\boldsymbol{\gamma} \cdot \mathbf{p} = -\not{p} + E_{\mathbf{p}}\gamma^0$  and we have then invoked the momentum space Dirac equation, eq. (3.14), as well as the orthogonality relations eqs. (3.18) and (3.20).

We encounter a serious problem here: if we interpret the operators  $a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s$  and  $b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s$  as particle number operators again, we are led to the conclusion that, the more particles we create using the operator  $b_{\mathbf{p}}^{s\dagger}$ , the lower the energy becomes. This seems like a terribly unstable system.

A discussion of additional problems with the above procedure is given in [1].

### 3.2.2 Quantizing the Dirac Field with Anticommutators

The goal of avoiding negative energy states can lead us to the proper way of quantizing the Dirac field.

A first attempt might be to replace  $b_{\mathbf{p}}^s$  by  $b_{\mathbf{p}}^{s\dagger}$  in eq. (3.31). This seems reasonable since also for the Klein–Gordon field, the negative frequency term in the field operator came with a creation operator rather than an annihilation operator. In this case, however, the second term in eq. (3.34) would get a minus sign, and we would not reproduce the desired commutation relations eq. (3.30). Also, the expression for  $H$ , eq. (3.41), would change:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s [E_{\mathbf{p}} a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^{s\dagger}] \quad (3.42)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s [E_{\mathbf{p}} a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s - E_{\mathbf{p}} (2\pi)^3 \delta^{(0)}(0)] . \quad (3.43)$$

Thus,  $H$  changes by an (infinite) constant from the commutator of  $b_{\mathbf{p}}^{s\dagger}$  and  $b_{\mathbf{p}}^s$ , but the minus sign in the term proportional to  $b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s$  would remain. The infinite constant is not a problem because, again, experiments measure only energy differences, so  $H$  can be renormalized to a finite expression by simply dropping the infinite constant.

Since the commutators eq. (3.30) didn't lead us anywhere, let us give up on them. Instead, note that the negative energy problem in the Hamiltonian is solved if we replace  $b_{\mathbf{p}}^s$  by  $b_{\mathbf{p}}^{s\dagger}$  in eq. (3.31) and postulate in addition that the field operators *anticommute*.

This means we write

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} (a_{\mathbf{p}}^s u^s(\mathbf{p}) e^{-ipx} + b_{\mathbf{p}}^{s\dagger} v^s(\mathbf{p}) e^{ipx}) , \quad (3.44)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum_{s=1,2} (b_{\mathbf{p}}^s \bar{v}^s(\mathbf{p}) e^{-ipx} + a_{\mathbf{p}}^{s\dagger} \bar{u}^s(\mathbf{p}) e^{ipx}) \quad (3.45)$$

and postulate

$$\begin{aligned} \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs} , \\ \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^s\} &= \{b_{\mathbf{p}}^r, b_{\mathbf{q}}^s\} = \{a_{\mathbf{p}}^r, b_{\mathbf{q}}^s\} = 0 , \\ \{a_{\mathbf{p}}^{r\dagger}, a_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^{r\dagger}, b_{\mathbf{q}}^{s\dagger}\} = \{a_{\mathbf{p}}^{r\dagger}, b_{\mathbf{q}}^{s\dagger}\} = 0 , \\ \{a_{\mathbf{p}}^r, b_{\mathbf{q}}^{s\dagger}\} &= \{b_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} = 0 . \end{aligned} \quad (3.46)$$

Then, the Hamilton operator is

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s [E_{\mathbf{p}} a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - E_{\mathbf{p}} b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}] \quad (3.47)$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s [E_{\mathbf{p}} a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + E_{\mathbf{p}} b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s] . \quad (3.48)$$

(We have already dropped the infinite constant.) The field operators  $\psi(\mathbf{x}, t)$  and  $\psi^\dagger(\mathbf{y}, t)$  now also satisfy *anticommutation* relations:

$$\begin{aligned} \{\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)\} &= \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}} \sqrt{2E_{\mathbf{q}}}} e^{i\mathbf{p}\mathbf{x} - i\mathbf{q}\mathbf{y}} \sum_{r,s} \\ &\quad \times \left( \{a_{\mathbf{p}}^r, a_{\mathbf{q}}^{s\dagger}\} u^r(\mathbf{p}) \bar{u}^s(\mathbf{q}) + \{b_{-\mathbf{p}}^{r\dagger}, b_{-\mathbf{q}}^s\} v^r(-\mathbf{p}) \bar{v}^s(-\mathbf{q}) \right) \gamma^0 \quad (3.49) \\ &= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} e^{i\mathbf{p}(\mathbf{x} - \mathbf{y})} (\gamma^0 E_{\mathbf{p}} - \gamma^i \mathbf{p} + m + \gamma^0 E_{\mathbf{p}} + \gamma^i \mathbf{p} - m) \gamma^0 \quad (3.50) \end{aligned}$$

$$= \delta^{(3)}(\mathbf{x} - \mathbf{y}) \mathbb{1}_{4 \times 4}. \quad (3.51)$$

As usual, the vacuum is defined to be the state that satisfies

$$a_{\mathbf{p}}^s |0\rangle = b_{\mathbf{p}}^s |0\rangle = 0. \quad (3.52)$$

The interpretation of the operators  $a_{\mathbf{p}}^s$  and  $b_{\mathbf{p}}^s$  is that both of them create (distinct) particles with momentum  $\mathbf{p}$  and positive energy  $E_{\mathbf{p}}$ .

### 3.2.3 Physical Significance of the Quantized Dirac Field

#### Particles and Antiparticles

We see that fermions always come in pairs. We call the one created by  $a_{\mathbf{p}}^{s\dagger}$  *particle*, the one created by  $b_{\mathbf{p}}^{s\dagger}$  *antiparticle*. (This is arbitrary—we could also interchange these definitions.) Note that the Hamiltonian eq. (3.48) tells us that particles and antiparticles have the same energy spectrum. In particular, they have identical mass.

#### The Pauli Exclusion Principle

The fact that the creation and annihilation operators anticommute allows us to derive another important result, namely the Pauli exclusion principle, which states that no two fermions can be in the same quantum state. In fact, consider a state

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{s\dagger} |0\rangle. \quad (3.53)$$

Assume we want to create another particle in the same state. Then, the operator that does this must contain  $a_{\mathbf{p}}^\dagger$ , and we would obtain an amplitude proportional to  $(a_{\mathbf{p}}^\dagger)^2 |0\rangle$ . The anticommutation relation  $\{a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger\} = 0$ , however, tells us that  $(a_{\mathbf{p}}^\dagger)^2$  vanished. Therefore, processes that create a particle in a state that is already occupied have zero amplitude.

## 3.3 The Feynman Propagator for the Dirac Field

As for the Klein–Gordon field, we can derive Green’s functions for the Dirac operator, i.e. functions that satisfy

$$\left[ i(\gamma^\mu)_{ab} \frac{\partial}{\partial x^\mu} - m\delta_{ab} \right] S_{bc}(x - y) = i\delta^{(4)}(x - y)\delta_{ac}, \quad (3.54)$$

where  $a$ ,  $b$  and  $c$  are spinor indices, which we have written out explicitly here. We go to Fourier space and write

$$S_{ac}(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{S}_{ac}(p). \quad (3.55)$$

Plugging this into eq. (3.54), we find

$$[(\gamma^\mu)_{ab}p_\mu - m\delta_{ab}]\tilde{S}_{bc}(p) = i\delta_{ac}. \quad (3.56)$$

Thus we can formally write

$$\tilde{S}(p) = \frac{i}{\not{p} - m}, \quad (3.57)$$

where the  $4 \times 4$  matrix (in spinor space) in the denominator means the inverse of that matrix. If one wishes to avoid matrix inverses, one can equivalently write

$$\tilde{S}(p) = \frac{i(\not{p} + m)}{p^2 - m^2}, \quad (3.58)$$

using that  $\not{p}\not{p} = p^2$ . With this  $\tilde{S}(p)$ , eq. (3.55) is infinite, so we again need to introduce a regularization scheme. As for the Klein–Gordon field, and by the same arguments, we use the Feynman prescription for shifting the poles away from the real axis. This defines the *Feynman propagator for the Dirac field*:

$$S_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}. \quad (3.59)$$

Like the Klein–Gordon propagator, also the Feynman propagator can be related to correlation functions. Indeed note that, by integrating over  $p^0$ , we can write

$$S_F(x - y) = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \left[ e^{-ip(x-y)} (\not{p} + m) \theta(x^0 - y^0) + e^{ip(x-y)} (-\not{p} + m) \theta(y^0 - x^0) \right]. \quad (3.60)$$

On the other hand, we also have

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \sum_s u^s(p) \bar{u}^s(p) e^{-ip(x-y)} \quad (3.61)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (\not{p} + m) e^{-ip(x-y)}. \quad (3.62)$$

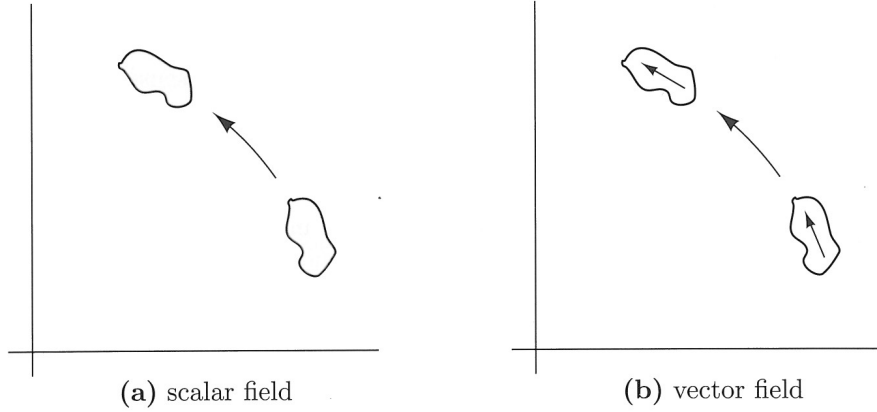
and

$$\langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} \sum_s v^s(p) \bar{v}^s(p) e^{ip(x-y)} \quad (3.63)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} (\not{p} - m) e^{ip(x-y)}. \quad (3.64)$$

Thus, we find again, similar to what we found for the Klein–Gordon field,

$$S_F(x - y) = \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle \theta(x^0 - y^0) - \langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle \theta(y^0 - x^0) \quad (3.65)$$



**Figure 3.1:** An active transformation (in this case a rotation) of a field configuration. Figure taken from [1].

$$= \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle. \quad (3.66)$$

For the Klein–Gordon field, there was a plus sign between the two terms. The fact that we now find a minus sign is a reflection of the anticommutating nature of fermion fields. In the second line of eq. (3.66), we have extended the definition of the time ordering symbol  $T$  to fermion fields. It is implied that any interchange of adjacent fermion field operators that is necessary to bring the field order into time order contributes a minus sign.

The physical interpretation of the Feynman propagator for Dirac fields is the following: it describes the propagation of a particle from  $y$  to  $x$  (if  $x^0 > y^0$ ), or the propagation of an antiparticle from  $x$  to  $y$  (if  $y^0 > x^0$ ).

## 3.4 Symmetries of the Dirac Theory

Symmetries are one of the most fundamental concepts in particle physics: the Noether theorem tells us that symmetries are related to conserved quantities, and conserved quantities are extremely useful in solving the equations of motion for particles and fields. More importantly still, symmetries dictate which terms are allowed in a Lagrangian and which are not. This will be crucial when we discuss interacting field theories, where the symmetry structure determines which interactions can exist.

### 3.4.1 Lorentz Invariance

#### Transformation Law for Spinor Fields

At the foundation of any relativistic theory should of course be Lorentz invariance, i.e. the principle that the equations of motion must be the same in any inertial frame. Let

us consider a Lorentz transformation<sup>1</sup>

$$x^\mu \rightarrow x^{\mu'} = \Lambda^\mu{}_\nu x^\nu. \quad (3.67)$$

We write the transformation of the spinor field  $\psi(x)$  under this transformation as

$$\psi(x) \rightarrow \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x), \quad (3.68)$$

where  $S(\Lambda)$  is a linear transformation matrix, yet to be determined. (The linearity of the transformation is for now an ansatz, but we will see shortly that it works.)

Lorentz invariance of the Dirac equation means that

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} \psi'(x) - m\psi'(x) = 0. \quad (3.69)$$

Plugging in the transformation property of  $\psi(x)$ , eq. (3.68), this becomes

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} S(\Lambda)\psi(\Lambda^{-1}x) - m S(\Lambda)\psi(\Lambda^{-1}x) = 0. \quad (3.70)$$

We proceed by making the substitution  $x \rightarrow \Lambda x$ , and obtain

$$i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu} S(\Lambda)\psi(x) - m S(\Lambda)\psi(x) = 0. \quad (3.71)$$

This can only be fulfilled if the  $\gamma$ -matrices transform according to

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \quad (3.72)$$

This consistency condition basically says that the index  $\mu$  on  $\gamma^\mu$  can indeed be treated like any other Lorentz index.

Let us now construct  $S(\Lambda)$  explicitly. To do so, consider an infinitesimal Lorentz transformation and write  $\Lambda^\mu{}_\nu$  as

$$\begin{aligned} \Lambda^\mu{}_\nu &= g^\mu{}_\nu + \omega^\mu{}_\nu, \\ (\Lambda^{-1})^\mu{}_\nu &= g^\mu{}_\nu - \omega^\mu{}_\nu, \end{aligned} \quad (3.73)$$

where  $\omega^\mu{}_\nu$  is infinitesimal. We know from the explicit form of  $\Lambda^\mu{}_\nu$ , familiar from special relativity, that  $\omega^\mu{}_\nu$  is antisymmetric.

We can write  $S(\Lambda)$  and its inverse as

$$\begin{aligned} S(\Lambda) &= \mathbb{1} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}, \\ S^{-1}(\Lambda) &= \mathbb{1} + \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}, \end{aligned} \quad (3.74)$$

---

<sup>1</sup>We work here with *active* transformations, i.e. we assume the field configuration to be rotated and/or boosted, see fig. 3.1. For a *passive* transformation, where the coordinate system is redefined instead, we would have to replace  $\Lambda$  by  $\Lambda^{-1}$  everywhere.

with a yet-to-be-determined tensor  $\sigma_{\mu\nu}$ . Since  $\omega^\mu{}_\nu$  is antisymmetric in  $\mu\nu$ ,  $\sigma_{\mu\nu}$  must be antisymmetric in  $\mu$  and  $\nu$  as well. Plugging the infinitesimal forms of  $\Lambda^\mu{}_\nu$  and  $S(\Lambda)$ , eqs. (3.73) and (3.74), into the transformation law for  $\gamma^\mu$ , eq. (3.72), we find

$$\frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta}\gamma^\mu - \gamma^\mu\frac{i}{4}\sigma_{\alpha\beta}\omega^{\alpha\beta} = \omega^\mu{}_\nu\gamma^\nu, \quad (3.75)$$

or, equivalently,

$$\omega^{\alpha\beta}[\sigma_{\alpha\beta}, \gamma^\mu] = -4i\omega^{\alpha\beta}g_\alpha{}^\mu\gamma_\beta = -2i\omega^{\alpha\beta}(g_\alpha{}^\mu\gamma_\beta - g_\beta{}^\mu\gamma_\alpha). \quad (3.76)$$

In the last step, we have used the antisymmetry of  $\omega^{\alpha\beta}$ . Since eq. (3.76) must be satisfied for any  $\omega^{\alpha\beta}$ , it reduces to

$$[\sigma_{\alpha\beta}, \gamma^\mu] = -2i(g_\alpha{}^\mu\gamma_\beta - g_\beta{}^\mu\gamma_\alpha). \quad (3.77)$$

We can show by explicit calculation that this is satisfied for

$$\sigma^{\alpha\beta} \equiv \frac{i}{2}[\gamma^\alpha, \gamma^\beta]. \quad (3.78)$$

Together with eq. (3.74), this defines the action of an infinitesimal Lorentz transformation on a spinor field. A finite transformation then has the form

$$S(\Lambda) = e^{-(i/4)\sigma_{\mu\nu}\omega^{\mu\nu}}, \quad (3.79)$$

where  $\omega^{\mu\nu}$  is now finite.

### The Generators of Lorentz Transformations

We recall from quantum mechanics that spatial rotations are *generated* by the angular momentum operator  $\mathbf{L} = \mathbf{x} \times (-i\nabla)$ . This means that, for an infinitesimal rotation described by a rotation matrix  $R_{ij} = \delta^{ij} + r^{ij}$ , a wave function  $\chi(x)$  transforms according to

$$\chi(\mathbf{x}, t) \rightarrow \chi'(\mathbf{x}, t) = \chi(R^{-1}\mathbf{x}, t) \quad (3.80)$$

$$= \chi(\mathbf{x}, t) - r^{ij}x^j\nabla^i\chi(\mathbf{x}, t). \quad (3.81)$$

If  $R$  describes a rotation around the  $z$  axis by an infinitesimal angle  $\alpha$ ,

$$r = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \\ & & 0 \end{pmatrix}, \quad (3.82)$$

this means for instance

$$\chi(\mathbf{x}, t) \rightarrow \chi'(\mathbf{x}, t) = \chi(\mathbf{x}, t) + \alpha(x^1\nabla^2 - x^2\nabla^1)\chi(\mathbf{x}, t) \quad (3.83)$$



$$= (1 + i\alpha L^3)\chi(\mathbf{x}, t). \quad (3.84)$$

A rotation about the  $z$  axis by a finite angle  $\alpha$  is then given by

$$\chi(\mathbf{x}, t) \rightarrow \chi'(\mathbf{x}, t) = e^{i\alpha L^3}\chi(\mathbf{x}, t). \quad (3.85)$$

In an analogous way, also the action of a Lorentz transformation on a spinor field  $\psi(x)$ , given by eq. (3.68), can be expressed in terms of a generating operator  $J_{\mu\nu}$ . Consider again an infinitesimal Lorentz transformation as in eq. (3.73) and write

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x) \quad (3.86)$$

$$= \left( \mathbb{1} - \frac{i}{2} J_{\mu\nu} \omega^{\mu\nu} \right) \psi(x). \quad (3.87)$$

(The factor  $i/2$  is mere convention.) Plugging in our expressions for  $S(\Lambda)$  and  $\Lambda$  in the infinitesimal case, eqs. (3.73) and (3.74), we obtain

$$\left( \mathbb{1} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \right) \psi(x^\mu - \omega^{\mu\nu} x_\nu) = \left( \mathbb{1} - \frac{i}{2} J_{\mu\nu} \omega^{\mu\nu} \right) \psi(x). \quad (3.88)$$

This leads to

$$-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} - \omega^{\mu\nu} x_\nu \partial_\mu = -\frac{i}{2} J_{\mu\nu} \omega^{\mu\nu}, \quad (3.89)$$

or

$$J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} + i(x_\mu \partial_\nu - x_\nu \partial_\mu). \quad (3.90)$$

In the last step, we have used the antisymmetry of  $\omega^{\mu\nu}$ . Note that the  $J_{\mu\nu}$  are block diagonal in the chiral basis for the Dirac matrices. This follows from the definition of  $\sigma_{\mu\nu}$ , eq. (3.78), together with the explicit form of the  $\gamma^\mu$  in the chiral basis, eq. (3.2). It means that the spinor representation of the Lorentz group is *reducible*: if the upper or lower two components in a 4-spinor are zero, they will remain zero under any Lorentz transformation.

## Spin

We know that angular momentum is the conserved quantity associated with symmetry under spatial rotations. Therefore, we can now use the transformation properties of the field operator under such rotations to derive the form of the angular momentum operator for a fermion field. This will in particular help us better understand the internal angular momentum (spin) of fermions.

Under an infinitesimal Lorentz transformation (of which spatial rotations are of course a special case), a fermion field transforms as

$$\psi(x) \rightarrow \psi(x) + \delta\psi(x) \quad (3.91)$$

with

$$\delta\psi(x) \equiv S(\Lambda)\psi(\Lambda^{-1}x) - \psi(x) \quad (3.92)$$

(see eq. (3.68)). Using the explicit form eq. (3.74) for  $S(\Lambda)$  and eq. (3.73) for  $\Lambda$ , this becomes

$$\delta\psi(x) = -\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\psi(x) - \omega^{\mu\nu}x_\nu\frac{\partial}{\partial x^\mu}\psi(x). \quad (3.93)$$

We now specialize to a rotation about the  $z$  axis ( $\omega^{12} = -\omega_{21} = \theta$ , all other  $\omega^{\mu\nu} = 0$ ), and use that

$$\sigma^{12} = \frac{i}{2}[\gamma^1, \gamma^2] \quad (3.94)$$

$$= \frac{i}{2} \begin{pmatrix} -\sigma^1\sigma^2 + \sigma^2\sigma^1 & \\ & -\sigma^1\sigma^2 + \sigma^2\sigma^1 \end{pmatrix} \quad (3.95)$$

$$= \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} \quad (3.96)$$

$$\equiv \Sigma^3. \quad (3.97)$$

Here, we have used the relation  $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$ . We get

$$\delta\psi(x) = -\frac{i}{2}\Sigma^3\theta\psi(x) - \left(x_2\frac{\partial}{\partial x^1} - x_1\frac{\partial}{\partial x^2}\right)\theta\psi(x) \equiv \theta\Delta\psi. \quad (3.98)$$

The associated conserved quantity (integral of the time-component of the associated Noether current) is

$$\int d^3x j^0 = \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_0\psi)}\Delta\psi = -i \int d^3x \bar{\psi}\gamma^0 \left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + \frac{i}{2}\Sigma^3\right)\psi(x). \quad (3.99)$$

The right hand side is thus the third component of the angular momentum operator for a fermion field. In exact analogy, also the other two components of the angular momentum operator can be found, leading to

$$\mathbf{J} = \int d^3x \psi^\dagger [\mathbf{x} \times (-i\nabla) + \frac{1}{2}\Sigma] \psi. \quad (3.100)$$

The first term in square brackets is just the angular momentum operator in quantum mechanics (dressed here with two field operators). For non-relativistic fermions, it can be interpreted as giving the orbital angular momentum. The second term in square brackets gives the internal angular momentum (spin). For relativistic fermions, this division into spin and orbital angular momentum is not so trivial due to spin-orbit coupling.

Nevertheless, eq. (3.100) allows us to check that a Dirac fermion had indeed spin 1/2. For this, it is sufficient to consider particles at rest (or very nearly so), so that the orbital

angular momentum term is negligible. In this case, we can plug the Fourier expansion of  $\psi(x)$ , eq. (3.44) into eq. (3.100) (omitting the orbital angular momentum term):

$$J^j = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}} \sqrt{2E_{\mathbf{p}'}}} e^{-i\mathbf{p}'\mathbf{x}} e^{i\mathbf{p}\mathbf{x}} \quad (3.101)$$

$$\cdot \sum_{r,r'} \left( a_{\mathbf{p}'}^{r'\dagger} u^{r'\dagger}(\mathbf{p}') + b_{-\mathbf{p}'}^{r'} v^{r'\dagger}(-\mathbf{p}') \right) \frac{\Sigma^j}{2} \left( a_{\mathbf{p}}^r u^r(\mathbf{p}) + b_{-\mathbf{p}}^{r\dagger} v^r(-\mathbf{p}) \right). \quad (3.102)$$

To apply this operator (or specifically its third component) to a one-particle state of zero momentum,  $a_0^{s\dagger}|0\rangle$ , we use the fact that the vacuum has zero spin, i.e.  $J^z|0\rangle = 0$  and therefore

$$J^z a_0^{s\dagger}|0\rangle = [J^z, a_0^{s\dagger}]|0\rangle. \quad (3.103)$$

We can then use the commutation relation  $[a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^r, a_0^{s\dagger}] = (2\pi)^3 \delta^{(3)}(\mathbf{p}) a_0^{r\dagger} \delta^{rs}$ , which is easily derived using the canonical anticommutators, as well as the explicit form eq. (3.15) of the  $u$  spinor, to find

$$J^z a_0^{s\dagger}|0\rangle = \frac{1}{2m} u^{s\dagger}(0) \frac{\Sigma^3}{2} u^s(0) a_0^{s\dagger}|0\rangle \quad (3.104)$$

$$= \xi^{s\dagger} \frac{\sigma^3}{2} \xi^s a_0^{s\dagger}|0\rangle. \quad (3.105)$$

(All other commutators of creation and annihilation operators appearing in eq. (3.103) vanish.) This shows that, for  $\xi^s \propto (1, 0)$ , the particle has spin  $+1/2$  along the  $z$  axis, for  $\xi^s \propto (0, 1)$ , it has spin  $-1/2$  along the  $z$  axis.

We can do a similar derivation for an antiparticle state  $b_0^{s\dagger}|0\rangle$ . In this case, the only relevant commutator is  $[b_{\mathbf{p}}^r b_{\mathbf{p}}^{r\dagger}, b_0^{s\dagger}] = -(2\pi)^3 \delta^{(3)}(\mathbf{p}) b_0^{r\dagger} \delta^{rs}$ . Note the extra minus sign! We thus find

$$J^z b_0^{s\dagger}|0\rangle = -\xi^{s\dagger} \frac{\sigma^3}{2} \xi^s b_0^{s\dagger}|0\rangle. \quad (3.106)$$

Therefore, for antiparticles, the association between  $\xi^s$  spinors and the orientation of the physical angular momentum is reversed: for  $\xi^s \propto (1, 0)$ , the particle has spin  $-1/2$  along the  $z$  axis, for  $\xi^s \propto (0, 1)$ , it has spin  $+1/2$  along the  $z$  axis.

### Helicity, Chirality and Weyl Fermions

Let us consider an ultrarelativistic particle or antiparticle moving in the positive  $z$  direction with momentum  $p_z$  and spin up along the  $z$  axis, i.e.  $\xi^s = (1, 0)$ . (Note that a particle moving along a line through the origin, such as a coordinate axis, has no orbital

angular momentum.) The  $u$  and  $v$  spinors for such a particle or antiparticle are

$$u(p) = \begin{cases} \begin{pmatrix} \sqrt{E - p_z \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p_z \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \stackrel{E \gg m}{\simeq} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{for spin up along the } z \text{ axis} \\ \begin{pmatrix} \sqrt{E - p_z \sigma^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E + p_z \sigma^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \stackrel{E \gg m}{\simeq} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{for spin down along the } z \text{ axis} \end{cases} \quad (3.107)$$

$$v(p) = \begin{cases} \begin{pmatrix} \sqrt{E - p_z \sigma^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ -\sqrt{E + p_z \sigma^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \stackrel{E \gg m}{\simeq} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & \text{for spin up along the } z \text{ axis} \\ \begin{pmatrix} \sqrt{E - p_z \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ -\sqrt{E + p_z \sigma^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \stackrel{E \gg m}{\simeq} \sqrt{2E} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} & \text{for spin down along the } z \text{ axis} \end{cases} \quad (3.108)$$

We say that a particle whose spin is pointing in the direction of motion has a *right-handed* (RH) helicity and a particle with its spin is pointing in the opposite direction has a *left-handed* (LH) helicity.

We observe that, in the ultrarelativistic limit, for spinors for LH particles and RH antiparticles, the lower components are zero, while for RH particles and LH antiparticles the upper components vanish. This shows again that the upper and lower components of a spinor are quite independent: they correspond to different spin orientations. We have argued previously, below eq. (3.90), on more formal grounds that the upper and lower components of a Dirac spinor are independent because the 4-dimensional spinor representation of the Lorentz group is reducible. It factorizes into two irreducible representations, one describing the transformation properties of the upper two components of the Dirac spinor, one describing the transformation properties of the lower two components.

It also makes sense that LH particles and RH antiparticles are in the same irreducible representation of the Lorentz group. In the Dirac hole theory, an antiparticle corresponds to the absence of a particle from the Dirac sea. If a left-handed particle is absent, the hole it leaves back effectively has opposite spin and corresponds to a right-handed antiparticle.

To highlight the factorization of the Lorentz representation, one sometimes splits up a Dirac spinor field into

$$\psi(x) = \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} \quad (3.109)$$

and works with the *left-chiral Weyl fermion*  $\psi_L(x)$  and the *right-chiral Weyl fermion*  $\psi_R(x)$  separately. Note that, according to eqs. (3.107) and (3.108), in the ultrarelativistic limit the Weyl fermions are helicity eigenstates, justifying the attribute “left” and “right”.

(Even though a left-chiral field corresponds to a LH particle, but a RH antiparticle.) In terms of the Weyl fields the Dirac Lagrangian eq. (3.11) then becomes

$$\mathcal{L} = \psi_L^\dagger (i\bar{\sigma} \cdot \partial) \psi_L + \psi_R^\dagger (i\sigma \cdot \partial) \psi_R - m\bar{\psi}_L \psi_R - m\bar{\psi}_R \psi_L. \quad (3.110)$$

We see that it is only the mass term that mixes the left- and right-chiral fields, thereby destroying the one-to-one correspondence between chirality and helicity. This is a reflection of the fact that for massive fermions, helicity is a frame-dependent quantity. One can always boost into a frame where the momentum of the particle is reversed.

When working with Weyl spinors, it is convenient to define the projection operators

$$P_L \equiv \frac{1 - \gamma^5}{2} \quad P_R \equiv \frac{1 + \gamma^5}{2}, \quad (3.111)$$

which project out the left-chiral and right-chiral components, respectively, of a Dirac spinor. They satisfy  $P_{L/R}^2 = P_{L/R}$  and  $P_L P_R = 0$ .

Weyl fermions play a crucial role in the theory of weak interactions, where the couplings of left- and right-chiral fields are completely different.

### 3.4.2 Parity ( $P$ )

In addition to the continuous Lorentz transformations discussed above, there are several discrete symmetries under which many quantum field theories are invariant. The first of these is parity or space inversion which transforms a Lorentz 4-vector  $(t, \mathbf{x})$  into  $(t, -\mathbf{x})$ . A parity transformation thus reverses the 3-momentum of a particle, but leaves its spin orientation unchanged. To see this, remember that orbital angular momentum  $\mathbf{L}$  is a cross product of  $\mathbf{x}$  and  $\mathbf{p}$ , and since both of these change sign under parity transformations,  $\mathbf{L}$  remains invariant. Spin and orbital angular momentum must transform the same way if spin is to be interpreted as a form of angular momentum, therefore also spin must remain invariant under parity. Consequently, the parity operator must transform a quantum state  $a_{\mathbf{p}}^s |0\rangle$  into  $a_{-\mathbf{p}}^s |0\rangle$ . This implies that the parity operation  $P$  in Hilbert space must act as

$$P a_{\mathbf{p}}^s P = \eta_a a_{-\mathbf{p}}^s \quad \text{and} \quad P b_{\mathbf{p}}^s P = \eta_b b_{-\mathbf{p}}^s, \quad (3.112)$$

where  $\eta_a$  and  $\eta_b$  are phase factors. We are in principle free to choose these phase factors arbitrarily—they are not restricted by the requirement that parity reverses space. They are, however, constrained if we demand that applying the parity operator twice should leave any physical observable unchanged. Since observables (such as the Hamiltonian) are constructed from even numbers of fermion field operators, this implies  $\eta_a, \eta_b = \pm 1$ .

When we discussed continuous Lorentz transformations, we found that their action of the field operator can be written as a multiplication by a matrix  $S(\Lambda)$  in spinor space, see eq. (3.68). We will now find a similar transformation law also for parity transformations. We write

$$P \psi(x) P = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum (\eta_a a_{-\mathbf{p}}^s u^s(p) e^{-ipx} + \eta_b^* b_{-\mathbf{p}}^{s\dagger} v^s(p) e^{ipx}). \quad (3.113)$$

We now substitute  $p = (p^0, \mathbf{p}) \rightarrow p' \equiv (p^0, -\mathbf{p})$  in the integral. In applying this substitution, we must express  $u^s(p)$  and  $v^s(p)$  in terms of  $p'$ . To do so, we use the explicit form of these spinors (see eq. (3.15)) and observe that  $p \cdot \sigma = p' \cdot \bar{\sigma}$  and  $p \cdot \bar{\sigma} = p' \cdot \sigma$ . This leads to

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{p' \cdot \bar{\sigma}} \xi^s \\ \sqrt{p' \cdot \sigma} \xi^s \end{pmatrix} = \gamma^0 u^s(p'), \quad (3.114)$$

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{p' \cdot \bar{\sigma}} \xi^s \\ -\sqrt{p' \cdot \sigma} \xi^s \end{pmatrix} = -\gamma^0 v^s(p'). \quad (3.115)$$

With these relations, we arrive at

$$P\psi(x)P = \int \frac{d^3p'}{(2\pi)^3 \sqrt{2E_{\mathbf{p}'}}} \sum (\eta_a a_{\mathbf{p}'}^s \gamma^0 u^s(p') e^{-ip'x'} - \eta_b^* b_{\mathbf{p}'}^{s\dagger} \gamma^0 v^s(p') e^{ip'x'}). \quad (3.116)$$

where we have defined  $x' \equiv (x^0, -\mathbf{x})$ . We now choose  $\eta_b^* = -\eta_a$ . Then, we have the transformation law

$$P\psi(x)P = \eta_a \gamma^0 \psi(x'). \quad (3.117)$$

In computing physical observables and in dealing with interacting field theories, we often encounter bilinearies of fermion fields, such as

$$\bar{\psi}\psi, \quad i\bar{\psi}\gamma^5\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \bar{\psi}\sigma^{\mu\nu}\psi. \quad (3.118)$$

Let us discuss the transformation properties of these expressions under parity. Since  $\bar{\psi}(x)$  appears in all of them, we need the result that

$$P\bar{\psi}(x)P = P\psi^\dagger(x)P\gamma^0 = \eta_a^* \psi^\dagger(x)(\gamma^0)^2 = \eta_a^* \bar{\psi}(x)\gamma^0. \quad (3.119)$$

We then find for instance

$$P\bar{\psi}(x)\psi(x)P = |\eta_a|^2 \bar{\psi}(x')\psi(x') = \bar{\psi}(x')\psi(x'). \quad (3.120)$$

Similarly, we can show that

$$P\bar{\psi}(x)\gamma^\mu\psi(x)P = \bar{\psi}(x')\gamma^0\gamma^\mu\gamma^0\psi(x') = \begin{cases} +\bar{\psi}(x')\gamma^\mu\psi(x') & \text{for } \mu = 0 \\ -\bar{\psi}(x')\gamma^\mu\psi(x') & \text{for } \mu = 1, 2, 3 \end{cases}. \quad (3.121)$$

In other words,  $\bar{\psi}(x)\gamma^\mu\psi(x)$  transforms like a Lorentz vector. The transformation properties of the other bilinearies can be derived in a similar way. They are summarized in table 3.1.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$
$P$	+1	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$
$T$	+1	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$
$C$	+1	+1	-1	+1	-1

**Table 3.1:** Transformation properties of Dirac field bilinearies under parity ( $P$ ), time reversal ( $T$ ) and charge conjugation ( $C$ ). The shorthand notation  $(-1)^\mu$  means +1 for  $\mu = 0$  and -1 for  $\mu = 1, 2, 3$ .

### 3.4.3 Time Reversal ( $T$ )

Next, we consider the time reversal operation that sends  $(t, \mathbf{x})$  to  $(-t, \mathbf{x})$ . A momentum 3-vector (which is a derivative of a coordinate 3-vector with respect to time) transforms under  $T$  as  $\mathbf{p} \rightarrow -\mathbf{p}$ . Since coordinate 3-vectors are invariant under  $T$  while momentum 3-vectors change sign, also orbital angular momentum  $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ , and thus also spin, changes sign.

This means that  $T$  acts on  $a_{\mathbf{p}}^s$  and  $b_{\mathbf{p}}^s$  according to

$$Ta_{\mathbf{p}}^sT = a_{-\mathbf{p}}^{-s} \quad \text{and} \quad Tb_{\mathbf{p}}^sT = b_{-\mathbf{p}}^{-s}, \quad (3.122)$$

In principle, we should again allow for arbitrary phase factors, but since they would not affect the following derivations, we omit them here. This is, however, not the full story yet: given some quantum mechanical transition amplitude  $\langle \psi_1 | \psi_2 \rangle$ , time reversal should interchange the initial and final states, i.e.  $\langle T\psi_1 | T\psi_2 \rangle = \langle \psi_2 | \psi_1 \rangle$ . (Such an operator is called *antiunitary*.)

This implies that  $T$  not only acts on Hilbert space states, but also on complex numbers by sending them to their complex conjugate. We thus have

$$T\psi(x)T = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum (a_{-\mathbf{p}}^{-s} u^{s*}(p) e^{+ipx} + b_{-\mathbf{p}}^{-s\dagger} v^{s*}(p) e^{-ipx}). \quad (3.123)$$

To write the right hand side as a linear transformation of  $\psi(x)$ , we need a way of writing  $u^s(p)$  in terms of  $u^{-s}(p')$ , where  $p' = (p^0, -\mathbf{p})$ .

To do so, we first note again that  $p' \cdot \sigma = p \cdot \bar{\sigma}$  and  $p' \cdot \bar{\sigma} = p \cdot \sigma$ . Moreover, note that a spin flip (sending  $s$  to  $-s$ ) is achieved by the transformation

$$\xi^s \rightarrow \xi^{-s} = -i\sigma^2(\xi^s)^*. \quad (3.124)$$

(We are not worrying about the inconsequential phase factors here.) To see this, assume  $\xi^s$  describes a spin along a unit vector  $\mathbf{n}$ , i.e.  $\xi^s$  is an eigenvector of the helicity operator  $\mathbf{n} \cdot \boldsymbol{\sigma}$ :

$$\mathbf{n} \cdot \boldsymbol{\sigma} \xi^s = +\xi^s. \quad (3.125)$$

Then, using  $\sigma^i \sigma^2 = -\sigma^2 \sigma^{i*}$ ,

$$(\mathbf{n} \cdot \boldsymbol{\sigma})[-i\sigma^2(\xi^s)^*] = -i\sigma^2(-\mathbf{n} \cdot \boldsymbol{\sigma})^*(\xi^s)^* \quad (3.126)$$

$$= -(-i\sigma^2(\xi^s)^*). \quad (3.127)$$

Therefore, we have

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad (3.128)$$

$$= \begin{pmatrix} \sqrt{p' \cdot \bar{\sigma}} [-i\sigma^2(\xi^{-s})^*] \\ \sqrt{p' \cdot \sigma} [-i\sigma^2(\xi^{-s})^*] \end{pmatrix}. \quad (3.129)$$

We would like to move the  $\sigma^2$  matrices to the left of the square roots. To this end, go to a reference frame where  $\mathbf{p}$  is aligned along the  $z$ -axis. In this frame  $\sqrt{p' \cdot \sigma} = \sqrt{E_{\mathbf{p}'} + |\mathbf{p}'|(\frac{1-\sigma^3}{2})} + \sqrt{E_{\mathbf{p}'} - |\mathbf{p}'|(\frac{1+\sigma^3}{2})}$ . Therefore, using again  $\sigma^i \sigma^2 = -\sigma^2 \sigma^{i*}$ , we find

$$\sqrt{p' \cdot \sigma} \sigma^2 = \sigma^2 \sqrt{p' \cdot \bar{\sigma}}. \quad (3.130)$$

Similarly,

$$\sqrt{p' \cdot \bar{\sigma}} \sigma^2 = \sigma^2 \sqrt{p' \cdot \sigma}. \quad (3.131)$$

This leads us to the relation

$$u^s(p) = \begin{pmatrix} -i\sigma^2 \sqrt{p' \cdot \sigma^*} (\xi^{-s})^* \\ -i\sigma^2 \sqrt{p' \cdot \bar{\sigma}^*} (\xi^{-s})^* \end{pmatrix} \quad (3.132)$$

$$= -i \begin{pmatrix} \sigma^2 & \\ & \sigma^2 \end{pmatrix} [u^{-s}(p')]^* \quad (3.133)$$

$$= -\gamma^1 \gamma^3 [u^{-s}(p')]^*. \quad (3.134)$$

An analogous relation holds for  $v^s(p)$ :

$$v^s(p) = -\gamma^1 \gamma^3 [v^{-s}(p')]^*. \quad (3.135)$$

We are now ready to rewrite eq. (3.123):

$$T\psi(x)T = -\gamma^1 \gamma^3 \int \frac{d^3 p'}{(2\pi)^3 \sqrt{2E_{\mathbf{p}'}}} \sum (a_{\mathbf{p}'}^s u^s(p) e^{-ip'x'} + b_{\mathbf{p}'}^{s\dagger} v^s(p) e^{ip'x'}) \quad (3.136)$$

$$= -\gamma^1 \gamma^3 \psi(x'). \quad (3.137)$$

Here, we have defined  $x' = (-t, \mathbf{x})$ , so that  $px = -p'x'$ .

As in section 3.4.2, we can again check the behavior of Dirac field bilinearies under the  $T$  transformation. The results are summarized in table 3.1.



### 3.4.4 Charge Conjugation ( $C$ )

The final discrete symmetry we wish to consider here is charge conjugation  $C$ , under which particles become antiparticles and vice-versa. In other words

$$Ca_{\mathbf{p}}^s C = b_{\mathbf{p}}^s \quad \text{and} \quad Cb_{\mathbf{p}}^s C = a_{\mathbf{p}}^s. \quad (3.138)$$

To write the action of  $C$  on the field  $\psi(x)$  as a linear operation as we did for the  $P$  and  $T$  transforms, we will need a transformation that converts a  $u$  spinor into a  $v$  spinor. Here it is:

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad (3.139)$$

$$= \begin{pmatrix} \sqrt{p \cdot \sigma} [-i\sigma^2 \xi^{-s}]^* \\ -\sqrt{p \cdot \bar{\sigma}} [-i\sigma^2 \xi^{-s}]^* \end{pmatrix} \quad (3.140)$$

$$= \begin{pmatrix} -i\sigma^2 \sqrt{p \cdot \bar{\sigma}} (\xi^{-s})^* \\ i\sigma^2 \sqrt{p \cdot \sigma} (\xi^{-s})^* \end{pmatrix} \quad (3.141)$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^{-s} \\ \sqrt{p \cdot \bar{\sigma}} \xi^{-s} \end{pmatrix}^* \quad (3.142)$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} [u^s(p)]^* \quad (3.143)$$

$$= -i\gamma^2 [u^s(p)]^*. \quad (3.144)$$

In the second equality, we have used eq. (3.124). Note that in the last step we have  $u^s(p)$  and not  $u^{-s}(p)$  because we have seen in section 3.4.1 that the association between the  $\xi^s$  2-spinors and the physical spin orientation is opposite for  $u$  and  $v$  spinors:  $\xi^s = (1, 0)$  corresponds to spin up along the  $z$  axis for a  $u$  spinor, but to spin down along the  $z$  axis for a  $v$  spinor. Similarly, we also have

$$u^s(p) = -i\gamma^2 [v^s(p)]^*. \quad (3.145)$$

(Note that to obtain eq. (3.145) directly from eq. (3.144), one has to take into account that, in our conventions from eq. (3.124),  $\xi^{-(-s)} = -\xi^s$ .) Thus,

$$C\psi(x)C = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum (b_{\mathbf{p}}^s u^s(p) e^{-ipx} + a_{\mathbf{p}}^{s\dagger} v^s(p) e^{ipx}) \quad (3.146)$$

$$= -i\gamma^2 \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum (b_{\mathbf{p}}^s [v^s(p)]^* e^{-ipx} + a_{\mathbf{p}}^{s\dagger} [u^s(p)]^* e^{ipx}) \quad (3.147)$$

$$= -i\gamma^2 [\psi(x)]^* \quad (3.148)$$

$$= -i(\bar{\psi} \gamma^0 \gamma^2)^T. \quad (3.149)$$

The transformation properties of the various Dirac field bilinearies under this transformation are again summarized in table 3.1.



# 4

## Interacting Fields and Feynman Diagrams

In the previous chapters, we have considered rather boring systems: non-interacting scalar and fermion fields. Of course, our ultimate goal is to study the interactions among fields, and in particular to compute the transition amplitude from some initial state to some final state as well as the associated cross section. We will now develop the tools to do that. We will first develop a formalism for evaluating so-called correlation functions in sections 4.1 to 4.3. Then, we will learn how to relate correlation functions to scattering matrix elements in section 4.4. In section 4.5, we will finally employ our formalism to evaluate scattering amplitudes. Along the way, we will get to know *Feynman diagrams* as a pictorial way of representing the mathematical expressions for correlation functions and scattering amplitudes. Up to this point, we will work in a theory containing only a real scalar field  $\phi(x)$ , but no fermions. We will generalize the formalism to include fermions in section 4.6.

### 4.1 Time-Dependent Perturbation Theory for Correlation Functions

#### 4.1.1 $\phi^4$ Theory

In this section and the following ones, we will consider an extension of the free real scalar field theory called “ $\phi^4$  theory”. It is perhaps the simplest interacting quantum field theory and will serve here as a proxy for more realistic theories like quantum electrodynamics (QED) and quantum chromodynamics (QCD) that we will consider later in this lecture.  $\phi^4$  theory itself describes for instance the Higgs boson and its self-interactions (but not its interactions with other elementary particles).

$\phi^4$  theory is obtained by adding a quartic interaction term to the Klein–Gordon La-

grangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (4.1)$$

With the Fourier expansion of  $\phi(x)$  in terms of creation and annihilation operators in mind, we realize that the  $\phi^4$  term describes for instance the scattering of two  $\phi$  particles, which can be viewed as the annihilation of two incoming particles, followed by the creation of two outgoing particles with possibly different momentum vectors. The quantity  $\lambda$  in eq. (4.1) is a coupling constant that determines the strength of the interaction between  $\phi$  particles, and the factor  $4!$  in the denominator is included by convention. The motivation for writing the interaction term in this way will become clear later in this chapter.

### 4.1.2 The Vacuum State of the Interacting Theory

At this point, we come across a first important subtlety. The vacuum state  $|0\rangle$  of the free (non-interacting) theory is *not* the vacuum state of the interacting theory. This is clear because the extra term  $H_{\text{int}}$  in the Hamiltonian should have some impact on the energy eigenstates of theory, and in particular on the lowest one, which is by definition the vacuum. We therefore have to be careful with our notation. We will henceforth call the vacuum state of the free theory  $|0\rangle$  and the vacuum state of the interacting theory  $|\Omega\rangle$ .  $|0\rangle$  has the property that  $a_{\mathbf{p}}|0\rangle = 0$  for all  $\mathbf{p}$ , i.e. it contains no particles. On the other hand,  $a_{\mathbf{p}}|\Omega\rangle \neq 0$  in general. The physical interpretation of  $|\Omega\rangle$  is that it is a complicated state containing *vacuum fluctuations*—short-lived particle–antiparticle pairs that are created from the vacuum and annihilate again within the time window allowed by the Heisenberg relation  $\Delta E \Delta t > 1/2$ . While they exist, they can interact.

### 4.1.3 Correlation Functions

As announced above, we cannot study transition matrix elements between multi-particle initial and final states right away. This has to do with the difficulties associated with the definition of appropriate initial and final states in the interacting theory. For instance, the one particle states of the form  $\sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle$  will not be energy eigenstates of the interacting theory, just as the vacuum states  $|0\rangle$  and  $|\Omega\rangle$  were different. We will instead first study *correlation functions*, i.e. expressions of the form

$$\langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle. \quad (4.2)$$

We have already encountered such a correlation function in the free Klein–Gordon theory: the Feynman propagator was given by the two-point correlation function

$$D_F(x-y) = \langle 0 | T \phi(x) \phi(y) | 0 \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}, \quad (\text{free theory}) \quad (4.3)$$

see eqs. (2.69) and (2.77). In the interacting theory, the expression will be more complicated of course. In fact, it is impossible to write the propagator of the interacting theory in closed form, but if  $\lambda$  is not too large, the interaction can be treated as a small perturbation, and we can use time-dependent perturbation theory to approximate it quite well. This is what we are going to do now. After we have studied the two point correlation function, we can easily generalize the formalism to more complicated correlation functions and then to transition matrix elements.

We will make the connection between correlation functions and transition matrix elements in section 4.4 when we discuss the LSZ reduction formula.

#### 4.1.4 Perturbation Theory

##### Splitting the Hamiltonian

In the spirit of perturbation theory, we write the Hamilton operator as

$$H = H_0 + H_{\text{int}} , \quad (4.4)$$

where

$$H_0 \equiv \int \frac{d^3 p}{(2\pi)^3} E_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}}) \quad (4.5)$$

(cf. eq. (2.48)) and

$$H_{\text{int}} \equiv \int d^3 x \frac{\lambda}{4!} \phi^4(x) . \quad (4.6)$$

(Remember that the Legendre transform that relates the Lagrangian and the Hamiltonian, eq. (2.13) acts nontrivially only on terms containing derivatives of the field, while those without derivatives just get a minus sign.)

##### Rewriting the Field Operator $\phi(\mathbf{x}, t)$

Our goal is to expand the correlation function  $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$  in  $\lambda$ , assuming that  $\lambda \ll 1$ . Many, but not all, of the coupling constants appearing in nature fortunately are indeed small. The main problem along the way is the fact that the time dependence of the field operator  $\phi(x)$  is non-trivial now, thank to the interaction. We therefore split this time dependence into two pieces: the one induced by  $H_0$  and the one due to  $H_{\text{int}}$ .

At a fixed reference time  $t_0$ , we can still Fourier expand the field operator as usual (cf. eqs. (2.34) and (2.39)):

$$\phi(\mathbf{x}, t_0) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}(t_0) e^{i\mathbf{p}\mathbf{x}} + a_{\mathbf{p}}^\dagger(t_0) e^{-i\mathbf{p}\mathbf{x}}) . \quad (4.7)$$

In quantizing the field, we impose as before that the creation and annihilation operators satisfy the canonical commutation relations eq. (2.45).<sup>1</sup> The full time-dependent field at arbitrary time  $t$  is then formally given by

$$\phi(\mathbf{x}, t) = e^{iH(t-t_0)} \phi(\mathbf{x}, t_0) e^{-iH(t-t_0)}. \quad (4.8)$$

We will define the *interaction picture field operator*  $\phi_I(\mathbf{x}, t)$ , which contains the time-dependence due to  $H_0$ , but not the one due to  $H_{\text{int}}$ . In other words, we define

$$\phi_I(\mathbf{x}, t) \equiv e^{iH_0(t-t_0)} \phi(\mathbf{x}, t_0) e^{-iH_0(t-t_0)} \quad (4.9)$$

$$= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}) \Big|_{x^0=t-t_0}. \quad (4.10)$$

This expression is of course (and not by chance) identical in form to the expression for the time-dependent field operator in the free Klein–Gordon theory, eq. (2.61). In the following, we will write the creation and annihilation operators at  $t = t_0$  simply as  $a_{\mathbf{p}}^\dagger$  and  $a_{\mathbf{p}}$ , omitting the explicit mentioning of  $t_0$ .  $\phi_I$  is an operator that we can work with: it contains only  $c$ -numbers and creation and annihilation operators that obey the canonical commutation relations. All the technology developed in chapter 2 can be reused. Therefore, we would like to rewrite the two point correlations  $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$  entirely in terms of  $\phi_I$ . We have

$$\phi(\mathbf{x}, t) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(\mathbf{x}, t) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \quad (4.11)$$

$$\equiv U^\dagger(t, t_0) \phi_I(\mathbf{x}, t) U(t, t_0). \quad (4.12)$$

In the last step, we have defined the time evolution operator

$$U(t, t_0) \equiv e^{iH_0(t-t_0)} e^{-iH(t-t_0)}, \quad (4.13)$$

which describes all the non-trivial time-dependence of the theory. It satisfies the Schrödinger-like equation

$$i \frac{\partial}{\partial t} U(t, t_0) = e^{iH_0(t-t_0)} (-H_0 + H) e^{-iH(t-t_0)} \quad (4.14)$$

$$= e^{iH_0(t-t_0)} (-H_0 + H) e^{-iH_0(t-t_0)} U(t, t_0) \quad (4.15)$$

$$\equiv H_{\text{int}, I}(t) U(t, t_0). \quad (4.16)$$

Explicitly, the interaction Hamiltonian in the interaction picture  $H_{\text{int}, I}$  is given by

$$H_{\text{int}, I}(t) = \int d^3x \frac{\lambda}{4!} e^{iH_0(t-t_0)} \phi^4(\mathbf{x}, t) e^{-iH_0(t-t_0)} \quad (4.17)$$

<sup>1</sup>It is sufficient to impose the canonical commutation relations at the reference time  $t = t_0$ . They then follow automatically also at other times because

$$[a_{\mathbf{p}}(t), a_{\mathbf{p}'}^\dagger(t)] = e^{iH(t-t_0)} [a_{\mathbf{p}}(t_0), a_{\mathbf{p}'}^\dagger(t_0)] e^{-iH(t-t_0)} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}').$$

$$= \int d^3x \frac{\lambda}{4!} \phi_I^4(\mathbf{x}, t). \quad (4.18)$$

The solution of the above Schrödinger equation eq. (4.16), with the initial condition  $U(t_0, t_0) = 1$ , can be written as

$$U(t, t_0) = 1 - i \int_{t_0}^t dt_1 H_{\text{int}, I}(t_1) U(t_1, t_0) \quad (4.19)$$

by simply integrating both sides. Plugging this expression into itself repeatedly, we obtain the infinite series

$$\begin{aligned} U(t, t_0) = & 1 + (-i) \int_{t_0}^t dt_1 H_{\text{int}, I}(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}, I}(t_1) H_{\text{int}, I}(t_2) \\ & + \dots + (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n H_{\text{int}, I}(t_1) \dots H_{\text{int}, I}(t_n) + \dots \end{aligned} \quad (4.20)$$

If  $\lambda$ , and thus  $H_{\text{int}, I}$ , is small, we can truncate the series after the first order or second order term. Note that the integration boundaries are such that the integration variables  $t_1 \dots t_n$  satisfy  $t_1 > t_2 > \dots > t_n$ . Therefore, using the time ordering symbol  $T$  introduced in section 2.4.3, we can simplify the second order term in eq. (4.20) into

$$(-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_{\text{int}, I}(t_1) H_{\text{int}, I}(t_2) = \frac{1}{2} (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T[H_{\text{int}, I}(t_1) H_{\text{int}, I}(t_2)]. \quad (4.21)$$

Here, we have used the fact that, with the time ordering symbol included, the integrand is symmetric under  $t_1 \leftrightarrow t_2$ , i.e. extending the domain of the  $t_2$  integral from  $[t_0, t_1]$  to  $[t_0, t]$  just double counts every point (see fig. 4.1). This is compensated by a prefactor  $1/2$ .

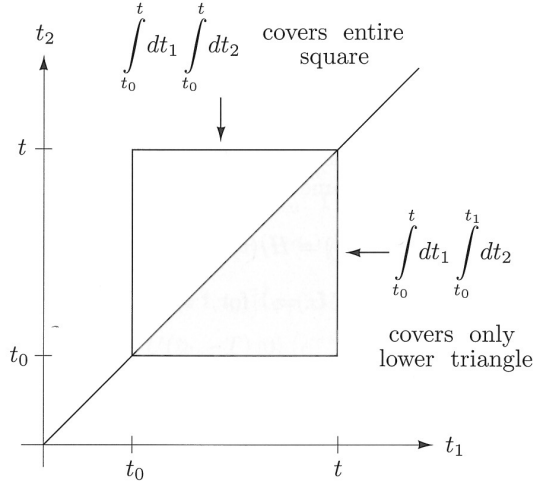
The above arguments can be extended for the higher order terms in the perturbation series:

$$\begin{aligned} & (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^{t_{n-1}} dt_n H_{\text{int}, I}(t_1) \dots H_{\text{int}, I}(t_n) \\ & = \frac{1}{n!} (-i)^n \int_{t_0}^t dt_1 \dots \int_{t_0}^t dt_n T[H_{\text{int}, I}(t_1) \dots H_{\text{int}, I}(t_n)]. \end{aligned} \quad (4.22)$$

Then, the perturbation series eq. (4.20) becomes an exponential series and we can write

$$U(t, t_0) = T \left[ \exp \left( -i \int_{t_0}^t dt' H_{\text{int}, I}(t') \right) \right], \quad (4.23)$$

where it is understood that the time ordering applies to each term in the exponential series.



**Figure 4.1:** Illustration of the arguments given below eq. (4.21), justifying why the integration domain of the  $t_2$  integral in that equation can be extended from  $[t_0, t_1]$  to  $[t_0, t]$ . Figure taken from [1].

It is useful to generalize the definition of  $U(t, t_0)$  to allow for both arguments to be different from the reference time  $t_0$  at which  $\phi(\mathbf{x}, t)$  and  $\phi_I(\mathbf{x}, t)$  coincide by definition. We define

$$U(t, t') \equiv T \left[ \exp \left( -i \int_{t'}^t dt'' H_{\text{int}, I}(t'') \right) \right]. \quad (t \geq t') \quad (4.24)$$

It is easy to check that also  $U(t, t')$  satisfies the Schrödinger-like eq. (4.16),

$$i \frac{\partial}{\partial t} U(t, t') = H_{\text{int}, I}(t) U(t, t'). \quad (4.25)$$

but now with the initial condition  $U(t, t') = 1$  at  $t = t'$ . Another way of writing  $U(t, t')$  is

$$U(t, t') = e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}. \quad (4.26)$$

This can be seen by checking that both eq. (4.24) and eq. (4.26) satisfy eq. (4.25) with the same initial condition. Equation (4.26) shows in particular that  $U(t, t')$  is unitary, i.e.  $[U(t, t')]^\dagger U(t, t') = 1$  and that

$$[U(t, t')]^\dagger = U(t', t). \quad (4.27)$$

It also shows that  $U(t, t')$  has the property that, for  $t_1 \geq t_2 \geq t_3$ ,

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3), \quad (4.28)$$

$$U(t_1, t_3) [U(t_2, t_3)]^\dagger = U(t_1, t_2). \quad (4.29)$$



### Rewriting the Vacuum $|\Omega\rangle$

We now have the tools to rewrite  $\phi(\mathbf{x}, t)$  in terms of  $\phi_I(\mathbf{x}, t)$ . To rewrite a correlation function, however, we also need to rewrite the vacuum state  $|\Omega\rangle$ . In particular, we would like to rewrite it in terms of the free vacuum  $|0\rangle$  because we know how  $a_{\mathbf{p}}$ ,  $a_{\mathbf{p}}^\dagger$  and thus  $\phi_I(\mathbf{x}, t)$  act on  $|0\rangle$ , while we have no idea how they act on  $|\Omega\rangle$ . We will now show that  $|\Omega\rangle$  can be obtained by starting with  $|0\rangle$  and evolving that state over a very long time interval.

Consider the time evolution of  $|0\rangle$ ,

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_n T} |n\rangle \langle n|0\rangle, \quad (4.30)$$

where  $n$  runs over a complete set of energy eigenstates of the interacting theory. The interacting vacuum  $|\Omega\rangle$  is of course an energy eigenstate, with eigenvalue  $E_0$ , and can therefore be separated from the sum:

$$e^{-iHT}|0\rangle = e^{-iE_0 T} |\Omega\rangle \langle \Omega|0\rangle + \sum_{n \neq 0} e^{-iE_n T} |n\rangle \langle n|0\rangle, \quad (4.31)$$

We would like to take the limit  $T \rightarrow \infty$ , but for the oscillating exponentials, this limit does not exist. We therefore have to regularize by taking the limit to a slightly imaginary direction:  $T \rightarrow \infty(1-i\epsilon)$ . Then, the term containing  $E_0$  is the slowest to die and therefore dominates. We thus get rid of all states  $|n\rangle$  except the desired vacuum state  $|\Omega\rangle$ . The latter is thus given by

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{-iHT}|0\rangle}{e^{-iE_0 T} \langle \Omega|0\rangle}. \quad (4.32)$$

Since  $H_0|0\rangle = 0$  (see the explicit form eq. (4.5)), we can insert a factor  $e^{iH_0 T}$  before  $|0\rangle$ . Moreover, since  $T$  is very large, we can impose a small shift  $T \rightarrow T + t_0$  without changing the result. This leads to

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{e^{-iH(T+t_0)} e^{iH_0(T+t_0)} |0\rangle}{e^{-iE_0(T+t_0)} \langle \Omega|0\rangle} \quad (4.33)$$

$$= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{U(t_0, -T)|0\rangle}{e^{-iE_0(T+t_0)} \langle \Omega|0\rangle}. \quad (4.34)$$

Similarly, we can also derive

$$\langle \Omega| = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0|U(T, t_0)}{e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle}. \quad (4.35)$$

Keeping in mind our goal of rewriting everything in terms of  $\phi_I(\mathbf{x}, t)$ , we see that the only piece that is still problematic is  $\langle \Omega|0\rangle$  in the denominator. To simplify it, use that

$$1 = \langle \Omega|\Omega\rangle \quad (4.36)$$

$$= \frac{\langle 0|U(T, t_0)U(t_0, -T)|0\rangle}{e^{-2iE_0 T} |\langle \Omega|0\rangle|^2}. \quad (4.37)$$

### Rewriting Correlation Functions

We have now all the tools to rewrite the two point correlation function  $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ . If we assume first  $x^0 > y^0 > t_0$ , we obtain using  $\phi(\mathbf{x})$  from eq. (4.12), the vacuum from eqs. (4.34), (4.35) and (4.37), and the properties eqs. (4.28) and (4.29) of the operator  $U(t, t')$ ,

$$\begin{aligned} \langle \Omega | T \phi(x) \phi(y) | \Omega \rangle &= \lim_{T \rightarrow \infty(1-i\epsilon)} [\langle 0 | U(T, t_0) U(t_0, -T) | 0 \rangle]^{-1} \\ &\quad \cdot \langle 0 | U(T, t_0) [U(x^0, t_0)]^\dagger \phi_I(\mathbf{x}, x^0) U(x^0, t_0) \\ &\quad \cdot [U(y^0, t_0)]^\dagger \phi_I(\mathbf{y}, y^0) U(y^0, t_0) U(t_0, -T) | 0 \rangle \\ &= \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | U(T, x^0) \phi_I(\mathbf{x}, x^0) U(x^0, y^0) \phi_I(\mathbf{y}, y^0) U(y^0, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}. \end{aligned} \quad (4.38)$$

If instead  $y^0 > x^0 > t_0$ , we obtain exactly the same expression, with  $x$  and  $y$  interchanged. (We do not need to consider the case  $t_0 > x^0$  or  $t_0 > y^0$  because we are free to choose  $t_0$  such that it is smaller than both  $x^0$  and  $y^0$ .) This allows us to write down the master formula

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \left[ \phi_I(\mathbf{x}, x^0) \phi_I(\mathbf{y}, y^0) \exp \left( -i \int_{-T}^T dt H_{\text{int}, I}(t) \right) \right] | 0 \rangle}{\langle 0 | T \left[ \exp \left( -i \int_{-T}^T dt H_{\text{int}, I}(t) \right) \right] | 0 \rangle}. \quad (4.40)$$

This is the final result of this section. As promised, it is written entirely in terms of the field operator  $\phi_I(\mathbf{x}, t)$  and the free vacuum  $|0\rangle$ . The field operator is made up of creation and annihilation operators whose action on each other and on the vacuum we know. The generalization from the two point correlation function to correlation functions with more than two field operator is straightforward: for each additional field operator  $\phi(z)$  on the left hand side, insert an extra factor of  $\phi_I(\mathbf{z}, z^0)$  in the numerator on the right hand side.

By expanding the exponential, we have a systematic way of approximately evaluating correlation functions. The  $n$ -th term in the perturbation series for the two point correlation function involves  $n$  spacetime integrals,  $n$  numerical factors of the form  $-i\lambda/4!$ , and (in the numerator) a correlation function of the form

$$\langle 0 | T \phi_I(x) \phi_I(y) \phi_I(z_1) \cdots \phi_I(z_{4n}) | 0 \rangle. \quad (4.41)$$

The correlation function in the denominator omits the factor  $\phi_I(x) \phi_I(y)$ . We will now develop a formalism that allows to compute such correlation functions in practice.

## 4.2 Wick's Theorem

Our goal in this section is to reduce a correlation function of the form  $\langle 0 | T \phi_I(x_1) \cdots \phi_I(x_n) | 0 \rangle$ , which still involves operators, to a purely algebraic expression. We already know that,

for  $n = 2$ , we obtain just the Feynman propagator:

$$\langle 0|T\phi_I(x)\phi_I(y)|0\rangle = D_F(x-y). \quad (4.42)$$

To generalize this to  $n > 2$ , it is useful to separate the field operator into a piece containing only creation operators and a piece containing only annihilation operators:

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x), \quad (4.43)$$

with

$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} e^{-ipx} a_{\mathbf{p}}, \quad (4.44)$$

$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} e^{+ipx} a_{\mathbf{p}}^\dagger. \quad (4.45)$$

(The superscripts  $+$  and  $-$  stand for positive and negative frequency.) This decomposition is useful because we know that

$$\phi_I^+(x)|0\rangle = 0 \quad \text{and} \quad \langle 0|\phi_I^-(x) = 0. \quad (4.46)$$

Decomposing the product of two fields yields

$$\phi_I(x)\phi_I(y) = \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \quad (4.47)$$

$$\begin{aligned} &= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \\ &\quad + [\phi_I^+(x), \phi_I^-(y)] \end{aligned} \quad (4.48)$$

In the second line, we have rewritten the expression in such a way that all terms except the commutator are in *normal order*, i.e. the creation and annihilation operators are ordered such that all annihilation operators are on the right and all creation operators are on the left. Normally ordered expressions have the advantage that they vanish when sandwiched between the vacuum  $|0\rangle$ . There is even a special notation for normal ordering: when operators are sandwiched between colons ( $:\cdots:$ ), it is implied that they should be put into normal order. For example,

$$:a_{\mathbf{p}}a_{\mathbf{k}}^\dagger a_{\mathbf{q}}: \equiv a_{\mathbf{k}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}. \quad (4.49)$$

The commutator  $[\phi_I^+(x), \phi_I^-(y)]$  is also very useful because it is just a  $c$ -number (times the identity operator). In fact, by virtue of the canonical commutation relations eq. (2.45),

$$[\phi_I^+(x), \phi_I^-(y)] = \int \frac{d^3p d^3p'}{(2\pi)^6 \sqrt{2E_{\mathbf{p}}} \sqrt{2E_{\mathbf{p}'}}} e^{-ipx+ip'y} [a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] \quad (4.50)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)}. \quad (4.51)$$

If we consider, instead of the simple product  $\phi_I(x)\phi_I(y)$  the time-ordered product  $T\phi_I(x)\phi_I(y)$ , we obtain for  $x^0 > y^0$  exactly the above expression eq. (4.48). For  $x^0 < y^0$ , the normal ordered terms are the same, but the commutator is  $[\phi_I^+(y), \phi_I^-(x)]$ . There is again a special symbol to simplify the notation in this situation. The *contraction* of two fields is defined as

$$\overline{\phi_I(x)\phi_I(y)} \equiv \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 \geq y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } x^0 < y^0 \end{cases}. \quad (4.52)$$

Writing out the commutators explicitly gives

$$\overline{\phi_I(x)\phi_I(y)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left[ e^{-ip(x-y)} \theta(x^0 - y^0) + e^{+ip(x-y)} \theta(y^0 - x^0) \right]. \quad (4.53)$$

Comparing to eq. (2.72), we see that the right hand side is just the Feynman propagator:

$$\overline{\phi_I(x)\phi_I(y)} = D_F(x - y). \quad (4.54)$$

This was to be expected because we know that  $\langle 0 | T\phi_I(x)\phi_I(y) | 0 \rangle = D_F(x - y)$  and we have argued that, in decomposition like eq. (4.48), only the commutator but not the normally ordered terms contribute.

To summarize the discussion so far, we have shown that

$$T\phi_I(x)\phi_I(y) = : \phi_I(x)\phi_I(y) : + \overline{\phi_I(x)\phi_I(y)}. \quad (4.55)$$

From here on, we will omit the subscript  $I$  on the field operators. Any field operator appearing in our expressions in this section will always be meant to be written in the interaction picture. In subsequent sections, it should be clear from the context where  $\phi(x)$  denotes the Heisenberg picture operator and where it denotes the interaction picture operator.

Let us now generalize eq. (4.55) to time-ordered products of more than two fields. The result we are going to prove is called *Wick's theorem* and reads

$$T\phi(x_1) \cdots \phi(x_n) = : \phi(x_1) \cdots \phi(x_n) : + \text{all possible contractions}, \quad (4.56)$$

where “all possible contractions” means one term for each way of contracting pairs of fields. For instance, for  $n = 4$ , Wick's theorem reads (with the shorthand notation  $\phi_a \equiv \phi(x_a)$ )

$$\begin{aligned} T\phi_1\phi_2\phi_3\phi_4 = & : \phi_1\phi_2\phi_3\phi_4 : + \overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi_1\phi_3}\phi_2\phi_4 + \overline{\phi_1\phi_4}\phi_2\phi_3 \\ & + \overline{\phi_2\phi_3}\phi_1\phi_4 + \overline{\phi_2\phi_4}\phi_1\phi_3 + \overline{\phi_3\phi_4}\phi_1\phi_2 \\ & + \overline{\phi_1\phi_2}\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\overline{\phi_2\phi_3} : . \end{aligned} \quad (4.57)$$

This expression means that all operators that are not contracted are in normal order, while each pair of contracted operators gives a factor  $D_F$ , irrespective of whether they are adjacent or not. For instance:

$$:T\overline{\phi_1\phi_2\phi_3\phi_4}: \equiv D_F(x_1 - x_3) : : \phi_2\phi_4 : . \quad (4.58)$$

Note that, in vacuum expectation values of the form  $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$ , only those terms where all field are contracted contribute.

Let us now prove Wick's theorem by induction. We have already proven it for  $n = 2$ . Assume now that the theorem has been proven for  $n - 1$  fields. Let us moreover assume without loss of generality that  $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$ . (If this condition is not satisfied, we can simply relabel the  $x_j$  to fulfill it.) Then,

$$\begin{aligned} T\phi(x_1)\cdots\phi(x_n) &= (\phi_1^+ + \phi_1^-) \cdot [ : \phi(x_2)\cdots\phi(x_n) \\ &\quad + [\text{all contractions not involving } \phi_1] : ] . \end{aligned} \quad (4.59)$$

We would like to pull  $\phi_1^+$  and  $\phi_1^-$  into the normally ordered expression. For  $\phi_1^-$ , this is easy because, according to the normal ordering prescription, it is anyway supposed to stay on the left. The  $\phi_1^+$  operator, on the other hand, has to be commuted past all other uncontracted fields. (Remember that fields that are contracted give just  $c$ -numbers that commute with everything.) In a term with  $m$  uncontracted fields (which we take to be  $\phi_2, \dots, \phi_m$  for definiteness), this leads to

$$\begin{aligned} \phi_1^+ : \phi_2 \cdots \phi_m : &= : \phi_2 \cdots \phi_m : \phi_1^+ + [\phi_1^+, : \phi_2 \cdots \phi_m :] \\ &= : \phi_1^+ \phi_2 \cdots \phi_m : + : [\phi_1^+, \phi_2^-] \phi_3 \cdots \phi_m + \phi_2 [\phi_1^+, \phi_3^-] \phi_4 \cdots \phi_m + \cdots : \\ &= : \phi_1^+ \phi_2 \cdots \phi_m : + \overline{\phi_1\phi_2}\phi_3 \cdots \phi_m + \overline{\phi_1\phi_3}\phi_2 \cdots \phi_m + \cdots . \end{aligned} \quad (4.60)$$

For  $m = n$ , the first term on the right hand side gives the first term on the right hand side of Wick's theorem, eq. (4.56). The other terms for  $n = m$  give all possible terms involving a contractions of  $\phi_1$  with one of the other fields, and no other contractions. For  $m < n$ , we obtain all terms involving an uncontracted  $\phi_1$  together with contractions of other pairs of fields, and all terms involving a contracted  $\phi_1$ , together with contractions of other pairs of fields. These are all the terms appearing on the right hand side of Wick's theorem, eq. (4.56), and the theorem is therefore proven.

### 4.3 Feynman Diagrams

The usefulness of Wick's theorem comes from the fact that it allows us to turn any correlation function  $\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$  into a sum of products of Feynman propagators. (Remember that all terms involving uncontracted fields vanish when sandwiched between  $\langle 0| \cdot |0\rangle$  thanks to the normal ordering.) The only difficulty is combinatorics, i.e. not missing any contractions. This is where *Feynman diagrams* help by graphically representing each combination of Feynman propagators.

### 4.3.1 Basic Idea and Application to a Simple 4-Point Function

In a Feynman diagram, each field operator  $\phi(x)$  is denoted by a point (called a *vertex*) in the drawing plane. Each contraction of two fields is depicted by a line (called a *propagator*) joining the two corresponding spacetime points. For instance, let us consider again the correlation function  $\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$ . The time-ordered product of field operators is rewritten according to eq. (4.57). Only the terms in the last line of that expression give a nonzero contribution to the correlation function. The Feynman diagram notation for these three terms is

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = \begin{array}{c} 1 \quad 2 \\ \bullet \text{---} \bullet \\ 3 \quad 4 \\ \bullet \text{---} \bullet \end{array} + \begin{array}{c} 1 \quad 2 \\ \bullet \text{---} \bullet \\ 3 \quad 4 \\ \bullet \text{---} \bullet \end{array} + \begin{array}{c} 1 \quad 2 \\ \bullet \text{---} \bullet \\ 3 \quad 4 \\ \bullet \text{---} \bullet \end{array} . \quad (4.61)$$

This diagrammatic notation also admits a straightforward physical interpretation: particles are created at two of the vertices, then each particle propagates to another vertex and is annihilated there. There are three different ways of picking pairs of vertices, corresponding to the three Feynman diagrams. Each diagram corresponds to a transition amplitude, and the amplitude for the overall process is the sum of the three Feynman diagrams.

### 4.3.2 An Example in $\phi^4$ Theory

So far, we have assumed the spacetime points appearing in the correlation function to be distinct. For correlation functions involving several field operators at the same spacetime point, the corresponding vertices in the Feynman diagram coincide as well. In this case, several propagators will connect to that vertex. Consider for instance the full propagator of the interacting theory,  $\langle \Omega|T\phi(x)\phi(y)|\Omega\rangle$ . According to our master formula, eq. (4.40), we have to evaluate specifically the expression<sup>2</sup>

$$\begin{aligned} \langle 0|T\phi(x)\phi(y)\exp\left(-i\int_{-T}^T dt H_{\text{int}}(t)\right)|0\rangle \\ = \langle 0|T\phi(x)\phi(y) + T\phi(x)\phi(y)\left(-i\int_{-T}^T dt H_{\text{int}}(t)\right) + \cdots|0\rangle, \end{aligned} \quad (4.62)$$

where on the right hand side, we have expanded the perturbation series up to first order in  $\lambda$ . The zeroth order term is simple:  $\langle 0|T\phi(x)\phi(y)|0\rangle = D_F(x-y)$ . The first order term reads, after plugging in the expression for  $H_{\text{int}}$

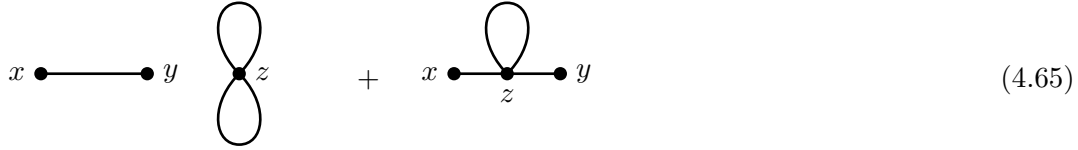
$$-i\frac{\lambda}{4!}\int d^4z \langle 0|T\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)|0\rangle. \quad (4.63)$$

<sup>2</sup>It is implicit here that the field operators  $\phi(x)$ ,  $\phi(y)$  appearing  $\langle \Omega|T\phi(x)\phi(y)|\Omega\rangle$  are the full Heisenberg fields, while those appearing in eq. (4.62) are the interaction picture operators which we denoted by  $\phi_I$  when we first introduced them.

According to Wick's theorem, we have to consider all possible pairwise contractions of the 6 field operators, and combinatorics tells us that there are 15 such contractions. They read

$$-i\frac{\lambda}{4!} \int d^4z \left( 3 D_F(x-y)[D_F(z-z)]^2 + 12 D_F(x-z)D_F(y-z)D_F(z-z) \right). \quad (4.64)$$

The diagrammatic representation of this expression is



Note that neither the prefactors nor the integral over  $d^4z$  is explicitly written out in the diagrammatic notation. It is implied that each *internal vertex* (i.e. each vertex that comes from  $H_{\text{int}}$ ) comes with a factor  $-i\lambda/4! \int d^4z$ . There is, however, no direct way of reading off the prefactors 3 and 12 from the diagrams. When translating Feynman diagrams into algebraic expressions, we therefore have to be careful not to forget these factors. This can sometimes be a headache.

### 4.3.3 A More Advanced Example

Let us now be more ambitious and apply our machinery to one of the contractions arising from the 3rd order term in the perturbative expansion of  $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ . In particular, we consider

$$\begin{aligned} & \langle 0 | T \phi(x) \phi(y) \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right)^3 \int d^4z \phi \phi \phi \phi \int d^4w \phi \phi \phi \phi \int d^4u \phi \phi \phi \phi | 0 \rangle \\ &= \frac{1}{3!} \left( \frac{-i\lambda}{4!} \right)^3 \int d^4z d^4w d^4u D_F(x-z) D_F(y-w) D_F(z-z) \\ & \quad \cdot D_F(z-w) [D_F(w-u)]^2 D_F(u-u). \end{aligned} \quad (4.66)$$

The corresponding Feynman diagram (“cactus diagram”)<sup>3</sup> is



<sup>3</sup>It has become a sort of sport to name Feynman diagrams according to plants, animals and other objects that they resemble. For instance: penguin diagram, lobster diagram, tadpole diagram, setting sun diagram—just google for them.

Of course, the contraction shown on the left hand side of eq. (4.66) is not the only one leading to the expression on the right hand side. For instance, interchanging the vertices  $z, w, u$  does not change the algebraic expression because the vertex coordinates are all integrated over. There are  $3!$  ways of interchanging the vertices. Moreover, the contractions that involve  $\phi(z)$  can connect to any of the four  $\phi(z)$  factors. This leads to a  $4! = 24$ -fold ambiguity. Since, however, one contraction has its beginning and its end on a  $\phi(z)$  factor, the ambiguity is only 12-fold in our case. Similarly, the ambiguity in how the contractions connect to the  $\phi(w)$  fields is 24-fold and the one for the  $\phi(u)$  fields is 12-fold. Finally, note that there are two contractions of the form  $D_F(w - u)$ :

$$\overbrace{\phi(w)\phi(w)\phi(u)\phi(u)} \quad (4.68)$$

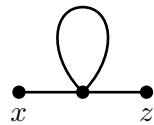
Interchanging the two  $\phi(w)$  factors and the two  $\phi(u)$  factors in this expression simultaneously does *not* change the contraction structure. Therefore, the multiplicity of the diagram is reduced by a factor of 2. In total, the multiplicity factor is thus

$$\underbrace{3!}_{\text{interchange of vertices}} \times \underbrace{\frac{4!}{2}}_{\text{interchange of } \phi(z) \text{ factors}} \times \underbrace{4!}_{\text{interchange of } \phi(w) \text{ factors}} \times \underbrace{\frac{4!}{2}}_{\text{interchange of } \phi(u) \text{ factors}} \times \underbrace{\frac{1}{2}}_{\text{interchange of the two } \phi(w)-\phi(u) \text{ contractions}}. \quad (4.69)$$

We see that the factor coming from the interchange of vertices cancels against the prefactor  $1/3!$  from the Taylor series. This is true at any order in perturbation theory: the  $n$ -th order term comes with a factor  $1/n!$  from the series expansion, and interchanging vertices leads to a multiplicity of  $n!$ . Also, each vertex has a prefactor  $1/4!$  that cancels against the factor  $4!$  from the different ways in which contractions connect to the four field operators in each vertex. (It now becomes clear why we have pulled the factor  $1/4!$  out of  $\lambda$  in defining the theory!) What remains is a factor  $1/S = 1/8$ , called a *symmetry factor* that accounts for permutations of vertices or fields that leave the contraction structure invariant.

#### 4.3.4 More Examples for Diagrams with Non-Trivial Symmetry Factors

In most practical calculations in elementary particle physics, symmetry factors do not even appear. For instance, in more realistic theories like quantum electrodynamics, the vertex connects three different field operators, so there is no ambiguity from permuting them. Nevertheless, it is important to keep in mind that sometimes, especially in theories with scalar fields, symmetry factors do show up. Therefore, we give here a few more examples:



$S = 2$

$(\text{interchange of vertices in } \overbrace{\phi(z)\phi(z)}) \quad (4.70)$





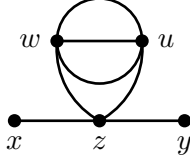
$$S = 8$$

(interchange of vertices within each contraction and interchange of contractions in  $\overbrace{\phi(z)\phi(z)}\overbrace{\phi(z)\phi(z)}$ , ) (4.71)



$$S = 3!$$

(interchange of the 3 contractions between  $\phi(x)$  and  $\phi(w)$ ) (4.72)



$$S = 3! \cdot 2$$

(symmetry factor of the “setting sun” diagram times factor 2 from interchanging  $u$  and  $w$  vertices.) (4.73)

### 4.3.5 Position Space Feynman Rules

We can summarize the results of this section so far in a simple cooking recipe for evaluating correlation function: to evaluate a correlation function

$$\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle \propto \langle 0 | T \phi_I(x_1) \cdots \phi_I(x_n) \exp \left( -i \int_{-T}^T dt H_{\text{int}}(t) \right) | 0 \rangle \quad (4.74)$$

up to a given order  $p$  in perturbation theory, draw all possible Feynman diagrams with  $n$  external vertices (each of which connects to exactly one propagator) and  $p$  internal vertices (each of which has four connection points). The algebraic expression for each diagram is then composed of factors determined by the following *Feynman rules*

$$1. \quad \text{For each propagator} \quad x \bullet \text{---} \bullet y = D_F(x - y) \quad (4.75)$$

$$2. \quad \text{For each internal vertex} \quad \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} = -i\lambda \int d^4z \quad (4.76)$$

$$3. \quad \text{For each external vertex} \quad x \bullet \text{---} = 1 \quad (4.77)$$

4. Figure out the symmetry factor and divide by it.

Let us briefly reiterate the physical interpretation of these rules: each propagator gives

the amplitude for a particle to propagate between two spacetime points. At each external vertex, one particle is created or annihilated. At each internal vertex, four propagators meet. This describes processes where 4 particles are created from the vacuum, or 1 particle is annihilated and 3 are created, or 2 are annihilated and 2 are created, or 3 are annihilated and 1 is created, or 4 are annihilated. The amplitude for these interaction processes is given by the factor  $-i\lambda$ . Finally, the integral over  $d^4z$  instructs us to integrate over all spacetime points where an interaction can happen. This is a manifestation of the superposition principle in quantum mechanics. If a process can happen in different ways (in this case: if the interaction can happen at different spacetime points  $z$ ), the amplitudes for all these different ways should be added up.

#### 4.3.6 Momentum Space Feynman Rules

The above cooking recipe was written entirely in coordinate space. In practice, it is usually more convenient to work in momentum space. We can easily translate the Feynman rules to momentum space by rearranging terms. In particular, remember that the Feynman propagator is given by (cf. eq. (2.69))

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}, \quad (4.78)$$

which is just the Fourier transform of

$$D_F(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon}. \quad (4.79)$$

In momentum space, we will therefore associate this expression with the propagator. The exponential factors will be associated with the vertices instead. To keep track of which vertex gets the  $e^{-ipx}$  factor and which one gets the  $e^{+ipy}$  factor, we endow each propagator with an arrow indicating the direction of momentum flow. The factor  $e^{+ipy}$  is associated with the vertex into which the momentum is flowing, the factor  $e^{-ipx}$  is associated with the vertex from which the momentum is flowing away. Since  $D_F(x - y) = D_F(y - x)$  we can choose the direction of the arrow arbitrarily, it is only important that the momentum on each propagator flows out of one vertex and into another one.

An external vertex, which gave just a factor 1 in coordinate space, will now contribute a factor of  $e^{\pm ipx}$  to the amplitude, where the  $+$  sign is relevant if the arrow of the attached propagator points away from the vertex and the  $-$  sign is relevant if the arrow points towards the vertex.

For an internal vertex, the exponential factors can be combined with the integral over  $d^4z$  to yield a delta function. If we call the momenta of the propagators attached to a given vertex  $p_1, p_2, p_3, p_4$  and assume for instance that  $p_1$  and  $p_2$  are incoming, while  $p_3, p_4$  are outgoing, an internal vertex becomes

$$\begin{array}{c} p_2 \quad p_4 \\ \swarrow \quad \searrow \\ \bullet \text{---} z \\ \nearrow \quad \nwarrow \\ p_1 \quad p_3 \end{array} = -i\lambda \int d^4z e^{-i(p_1+p_2+p_3+p_4)z} = -i\lambda (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4). \quad (4.80)$$

If any of the arrows was reversed, the sign of the corresponding momentum would be flipped on the right and side. The  $\delta$ -function simply implies energy-momentum conservation at the vertex: the 4-momentum carried into the vertex by the incoming particles must be equal to the 4-momentum carried away by the outgoing particles.

The only pieces that are left are the 4-momentum integrals from the propagators. Many of these can be carried out directly using the  $\delta$ -functions from the vertices. The remaining ones have to be included in the amplitude explicitly.

To summarize, the momentum space Feynman rules read

$$1. \quad \text{For each propagator} \quad \longrightarrow = \frac{i}{p^2 - m^2 + i\epsilon} \quad (4.81)$$

$$2. \quad \text{For each internal vertex} \quad \begin{array}{c} \swarrow \quad \searrow \\ \bullet \\ \nearrow \quad \nwarrow \end{array} = -i\lambda \quad (4.82)$$

$$3. \quad \text{For each external vertex} \quad x \bullet \longrightarrow = e^{\pm ipx} \quad (4.83)$$

4. Impose 4-momentum conservation at each vertex.

$$5. \quad \text{Integrate over momenta not determined by 4.:} \quad \int \frac{d^4p}{(2\pi)^4} \quad (4.84)$$

6. Figure out the symmetry factor and divide by it.

### 4.3.7 Disconnected Feynman Diagrams

Most Feynman diagrams we have considered so far were *connected*, i.e. each vertex in the diagram could be reached from any other vertex by following propagator lines. However, the Feynman rules and the underlying algebraic structures admit also diagrams where this is not the case. Perhaps the simplest example for a *disconnected* diagram arises already at first order in the perturbative expansion of the two-point function, see eq. (4.65):



$$= \exp \left( \sum_i V_i \right). \quad (4.89)$$

### 4.3.8 The Denominator of the Correlation Function

Up to this point we focused on computing the numerator in our master formula for evaluating correlation functions eq. (4.40). Of course, the denominator  $\langle 0 | T \left[ \exp \left( -i \int_{-T}^T dt H_{\text{int},I}(t) \right) \right] | 0 \rangle$  can be obtained in complete analogy. The only difference compared to the numerator is that the Feynman diagrams arising from the denominator have no external vertices, i.e. they consist only of vacuum bubbles. The results from the previous section (eqs. (4.86) and (4.89)) immediately tell us what the denominator looks like. It is simply given by

$$\langle 0 | T \left[ \exp \left( -i \int_{-T}^T dt H_{\text{int},I}(t) \right) \right] | 0 \rangle = \exp \left( \sum_i V_i \right). \quad (4.90)$$

This cancels exactly the corresponding exponential from the numerator!

In other words, a general correlation function can be evaluated by computing only Feynman diagrams containing *no* vacuum bubbles

$$\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle = \sum_j E_j \quad (4.91)$$

$$= \left( \begin{array}{l} \text{sum over all diagrams in which each connected} \\ \text{piece contains at least one external vertex} \end{array} \right). \quad (4.92)$$

## 4.4 The LSZ Reduction Formula

Now that we know how to evaluate a correlation function in an interacting quantum field theory, we should proceed to discuss the connection to scattering amplitudes, which are physically much more interesting.

A highly non-trivial problem in this context is to define appropriate initial and final states (*asymptotic state*). We know that the creation operator  $a_{\mathbf{p}}^\dagger$ , when acting on the vacuum of the *free theory*  $|0\rangle$ , creates a one-particle state in the free theory. However, this state will in general not be a momentum eigenstate of the interacting theory and is therefore not suitable for constructing initial or final states for a scattering process. Moreover, in the general Fourier expansion of the Heisenberg field  $\phi(x)$ ,

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}}(t) e^{i\mathbf{p}\mathbf{x}} + a_{\mathbf{p}}^\dagger(t) e^{-i\mathbf{p}\mathbf{x}}), \quad (4.93)$$

the creation and annihilation operators  $a_{\mathbf{p}}^\dagger(t)$  and  $a_{\mathbf{p}}(t)$  acquire a time dependence. In section 4.1.4, we have imposed the canonical commutation relations at  $t = t_0$  to obtain an object whose action on the free vacuum  $|0\rangle$  we know. However, this does not help us

define one-particle states in the interacting theory because we do not know how  $a_{\mathbf{p}}^\dagger(t_0)$  acts on  $|\Omega\rangle$ .

We will *assume* that, in the interacting field theory, asymptotic one-particle states with definite momentum are given by

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger(\pm\infty)|\Omega\rangle, \quad (4.94)$$

in analogy to the definition of one-particle momentum eigenstates in the free theory. We use the creation operator at  $a_{\mathbf{p}}^\dagger(-\infty)$  for initial states in scattering process and  $a_{\mathbf{p}}^\dagger(+\infty)$  for final states. The justification for this assumption is given for instance in Srednicki [2], sec. 5. Since a one-particle state should be orthogonal to the zero-particle state—the vacuum  $|\Omega\rangle$ —it follows from eq. (4.94) that

$$a_{\mathbf{p}}(\pm\infty)|\Omega\rangle = 0. \quad (4.95)$$

We will assume that the one-particle states defined by eq. (4.94) are normalized in the same way as in the free field theory:

$$\langle\mathbf{p}|\mathbf{p}'\rangle = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \quad (4.96)$$

A further complication arises because the initial state of a scattering process involves more than one particle. As soon as the wave functions of these particles overlap, they will start to interact. If particles were described by plane wave states (momentum eigenstates), their wave functions would overlap at all times. This would make it difficult to define suitable initial and final states as combinations of one-particle states. In particular, even though one particle states  $a_{\mathbf{p}_1}^\dagger(\pm\infty)|\Omega\rangle$  and  $a_{\mathbf{p}_2}^\dagger(t = \pm\infty)|\Omega\rangle$  are momentum eigenstates, the two particle state  $a_{\mathbf{p}_2}^\dagger a_{\mathbf{p}_1}^\dagger(\pm\infty)|\Omega\rangle$  is not a momentum eigenstate because of the interaction. The way out is to consider localized wave packets instead of plane waves. Then, at  $t = \pm\infty$ , the wave packets will have negligible overlap. The different particles in the initial and final states will know of each other only when they get close to the origin, so their interaction does not hinder the definition of the asymptotic states. We define the creation operator for a wave packet state centered around a momentum  $\mathbf{p}_j$  as

$$\tilde{a}_j^\dagger(t) \equiv \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p}; \mathbf{p}_j) a_{\mathbf{p}}^\dagger(t), \quad (4.97)$$

where  $f(\mathbf{p}; \mathbf{p}_j)$  is the shape factor of the wave packet. Its exact form is unimportant, but we could for instance choose a Gaussian:

$$f(\mathbf{p}; \mathbf{p}_j) = \frac{(2\pi)^{3/4}}{\sigma^{3/2}} \exp\left[-\frac{(\mathbf{p} - \mathbf{p}_j)^2}{4\sigma^2}\right]. \quad (4.98)$$

(The normalization is chosen such that  $\langle\Omega|\tilde{a}_j\tilde{a}_j^\dagger|\Omega\rangle = 1$ .) To be specific, let us consider a  $2 \rightarrow 2$  scattering process in which two initial state wave packets with central momenta

$\mathbf{p}_1$  and  $\mathbf{p}_2$  scatter into two final state wave packets with central momenta  $\mathbf{p}_3$  and  $\mathbf{p}_4$ . The initial and final states are thus

$$|i\rangle = \tilde{a}_1^\dagger(-\infty) \tilde{a}_2^\dagger(-\infty) |\Omega\rangle \quad (4.99)$$

and

$$\langle f| = \langle \Omega| \tilde{a}_3(+\infty) \tilde{a}_4(+\infty). \quad (4.100)$$

We would ultimately like to relate scattering amplitudes  $\langle f|i\rangle$  to correlation functions  $\langle \Omega|\phi(x_1) \cdots \phi(x_n)|\Omega\rangle$ , therefore we should express the particle creation and annihilation operators appearing in the definitions of  $|i\rangle$  and  $|f\rangle$  in terms of the field operators. The relation that achieves this is

$$a_{\mathbf{p}}^\dagger(t \rightarrow \pm\infty) = -\frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ipx} \overleftrightarrow{\partial}_0 \phi(x) \Big|_{t \rightarrow \pm\infty}, \quad (4.101)$$

where  $f(x) \overleftrightarrow{\partial}_0 g(x) \equiv f(x)(\partial g(x)/\partial x^0) - (\partial f(x)/\partial x^0)g(x)$ . The construction with the derivative is necessary to extract only  $a_{\mathbf{p}}^\dagger(t)$  instead of a combination of  $a_{\mathbf{p}}^\dagger(t)$  and  $a_{\mathbf{p}}(t)$ , as can be easily shown by direct computation:

$$\begin{aligned} & -\frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x e^{-ipx} \overleftrightarrow{\partial}_0 \phi(x) \Big|_{t \rightarrow \pm\infty} \\ &= -\frac{i}{\sqrt{2E_{\mathbf{p}}}} \int d^3x \int \frac{d^3q}{(2\pi)^3 \sqrt{2E_{\mathbf{q}}}} e^{-ipx} \overleftrightarrow{\partial}_0 \left( a_{\mathbf{q}} e^{-iqx} + a_{\mathbf{q}}^\dagger e^{iqx} \right) \Big|_{t \rightarrow \pm\infty} \\ &= -\frac{i}{2E_{\mathbf{p}}} e^{-iE_{\mathbf{p}}t} \left( -iE_{\mathbf{p}} a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} + iE_{\mathbf{p}} a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} + iE_{\mathbf{p}} a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} + iE_{\mathbf{p}} a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} \right) \Big|_{t \rightarrow \pm\infty} \\ &= a_{\mathbf{p}}^\dagger(t \rightarrow \pm\infty). \end{aligned} \quad (4.102)$$

Note that here we have expanded the full Heisenberg field  $\phi(x)$  in the same way as in the free theory. This is *not* possible in general—the general time dependence of the field operator is more complicated than that. However, by our assumption, at  $t \rightarrow \pm\infty$ , the operators  $a_{\mathbf{p}}^\dagger(t \rightarrow \pm\infty)$  and  $a_{\mathbf{p}}(t \rightarrow \pm\infty)$  create and annihilate energy-momentum eigenstates of the full theory. This means that they commute with the Hamiltonian at  $t \rightarrow \pm\infty$ , and consequently, according to eq. (4.8) (or according to the derivation from section 2.3.3) the time evolution of the field operator at  $t \rightarrow \pm\infty$  is indeed as in the free theory.

To rewrite a scattering amplitude, it is moreover useful to be able to rewrite  $\tilde{a}_j^\dagger(-\infty)$  in terms of  $\tilde{a}_j^\dagger(+\infty)$  and vice versa. Here is the relevant relation:

$$\tilde{a}_j^\dagger(+\infty) - \tilde{a}_j^\dagger(-\infty) = \int_{-\infty}^{\infty} dt \partial_0 \tilde{a}_j^\dagger(t)$$

$$\begin{aligned}
 &= \int \frac{d^3 p}{(2\pi)^3} f(\mathbf{p}; \mathbf{p}_j) \int d^4 x \frac{-i}{\sqrt{2E_{\mathbf{p}}}} \underbrace{\partial_0 \left( e^{-ipx} \overleftrightarrow{\partial}_0 \phi(x) \right)}_{\substack{= -ip^0 [e^{-ipx} \overleftrightarrow{\partial}_0 \phi(x)] + e^{-ipx} \overleftrightarrow{\partial}_0 (\partial_0 \phi(x)) \\ = e^{-ipx} [(p^0)^2 + \partial_0^2] \phi(x)}} \\
 &= \int \frac{d^3 p}{(2\pi)^3} f(\mathbf{p}; \mathbf{p}_j) \int d^4 x \frac{-i}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx} [\partial_0^2 + \mathbf{p}^2 + m^2] \phi(x) \\
 &= \int \frac{d^3 p}{(2\pi)^3} f(\mathbf{p}; \mathbf{p}_j) \int d^4 x \frac{-i}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx} [\partial_0^2 - \overleftarrow{\nabla}^2 + m^2] \phi(x) \\
 &= \int \frac{d^3 p}{(2\pi)^3} f(\mathbf{p}; \mathbf{p}_j) \int d^4 x \frac{-i}{\sqrt{2E_{\mathbf{p}}}} e^{-ipx} [\partial^2 + m^2] \phi(x). \quad (4.103)
 \end{aligned}$$

In the last step, we have integrated by parts twice to turn the left-acting derivative operator  $\overleftarrow{\nabla}^2$  into a right-acting operator  $\nabla^2$ . Note that  $(\partial^2 + m^2)\phi(x)$  would vanish in the free theory due to the Klein–Gordon equation. In the interacting theory, however, this expression is in general nonzero.

On the right hand side of eq. (4.103), the shape factor  $f(\mathbf{p}, \mathbf{p}')$  of the wave packet no longer plays an important (we only needed it to give meaning to the asymptotic states). Therefore, we will from now on assume the momentum distribution of the wave packets to be very narrow, so that we can approximate  $f(\mathbf{p}, \mathbf{p}_j) \simeq (2\pi)^3 \sqrt{2E_{\mathbf{p}_j}} \delta^{(3)}(\mathbf{p} - \mathbf{p}_j)$ . (The normalization here is such that the states  $\tilde{a}_j^\dagger |\Omega\rangle$  are normalized in the same way as the states  $|\mathbf{p}_j\rangle$ , see eq. (4.96).

We can now go back to the scattering amplitude and write

$$\langle f|i\rangle = \langle \Omega | T \tilde{a}_4(+\infty) \tilde{a}_3(+\infty) \tilde{a}_1^\dagger(-\infty) \tilde{a}_2^\dagger(-\infty) | \Omega \rangle. \quad (4.104)$$

On the right hand side, we have added a time ordering symbol  $T$ . We can do this because the operators that follow it are in time order already. We now use eq. (4.103) and its hermitian conjugate to turn all  $\tilde{a}_j(+\infty)$  operators into  $\tilde{a}_j(-\infty)$  operators and all  $\tilde{a}_j^\dagger(-\infty)$  into  $\tilde{a}_j^\dagger(+\infty)$ . The time ordering symbol then moves the annihilation operators to the right, where they act on the vacuum to yield zero. What is left is only a term containing field operators, but no  $\tilde{a}_j$  or  $\tilde{a}_j^\dagger$  operators any more. This term reads

$$\begin{aligned}
 \langle f|i\rangle &= \int d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 e^{-ip_1 x_1} (\partial_{x_1}^2 + m^2) \cdot e^{-ip_2 x_2} (\partial_{x_2}^2 + m^2) \\
 &\quad \cdot e^{ip_3 x_3} (\partial_{x_3}^2 + m^2) e^{ip_4 x_4} (\partial_{x_4}^2 + m^2) \langle \Omega | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | \Omega \rangle. \quad (4.105)
 \end{aligned}$$

This is the desired relation between a scattering amplitude and a correlation function. It is called the *Lehmann-Symanzik-Zimmermann (LSZ) reduction formula*. Generalizing it from the  $2 \rightarrow 2$  case to an arbitrary process with  $n$  particles in the initial state (momenta  $\mathbf{p}_1, \dots, \mathbf{p}_n$ ) and  $m$  particles in the final state (momenta  $\mathbf{p}_{n+1}, \dots, \mathbf{p}_{n+m}$ ) is



straightforward:

$$\begin{aligned}
 \langle f|i \rangle &= i^{n+m} \int d^4x_1 \cdots d^4x_{n+m} \\
 &\cdot e^{-ip_1x_1}(\partial_{x_1}^2 + m^2) \cdots e^{-ip_nx_n}(\partial_{x_n}^2 + m^2) \\
 &\cdot e^{ip_{n+1}x_{n+1}}(\partial_{x_{n+1}}^2 + m^2) \cdots e^{ip_{n+m}x_{n+m}}(\partial_{x_{n+m}}^2 + m^2) \\
 &\cdot \langle \Omega | T \phi(x_1) \cdots \phi(x_{n+m}) | \Omega \rangle .
 \end{aligned} \tag{4.106}$$

Let us finally reiterate the main assumptions that went into the LSZ reduction formula. The formula hinges on the fact that  $a_{\mathbf{p}}^\dagger(\pm\infty)|\Omega\rangle$  is indeed a one-particle state. This can only be the case if

$$\langle \Omega | \phi(x) | \Omega \rangle = 0 , \tag{4.107}$$

i.e. if the *vacuum expectation value* of  $\phi(x)$  vanishes. If eq. (4.107) was not satisfied, this would mean that one or several of the  $a_{\mathbf{p}}^\dagger$  create a superposition of a one-particle state and the vacuum.

The LSZ formula also relied on the normalization of the one-particle states, see eq. (4.96). If the normalization was different, the numerical factors in the LSZ formula would change. The normalization condition eq. (4.96) implies

$$\langle \mathbf{p} | \phi(x) | \Omega \rangle = e^{ipx} . \tag{4.108}$$

A more detailed discussion of these conditions is given in Srednicki [2], sec. 5.

Equations (4.107) and (4.108) are crucial for our formalism to work, and we will go to great lengths to make sure they are satisfied. In particular, we will have to *renormalize* QFTs to remove violations of eqs. (4.107) and (4.108) that arise when computing loop diagrams.

## 4.5 Computing $S$ -Matrix Elements from Feynman Diagrams

We have come along way already: we have learned how to turn a general correlation function  $\langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$ , involving the unknown vacuum state  $|\Omega\rangle$  and the unknown time-dependent field operator  $\phi(x)$ , into an expression we can compute perturbatively using Feynman diagrams. The LSZ formula tells us how to turn the result of such a computation into a scattering amplitude. However, the LSZ formula is a bit clumsy. Therefore, we will now use it only to derive a new set of *Feynman rules for scattering amplitudes*. From then on, we can directly use these rules to evaluate scattering amplitudes at our leisure.

Consider the momentum space Feynman rules derived in section 4.3.6. For each field operator  $\phi(x_j)$  appearing in the correlation function, they yield a factor  $e^{ipx_j}$  for an incoming particle (arrow pointing into the diagram away from the external vertex) with

4-momentum  $p$  and a factor  $e^{-ipx_j}$  for an outgoing particle (arrow pointing out of the diagram towards the external vertex) with 4-momentum  $p$ . This factor is the only dependence of the correlation function on  $x_j$ . Therefore, applying the Klein–Gordon operator  $\partial_{x_j}^2 + m^2$  from the LSZ formula to these exponential factors yields simply  $(-p^2 + m^2)e^{\pm ipx_j}$ . The term in parentheses (together with one of the factors of  $i$  in the LSZ formula) *cancels exactly* the  $i/(p^2 - m^2 + i\epsilon)$  from the propagator connected to that external vertex.

The remaining exponential  $e^{\pm i(p_j - p)x_j}$ , together with the integral over  $d^4x_j$ , yields a delta function  $(2\pi)^4\delta^{(4)}(p_j - p)$ . These delta function can be used to evaluate some of the momentum integrals  $\int d^4p/(2\pi)^4$ . Only one of them remains, enforcing overall 4-momentum conservation among the external particles:  $(2\pi)^4\delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm \dots)$ , where the plus signs stand for incoming particles and the minus signs for outgoing particles (or vice-versa). In practice, this delta function is usually not associated with the matrix element, but is reintroduced in the formula for the cross section. We will follow this convention here.

Beyond the overall delta function, neither external vertices nor propagators connected to them contribute to the Feynman diagrams for the scattering amplitude. Propagator factors need to be included only for internal propagators, i.e. propagators connecting two internal vertices. To emphasize this point, incoming and outgoing particles in the Feynman diagram for a scattering amplitude are depicted as lines with only one end attached to a vertex.

These rules can be summarized as follows

$$1. \quad \text{For each (internal) propagator} \quad \text{---} = \frac{i}{p^2 - m^2 + i\epsilon} \quad (4.109)$$

$$2. \quad \text{For each internal vertex} \quad \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = -i\lambda \quad (4.110)$$

$$3. \quad \text{For each incoming/outgoing particle} \quad \text{---} = 1 \quad (4.111)$$

4. Impose 4-momentum conservation at each vertex.

$$5. \quad \text{Integrate over momenta not determined by 4.:} \quad \int \frac{d^4p}{(2\pi)^4} \quad (4.112)$$

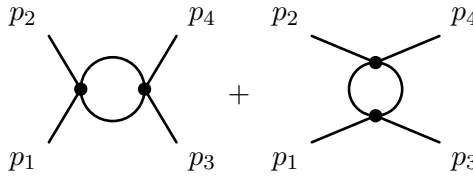
6. Figure out the symmetry factor and divide by it.

Here are two examples: first, the amplitude for the scattering of a  $\phi$  particle off another  $\phi$  particle a lowest order in  $\lambda$  is simply



$$= -i\lambda \quad (4.113)$$

Here  $p_j$  are the momenta of the external particles. It is implied that  $p_1 + p_2 = p_3 + p_4$ . Going one order higher in  $\lambda$ , there are two diagrams:



$$= (-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{(p_1 + p_2 + q)^2 - m^2 + i\epsilon} \cdot \frac{i}{q^2 - m^2 + i\epsilon} + (-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{(p_1 - p_3 + q)^2 - m^2 + i\epsilon} \cdot \frac{i}{q^2 - m^2 + i\epsilon}. \quad (4.114)$$

We call diagrams like eq. (4.113), where no path taken through the diagrams hits a vertex twice, a *tree level* diagram. Diagrams with closed loops, i.e. paths that return to the vertex at which they originated are called *loop diagrams*. We will develop the technology to evaluate loop diagrams, i.e. to simplify algebraic expressions like the right hand side of eq. (4.114), in the second part of this course.

In principle, when considering the second order (in  $\lambda$ ) corrections to the scattering rate of two  $\phi$  particles, one could also draw diagrams with loops connected to the external legs, for instance



$$(4.115)$$

We will see later, when discussing renormalization, that such diagrams in which loops are attached to one of the external legs, need not be considered in many cases. For the moment, we simply accept this result and will therefore drop such contributions henceforth.

A few comments are in order:

- Note that we have omitted the arrows on the lines. Since the only place where the 4-momenta appear is in the propagators, where they appear as  $p^2$ , the direction of momentum flow is irrelevant.

- In computing scattering amplitudes, we are only interested in connected diagrams. We have already seen in section 4.3.8 that diagrams containing vacuum bubbles do not contribute to correlation functions. The possibility remained, however, that a diagram consists of several disconnected pieces, each of them containing external vertices. However, such process are not scattering processes in the usual sense. For instance, in the diagrams

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} | \\ | \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array}, \quad (4.116)$$

contributing to the 4-point function, no momentum is exchanged.

- In a typical Feynman diagram, most momenta are determined by the fixed momenta of the external particle, together with 4-momentum conservation at the vertices. The undetermined momenta that we have to integrate over according to rule 5. correspond to momentum flows inside the diagram along closed loops, i.e. beginning and ending on the same vertex. For instance, in the diagram

$$\text{---} \bigcirc \text{---} \quad (4.117)$$

there are two such *loop momenta*. To see this, note that there are three internal propagators with per se undetermined momenta that should be integrated over. Momentum conservation at the vertices imposes two constraints (in the form of 4d  $\delta$ -functions). One of them becomes the overall energy-momentum-conserving  $\delta$ -factor that we have pulled out of the matrix element, the other eliminates the integral over one of the internal momenta. We are left with two momentum integrations, and we say that the diagram is a *2-loop diagram*.

## 4.6 Feynman Rules for Fermions

Up to now, we have only considered interacting scalar field theories to keep things as simple as possible. Unfortunately, most elementary particles are fermions. Therefore, let us now generalize what we have learned to include fermions.

### 4.6.1 The Master Formula for Correlation Functions Involving Fermions

The derivation of our master formula eq. (4.40), which allows us to rewrite correlation functions in the interacting theory in terms of correlation functions of the corresponding free theory goes through also for fermions. There is only one small modification: in several places along the way, we reordered the field operators in the correlation function

to bring them into time order (see eq. (4.21), where we rewrote  $\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2$  in terms of  $\int_{t_0}^t dt_1 \int_{t_0}^t dt_2$  and the last step leading to the master formula eq. (4.40), where we pulled the various  $U$ -factors together into one exponential). For fermions, reordering field operators leads to extra minus signs due to their anticommuting nature. However, these minus signs can be fully accounted for if we define the time ordering symbol  $T$  such that any interchange of two fermion fields required to restore time ordering yields one extra minus sign. For instance:

$$T\psi_1\bar{\psi}_2\psi_3\psi_4 = (-1)^3\psi_3\psi_1\psi_4\bar{\psi}_2 \quad \text{if } x_3^0 > x_1^0 > x_4^0 > x_2^0. \quad (4.118)$$

Here, we have again used the shorthand notation  $\bar{\psi}_j \equiv \bar{\psi}(x_j)$ . For the special case of only two fermion fields, we have already used this definition in section 3.3.

### 4.6.2 Wick's Theorem for Fermions

For Wick's theorem to hold for fermions, we should in a similar way also define the normal ordering symbol  $:\cdot:$  to yield an extra minus sign if an odd permutation of the fermion fields is required to put them into normal order. For instance:

$$:a_{\mathbf{p}}b_{\mathbf{k}}a_{\mathbf{q}}^\dagger: = (-1)^2a_{\mathbf{q}}^\dagger a_{\mathbf{p}}b_{\mathbf{k}} = (-1)^3a_{\mathbf{q}}^\dagger b_{\mathbf{k}}a_{\mathbf{p}}. \quad (4.119)$$

With this definition, we can easily generalize Wick's theorem. Consider first again the case of just two fermion fields (cf. eq. (4.48) et seq.):

$$T\psi(x)\bar{\psi}(y) = :\psi(x)\bar{\psi}(y): + \overline{\psi(x)\bar{\psi}(y)} \quad (4.120)$$

Remember that the contraction term arises from those contributions that require an interchange of fields in order to bring them into normal order. For scalar fields, this was achieved by a commutator, for fermion fields we should use the anticommutator instead. Since the time ordering symbol and the normal ordering symbols imply an extra minus sign if  $y^0 > x^0$ , also the definition of the contraction has to include such an extra minus sign:

$$\overline{\psi(x)\bar{\psi}(y)} \equiv \begin{cases} \{\psi^+(x), \bar{\psi}^-(y)\} & \text{for } x^0 \geq y^0 \\ -\{\bar{\psi}^+(y), \psi^-(x)\} & \text{for } x^0 < y^0 \end{cases} \quad (4.121)$$

$$= S_F(x - y). \quad (4.122)$$

Here,  $\psi^+(x)$  and  $\psi^-(x)$  are again defined as the positive and negative frequency components of  $\psi(x)$ , respectively. The fact that the contraction of  $\psi(x)$  and  $\bar{\psi}(y)$  equals  $S_F(x - y)$  is completely analogous to the proof of the corresponding identity for scalar fields, eq. (4.54). Note that, unlike for the scalar propagator, for fermions  $S_F(x - y) \neq S_F(y - x)$ , so care must be taken to put  $\psi(x)$  and  $\bar{\psi}(y)$  into the appropriate order before evaluating the contraction.

The time ordered product of two  $\psi$  fields or two  $\bar{\psi}$  fields is equal to the normally ordered product of these fields because their anticommutator vanishes. In order for our equations to have the same form in this case as in eq. (4.120), we define

$$\overline{\psi(x)\psi(y)} = 0 \quad (4.123)$$

$$\overline{\bar{\psi}(x)\bar{\psi}(y)} = 0. \quad (4.124)$$

To carry out the induction step in the proof of Wick's theorem for fermions, we need to worry about one more source of minus signs. The analogue of eq. (4.60) for fermions reads

$$\begin{aligned} \psi_1^+ : \psi_2 \cdots \psi_m : &= - : \psi_2 \cdots \psi_m : \psi_1^+ + \{ \psi_1^+, : \psi_2 \cdots \psi_m : \} \\ &= : \psi_1^+ \psi_2 \cdots \psi_m : + : \{ \psi_1^+, \psi_2^- \} \psi_3 \cdots \psi_m - \psi_2 \{ \psi_1^+, \psi_3^- \} \psi_4 \cdots \psi_m + \cdots : \\ &= : \psi_1^+ \psi_2 \cdots \psi_m : + : \overline{\psi_1 \psi_2 \psi_3} \cdots \psi_m : + : \overline{\psi_1 \psi_2 \psi_3} \cdots \psi_m : + \cdots . \end{aligned} \quad (4.125)$$

(Replacing any of the  $\psi$  fields by  $\bar{\psi}$  fields would not change this result.) This implies that a contraction of two non-adjacent field operators comes with a minus sign if an odd number of anticommutations is necessary to move these field operators next to each other. For example:

$$: \overline{\psi_1 \psi_2 \psi_3 \bar{\psi}_4} : = - \overline{\psi_1 \bar{\psi}_3} : \psi_2 \bar{\psi}_4 : = -S_F(x_1 - x_3) : \psi_2 \bar{\psi}_4 : . \quad (4.126)$$

To summarize, Wick's theorem for fermions takes exactly the same form as for scalars:

$$T\psi(x_1) \cdots \psi(x_n) = : \psi(x_1) \cdots \psi(x_n) : + \text{all possible contractions} : . \quad (4.127)$$

For fermions, however, it is understood that an extra minus sign appears on the left hand side when  $x_1, \dots, x_n$  are such that bringing the field operators into time order requires an odd permutation. Similarly, in the normally ordered terms on the right hand side, any summand that requires an odd permutation to restore normal order gets a minus sign. Finally in replacing contracted field operators by Feynman propagators, we should first move the contracted fields next to each other, possibly paying the price of another minus sign if an odd number of anticommutations is required.

### 4.6.3 The LSZ Formula for Fermions

The LSZ reduction formula can also be generalized to theories with fermions. Here, we only quote the result. The full proof is given in Srednicki, [2], sec. 41. For a scattering process with  $n$  incoming fermions (4-momenta  $p_1, \dots, p_n$  and spins  $s_1, \dots, s_n$ ) and  $m$  outgoing fermions (4-momenta  $p_{n+1}, \dots, p_{n+m}$  and spins  $s_{n+1}, \dots, s_{n+m}$ ) it reads:

$$\langle f|i \rangle = (-i)^{n+m} \int d^4x_1 \cdots d^4x_{n+m}$$

$$\begin{aligned}
 & \cdot e^{ip_{n+1}x_{n+1}} [\bar{u}^{s_{n+1}}(p_{n+1})(i\vec{\partial}_{x_{n+1}} - m)]_{\alpha_{n+1}} \cdots e^{ip_{n+m}x_{n+m}} [\bar{u}^{s_{n+m}}(p_{n+m})(i\vec{\partial}_{x_{n+m}} - m)]_{\alpha_{n+m}} \\
 & \cdot \langle \Omega | T \psi_{\alpha_{n+m}}(x_{n+m}) \cdots \psi_{\alpha_{n+1}}(x_{n+1}) \bar{\psi}(x_1)_{\alpha_1} \cdots \bar{\psi}(x_n)_{\alpha_n} | \Omega \rangle \\
 & \cdot [(-i\overleftarrow{\vec{\partial}}_{x_1} - m)u^{s_1}(p_1)]_{\alpha_1} e^{-ip_1x_1} \cdots [(-i\overleftarrow{\vec{\partial}}_{x_n} - m)u^{s_n}(p_n)]_{\alpha_n} e^{-ip_nx_n}.
 \end{aligned} \tag{4.128}$$

Here,  $\alpha_1, \dots, \alpha_{n+m}$  are spinor indices. If any of the incoming fermions is replaced by an antifermion (created by  $\psi$  instead of  $\bar{\psi}$ ), the corresponding factor is replaced according to

$$-i(-i\overleftarrow{\vec{\partial}}_{x_j} - m)u^{s_j}(p_j)e^{-ip_jx_j} \rightarrow ie^{-ip_jx_j}\bar{v}^{s_j}(p_j)(i\vec{\partial}_{x_j} - m) \tag{4.129}$$

and is moved from the right of the correlation function to the left. Similarly, if an outgoing fermion is replaced by an antifermion (annihilated by  $\bar{\psi}$  instead of  $\psi$ ), the corresponding factor is replaced according to

$$-ie^{ip_jx_j}\bar{u}^{s_j}(p_j)(i\vec{\partial}_{x_j} - m) \rightarrow i(-i\overleftarrow{\vec{\partial}}_{x_j} - m)v^{s_j}(p_j)e^{ip_jx_j} \tag{4.130}$$

and is moved from the left of the correlation function to the right.

There are two main differences between the LSZ formula for fermions compared to the one for scalars: first, the replacement of the Klein–Gordon operator by the Dirac operator comes about because the analogue of eq. (4.101) which allows us to write a creation operator in terms of a field operator does not involve a derivative in the case of fermions. Instead, one exploits the orthogonality relations of the  $u$  and  $v$  spinors, which also explains the appearance of such spinors in the LSZ formula for fermions.

#### 4.6.4 Yukawa Theory

To put the master formula and Wick’s theorem for fermions to good use, we need an actual interacting field theory involving fermions. The simplest one of its kind is *Yukawa theory*, a toy model involving one Dirac fermion  $\psi$ , one scalar field  $\phi$ , and an interaction between these fields. This theory was originally introduced by Hideki Yukawa (Nobel Prize 1949) to describe the interaction between nucleons and mesons. In modern particle physics, it also describes the interactions of the Higgs boson with quarks and electrons. Finally, it can be considered a simplified toy model of quantum electrodynamics. Here is the Lagrangian

$$\mathcal{L}_{\text{Yukawa}} = \bar{\psi}(i\vec{\partial} - m_\psi)\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_\phi^2\phi^2 - g\bar{\psi}\psi\phi. \tag{4.131}$$

Here,  $m_\psi$  and  $m_\phi$  are the fermion and scalar masses, respectively, and  $g$  is a dimensionless coupling constant describing the strength of the Yukawa interactions. For perturbation theory to work,  $g$  had better not be too large. The interaction Hamiltonian in Yukawa theory is thus  $H_{\text{int}} = \int d^3x g \bar{\psi}\psi\phi$ .

Let us evaluate the fermion–fermion scattering rate in this model:

$$\text{fermion}(p) + \text{fermion}(k) \rightarrow \text{fermion}(p') + \text{fermion}(k'). \quad (4.132)$$

The matrix element for this process can be obtained from the LSZ formula applied to the correlation function  $\langle \Omega | T \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) | \Omega \rangle$ . This correlation function receives its lowest order non-trivial contribution at order  $g^2$  in perturbation theory:

$$\frac{(-ig)^2}{2!} \langle 0 | T \psi_\alpha(x_1) \psi_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) \int d^4x \bar{\psi}_\tau(x) \psi_\tau(x) \phi(x) \int d^4y \bar{\psi}_\rho(y) \psi_\rho(y) \phi(y) | 0 \rangle. \quad (4.133)$$

Here,  $\alpha, \beta, \gamma, \delta, \tau, \rho$  are spinor indices. One contraction structure that is nonzero is the following

$$\frac{(-ig)^2}{2!} \langle 0 | T \psi_\alpha(x_1) \psi_\beta(x_2) \bar{\psi}_\gamma(x_3) \bar{\psi}_\delta(x_4) \int d^4x \bar{\psi}_\tau(x) \psi_\tau(x) \phi(x) \int d^4y \bar{\psi}_\rho(y) \psi_\rho(y) \phi(y) | 0 \rangle. \quad (4.134)$$

Contracting  $\psi_\alpha(x_1)$  with  $\bar{\psi}_\tau(x)$  and  $\psi_\beta(x_2)$  with  $\bar{\psi}_\rho(y)$  instead yields another nonzero contribution. As usual, interchanging the two vertices  $x$  and  $y$  is always possible, and the resulting factor  $2!$  cancels the  $1/2!$  prefactor. Therefore we can drop this prefactor from now on. Note that, in Yukawa theory, the vertex couples three different fields, so no extra factors arise from interchanging those. In other words, Yukawa theory does not have symmetry factors!. Up to a possible minus sign, eq. (4.134) yields

$$(-ig)^2 \int d^4x \int d^4y [S_F(x_1 - y)]_{\alpha\rho} [S_F(x_2 - x)]_{\beta\tau} [S_F(y - x_3)]_{\rho\gamma} [S_F(x - x_4)]_{\tau\delta} D_F(x - y). \quad (4.135)$$

We now apply the LSZ reduction formula to this expression. The terms for the incoming fermions have the structure

$$\begin{aligned} & -i \int d^4x_3 S_F(y - x_3) (-i \overleftarrow{\not{\partial}}_{x_3} - m_\psi) u^{s_3}(p) e^{-ipx_3} \\ &= -i \int d^4x_3 \int \frac{d^4q}{(2\pi)^4} \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2 + i\epsilon} e^{-iq(y-x_3)} (-i \overleftarrow{\not{\partial}}_{x_3} - m_\psi) u^{s_3}(p) e^{-ipx_3} \\ &= \int d^4x_3 \int \frac{d^4q}{(2\pi)^4} \underbrace{\frac{(\not{q} + m_\psi)(\not{q} - m_\psi)}{q^2 - m_\psi^2 + i\epsilon}}_{=1} e^{-iq(y-x_3)} u^{s_3}(p) e^{-ipx_3} \\ &= e^{-ipy} u^{s_3}(p). \end{aligned} \quad (4.136)$$

Similarly, for each outgoing fermion we obtain a factor of the form  $e^{+ip'y} \bar{u}^{s_1}(p')$  or  $e^{+ik'x} \bar{u}^{s_2}(k')$ . The overall expression for the matrix element is thus, after plugging in the Fourier expansion of  $D_F(x - y)$ ,

$$i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)}(p + k - p' + k')$$



$$\begin{aligned}
 &= (-ig)^2 [\bar{u}^{s_1}(p') u^{s_3}(p)] [\bar{u}^{s_2}(k') u^{s_4}(k)] e^{-i(p-p')y} e^{-i(k-k')x} \int \frac{d^4 q}{(2\pi)^4} \frac{i e^{-iq(x-y)}}{q^2 - m_\phi^2 + i\epsilon} \\
 &= (-ig)^2 [\bar{u}^{s_1}(p') u^{s_3}(p)] [\bar{u}^{s_2}(k') u^{s_4}(k)] \frac{i}{(p' - p)^2 - m_\phi^2} \cdot (2\pi)^4 \delta^{(4)}(p + k - p' + k')
 \end{aligned}
 \tag{4.137}$$

As mentioned earlier, we pull the overall 4-momentum conserving delta function out of the matrix element by convention. The procedure we have followed can obvious also be applied to more general processes, and to higher order terms in the perturbation series. What we have found can be summarized in the Feynman rules for Yukawa theory:

$$1. \quad \text{Incoming fermion} \quad \text{---}\!\!\!\!\!\rightarrow\!\!\!\!\!\bullet \quad \underset{p}{=} \quad u^s(p) \quad (4.138)$$

$$2. \quad \text{Incoming antifermion} \quad \text{---}\!\!\!\!\!\leftarrow\!\!\!\!\!\bullet \quad \overset{p}{=} \quad \bar{v}^s(p) \quad (4.139)$$

$$3. \quad \text{Outgoing fermion} \quad \bullet \text{---}\!\!\!\!\!\rightarrow \quad = \quad \bar{u}^s(p) \quad (4.140)$$

$$4. \quad \text{Outgoing antifermion} \quad \bullet \text{---}\!\!\!\!\!\leftarrow \quad = \quad v^s(p) \quad (4.141)$$

$$5. \quad \text{Incoming scalar} \quad \text{---}\!\!\!\!\!\bullet \quad \underset{p}{=} \quad 1 \quad (4.142)$$

$$6. \quad \text{Outgoing scalar} \quad \bullet \text{---}\!\!\!\!\! \quad \underset{p}{=} \quad 1 \quad (4.143)$$

$$7. \quad \text{Scalar propagator} \quad \text{---}\!\!\!\!\! \quad \underset{p}{=} \quad \frac{i}{p^2 - m_\phi^2 + i\epsilon} \quad (4.144)$$

$$8. \quad \text{Fermion propagator} \quad \text{---}\!\!\!\!\!\rightarrow \quad \underset{p}{=} \quad \frac{i(\not{p} + m_\psi)}{p^2 - m_\psi^2 + i\epsilon} \quad (4.145)$$

$$9. \quad \text{Vertex} \quad \text{---}\!\!\!\!\!\bullet \begin{array}{l} \nearrow \\ \searrow \end{array} \quad = \quad -ig \quad (4.146)$$

10. Impose 4-momentum conservation at each vertex.

$$11. \quad \text{Integrate over momenta not determined by 10.:} \quad \int \frac{d^4 p}{(2\pi)^4} \quad (4.147)$$

12. Figure out the overall sign of the diagram.

As mentioned above, there are no symmetry factors in Yukawa theory. With these rules, the scattering amplitude we just computed corresponds to the diagram


(4.148)

and the second diagram relevant for the process of fermion–fermion scattering is


(4.149)

A few notes on the Feynman rules for fermions are in order

- By convention, the arrows on the fermion lines do not represent momentum flow, but particle number flow: particle number flows into the diagram along an incoming fermion line and out of the diagram along an outgoing fermion line. For antiparticles, the flow is reversed: for each incoming antifermion, one unit of particle number flows *out* of the diagram (or, equivalently, a unit of *negative* particle number flows into the diagram) and for outgoing antifermions, particle number flows into the diagram. On the external lines, the arrows remind us whether the line corresponds to a barred spinor or an unbarred spinor. At the vertices, they remind us of which propagator is contracted with the  $\bar{\psi}$  factor from the vertex and which one is contracted with the  $\psi$  factor. The arrows thus also help us avoid mistakes: if we end up with a vertex that has two arrows pointing towards it or two arrows pointing away from it, that diagram is forbidden.
- Especially in the experimental literature, one sometimes draws an arrow of time (typically pointing left to right) into a Feynman diagram and puts the initial particles on the left, the final state particles on the right of the diagram. We will refrain from doing so and not associate any physical meaning with the orientation of the Feynman diagram in the drawing plane.
- Spinor indices are always contracted along fermion lines. This can be understood by noting that fermion lines are joint together at vertices, where the indices of the  $\psi$  field corresponding to the line pointing towards the vertex and the  $\bar{\psi}$  field corresponding to the line pointing away from the vertex are contracted. In practice, when translating a Feynman diagram to an algebraic expression, we start at the end

of a fermion lines (which gives  $\bar{u}$  or  $\bar{v}$  spinor) and then work our way backwards along the lines, in the direction opposite to the arrows, applying the rules for vertices and propagators as we encounter them.

- For fermion propagators, the direction of momentum flow matters. The above rule holds if the momentum  $p$  flows along the arrow. (In case the momentum flows in the opposite direction  $p$  should be replaced by  $-p$ .) This rule arise from the way the oscillating exponential combine into 4-momentum conserving delta functions. To see this explicitly, let us work out another example from first principles. Consider the process of fermion–antifermion annihilation into two scalars:


(4.150)

The relevant correlation function (with contractions already indicated) is

$$\begin{aligned}
 & \langle 0 | T \bar{\psi}(x_1) \psi(x_2) \phi(x_3) \phi(x_4) (-ig) \int d^4x \bar{\psi} \psi \phi (-ig) \int d^4y \bar{\psi} \psi \phi | 0 \rangle \\
 &= (-ig)^2 \int d^4x \int d^4y (-1)^5 S_F(x-x_1) S_F(x_2-y) S_F(y-x) D_F(x-x_4) D_F(y-x_3),
 \end{aligned}
 \tag{4.151}$$

where the coordinate  $x_1$  corresponds to the incoming fermion,  $x_2$  to the incoming antifermion,  $x_3$  to the upper outgoing scalar, and  $x_4$  to the lower outgoing scalar. Note the factor of  $(-1)^5$  coming from the interchange of fermion fields necessary to put the operators contracted to  $S_F(x-x_1)$ ,  $S_F(x_2-y)$  and  $S_F(y-x)$  into the right order ( $\psi$  before  $\bar{\psi}$ ). Applying the LSZ formula, we obtain

$$\begin{aligned}
 & i\mathcal{M} \cdot (2\pi)^4 \delta^{(4)}(p + p' - k - k') \\
 &= (-ig)^2 \int d^4x \int d^4y \int \frac{d^4q}{(2\pi)^4} \left[ \bar{v}(p') \frac{i(\not{q} + m_\psi)}{q^2 - m_\psi^2} u(p) \right] e^{-i(p-k-q)x} e^{-i(p'-k'+q)y}.
 \end{aligned}
 \tag{4.152}$$

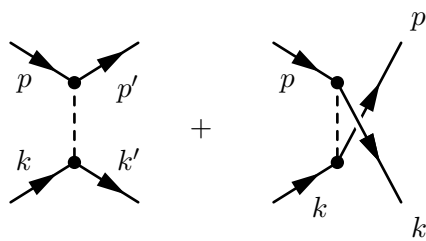
We see that the  $d^4x$  and  $d^4y$  integrals give delta functions that require the momentum in the internal fermion propagator to flow from  $x$  to  $y$ , i.e. along the arrow.

- Perhaps the least useful Feynman rule is #12 (figure out the overall sign) because it doesn't really tell us what to do. In the above example of fermion–antifermion annihilation, we have already seen what is needed: we have to count the number of

anticommutation operations necessary to put all pairs of contracted fermion fields next to each other, with the  $\psi$  field to the right of the  $\bar{\psi}$  field. In eq. (4.151), 5 such operations were required. In other words, we have to write down the correlation function leading to the Feynman diagram in question, together with the contraction structure leading to that diagram. Then, we need to figure out the number of required contractions.

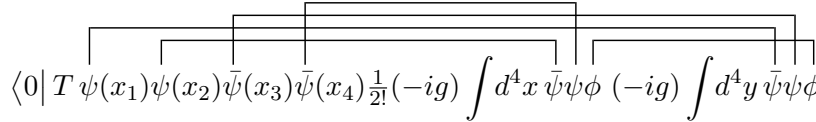
At this point, an ambiguity arises because the field operators corresponding to the external fermions can be in arbitrary order in the correlation function. Interchanging two of them would lead to an extra minus sign. However, this minus sign would be common to all diagrams contributing to a given scattering process, and since physical observables depend only on the absolute value of the total matrix element and on the *relative* signs of individual diagrams, such an overall minus sign is irrelevant. In other words, we are free to put the operators corresponding to incoming and outgoing fields in arbitrary order, as long as we pick the same order for all diagrams.

To illustrate this point, let us go back to the example of fermion–fermion scattering,



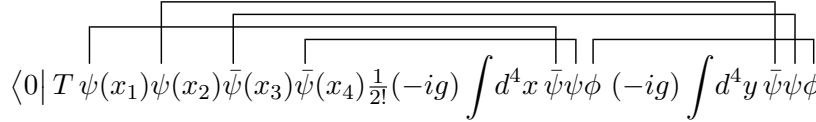
$$(4.153)$$

We repeat here for convenience the contractions from eq. (4.134), corresponding to the first diagram:



$$\langle 0 | T \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) \frac{1}{2!} (-ig) \int d^4x \bar{\psi} \psi \phi (-ig) \int d^4y \bar{\psi} \psi \phi | 0 \rangle. \quad (4.154)$$

As it is written here, this contraction structure does not come with an extra minus sign. The contractions corresponding to the second diagram, however, are



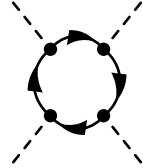
$$\langle 0 | T \psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4) \frac{1}{2!} (-ig) \int d^4x \bar{\psi} \psi \phi (-ig) \int d^4y \bar{\psi} \psi \phi | 0 \rangle. \quad (4.155)$$

Therefore, when applying the Feynman rules to write down the expression for the amplitude of fermion–fermion scattering, the two diagrams should come with a relative minus sign:

$$i\mathcal{M} = (-ig)^2 \left( [\bar{u}^{s_1}(p') u^{s_3}(p)] \frac{i}{(p' - p)^2 - m_\phi^2} [\bar{u}^{s_2}(k') u^{s_4}(k)] \right)$$

$$- [\bar{u}^{s_1}(p') u^{s_4}(k)] \frac{i}{(k' - p)^2 - m_\phi^2} [\bar{u}^{s_2}(k') u^{s_3}(p)] \Bigg). \quad (4.156)$$

One special case is that of closed fermion loops. Consider for instance the leading order diagram contributing to the scattering of two scalars:



$$\sim \overline{\psi\psi\psi\psi\psi\psi}. \quad (4.157)$$

We see that a closed fermion loop in a diagram *always* leads to an extra minus sign.

#### 4.6.5 The Yukawa Potential

We now have all the tools to compute scattering amplitudes in a straightforward way, it is perhaps time to finally compute some actual physical observable. We will consider the scattering of non-relativistic fermions (e.g. nucleons) through the Yukawa interaction (e.g. mediated by mesons) and compute the associated nucleon–nucleon potential  $V(r)$ . Our strategy is to compute the scattering matrix element first in QFT based on Feynman diagrams, then in QM based on a Hamiltonian  $H = H_0 + V(r)$ . By comparing the two expressions, we can determine  $V(r)$ .

We assume one of the two fermions to be much heavier than the other so that its 4-momentum remains essentially unchanged in the scattering process and the approximation of a perfect central potential holds. The only way in which the mass difference of the two fermions alters the process depicted in eq. (4.153) is by making the two particles distinguishable, so that the second Feynman diagrams (with crossed outgoing legs) is absent. To evaluate the first one (first line of eq. (4.156)), we need to compute spinor products of the form  $\bar{u}(p')u(p)$  in the non-relativistic limit. This is fairly easy because in that limit,  $u^s(p) \simeq \sqrt{m}(\xi^s, \xi^s)$ , i.e. the 3-momentum dependence in the spinors is negligible. We therefore have

$$\bar{u}^s(p')u^{s'}(p) = 2m\xi^{s\dagger}\xi^{s'} = 2m\delta^{ss'} \quad (4.158)$$

For the propagator, we also need

$$(p' - p)^2 \simeq -|\mathbf{p}' - \mathbf{p}|^2 + \mathcal{O}(\mathbf{p}^4). \quad (4.159)$$

The matrix element is thus

$$i\mathcal{M} = \frac{2m_{\psi 1} 2m_{\psi 2} i g^2 \delta^{ss'} \delta^{rr'}}{|\mathbf{p}' - \mathbf{p}|^2 + m_\phi^2}, \quad (4.160)$$

where, by assumption, the masses of the two fermions satisfy  $m_{\psi_2} \gg m_{\psi_1}$ . The scattering amplitude is as usual

$$\langle f|i \rangle = i\mathcal{M}(2\pi)^4 \delta^{(4)}(p + k - p' - k'). \quad (4.161)$$

In non-relativistic quantum mechanics, the amplitude for scattering of a particle off a potential  $V(r)$  is

$$\langle f|i \rangle = \lim_{T \rightarrow \infty} {}_T\langle \mathbf{p}' | e^{-i \int_{-T}^T dt [-\nabla^2/(2m_{\psi_1}) + V(t, \mathbf{x})]} | \mathbf{p} \rangle_{-T} \cdot {}_T\langle \mathbf{k}' | e^{-i \int_{-T}^T dt [-\nabla^2/(2m_{\psi_2})]} | \mathbf{k} \rangle_{-T}. \quad (4.162)$$

The subscripts  $\pm T$  on the bra and ket vectors indicate that the states are taken at  $t = T \rightarrow \pm\infty$  and then evolved in time with the Hamiltonian  $H = H_0 + V(t, \mathbf{x})$ . In the second scalar product in eq. (4.162), which comes from the very heavy fermion, we have taken into account that this fermion is too heavy to be affected by the scattering process, so the potential can be neglected in its Hamiltonian. Note also that, for very heavy  $m_{\psi_2}$ , the particle can absorb any momentum without its state changing. Therefore, we can without loss of generality replace  $\langle \mathbf{k}' |$  by  $\langle \mathbf{k}' - \mathbf{k} - \mathbf{p} |$ . (We do this in order to bring the final form of  $\langle f|i \rangle$  to the same form as eq. (4.161), see below.)

The first factor on the right hand side of eq. (4.162) can be evaluated perturbatively by expanding the exponential in  $V(t, \mathbf{x})$ . The zeroth order term in this expansion is uninteresting because no momentum is exchanged. We thus focus on the first order term. Assuming that the potential is time-independent, it yields

$$\lim_{T \rightarrow \infty} {}_T\langle \mathbf{p}' | -i \int dt e^{-i \int_t^T dt' H_0} V(\mathbf{x}) e^{-i \int_{-T}^t dt' H_0} | \mathbf{p} \rangle_{-T} \quad (4.163)$$

$$= \lim_{T \rightarrow \infty} -i \int dt \int d^3x e^{-i(E_{\mathbf{p}} - E_{\mathbf{p}'})t + i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} V(\mathbf{x}) \quad (4.164)$$

$$= -i\tilde{V}(\mathbf{p}' - \mathbf{p}) \cdot 2\pi \delta(E_{\mathbf{p}'} - E_{\mathbf{p}}) \quad (4.165)$$

$$= -i\tilde{V}(\mathbf{p}' - \mathbf{p}) \cdot 2\pi \delta(E_{\mathbf{p}} + E_{\mathbf{k}} - E_{\mathbf{p}'} - E_{\mathbf{k}'}). \quad (4.166)$$

In the second line, we have used that  $|\mathbf{p}\rangle$  and  $|\mathbf{p}'\rangle$  are eigenstates of the unperturbed Hamiltonian  $H_0$ , and we have replaced the bra and ket vectors by the (time-dependent) wave functions of the initial and final states. In the third line, we have introduced the 3-dimensional Fourier transform  $\tilde{V}(\mathbf{p}' - \mathbf{p})$  of the potential  $V(\mathbf{x})$ . In the fourth line, we have rewritten the  $\delta$ -function to bring it to the form appearing in the QFT expression eq. (4.161). In doing so, we have used that  $m_{\psi_2}$  is very large, so that  $E_{\mathbf{k}} - E_{\mathbf{k}'} \simeq 0$ . After a similar calculation for the second term in eq. (4.162), we obtain

$$\langle f|i \rangle = -i\tilde{V}(\mathbf{p}' - \mathbf{p}) \cdot (2\pi)^4 \delta^{(4)}(p + k - p' - k'). \quad (4.167)$$

Before directly comparing eq. (4.167) to the QFT expressions eqs. (4.160) and (4.161), we need to remember that in QM a one-particle state  $|\mathbf{p}\rangle$  is usually normalized according

to  $\langle \mathbf{p} | \mathbf{p}' \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ , whereas in QFT we have used the Lorentz-invariant normalization  $\langle \mathbf{p} | \mathbf{p}' \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ . Therefore, to make the comparison, we should drop the factor  $2m_{\psi_1} 2m_{\psi_2}$  in eq. (4.160). We can then conclude that

$$\tilde{V}(\mathbf{p}' - \mathbf{p}) = \frac{-g^2}{|\mathbf{p}' - \mathbf{p}|^2 + m_\phi^2}. \quad (4.168)$$

All that's left to do to obtain  $V(r)$  is to carry out an inverse Fourier transform:

$$\begin{aligned} V(\mathbf{x}) &= \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{\mathbf{p}^2 + m_\phi^2} e^{i\mathbf{q}\mathbf{x}} \\ &= \frac{-g^2}{4\pi^2} \int_0^\infty dq q^2 \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2 + m_\phi^2} \\ &= \frac{-g^2}{4\pi^2 i r} \int_{-\infty}^\infty dq \frac{q e^{iqr}}{q^2 + m_\phi^2} \end{aligned} \quad (4.169)$$

This integral can be carried out using the residual theorem. The integration contour can be closed in the complex  $q$ -plane by an infinite half-circle in the upper half-plane. Thanks to the exponential, the integrand vanishes along that half-circle. We then pick up the pole at  $q = +im_\phi$  and get

$$\begin{aligned} V(\mathbf{x}) &= \frac{-g^2}{4\pi^2 i r} \cdot 2\pi i \frac{im_\phi e^{-rm_\phi}}{2im_\phi} \\ &= -\frac{g^2}{4\pi} \frac{1}{r} e^{-rm_\phi}, \end{aligned} \quad (4.170)$$

where  $r \equiv |\mathbf{x}|$ . The minus sign indicates that the Yukawa interaction leads to an *attractive force*. The exponential limits the range of the force to distances  $\lesssim m_\phi^{-1}$ . For instance, for nucleon–nucleon interactions mediated by pions, the range of the interaction is  $\simeq m_\pi^{-1} \sim (100 \text{ MeV})^{-1} \sim \mathcal{O}(\text{fm})$ .



# 5

## Quantum Electrodynamics

After lots of formal developments, this section will be somewhat more practical. We will abandon the toy models considered so far and move on to a theory of great practical relevance: quantum electrodynamics (QED). In the context of QED, we will develop the technology to efficiently calculate tree level Feynman diagrams and the associated cross sections and decay rates.

### 5.1 The QED Lagrangian from Symmetry Arguments

Quantum electrodynamics is the simplest example of a *gauge theory*—a theory that is invariant under a certain kind of symmetry called a *gauge symmetry*. In fact, one can (and, in particle physics, usually does) argue that QED is *defined* through its gauge symmetry. Let’s see how this argument goes.

We start from the theory of a free fermion, let’s called it the electron:

$$\mathcal{L}_\psi \equiv \bar{\psi}(i\cancel{\partial} - m)\psi. \quad (5.1)$$

We demand that the theory of electrons be invariant under  $U(1)$  *gauge transformations* of the form

$$\psi(x) \rightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x), \quad (5.2)$$

where  $\alpha(x)$  is an arbitrary function of  $x$ . The crucial point here is the  $x$ -dependence of  $\alpha$ . This is what makes the symmetry a gauge symmetry or *local symmetry*, as opposed to a global symmetry, for which the transformation law is the same at each spacetime point. The symmetry is called a  $U(1)$  gauge symmetry because, at fixed  $x$ , the set of all possible transformations forms a mathematical group called  $U(1)$  (for “unitary transformations in one dimension”).

Obviously, the free Dirac Lagrangian eq. (5.1) is not invariant under  $U(1)$  gauge transformations because of the derivative. Rather,  $\mathcal{L}_\psi$  transforms as

$$\mathcal{L}_\psi \rightarrow \mathcal{L}'_\psi \equiv \bar{\psi}(i\not{\partial} - m)\psi + \bar{\psi}\gamma^\mu\psi(\partial_\mu\alpha). \quad (5.3)$$

Gauge invariance could be achieved if the theory contained another field  $A^\mu$  (judiciously called the photon) that couples to  $\psi$  through a term

$$\mathcal{L}_{\text{int}} \equiv -e\bar{\psi}\gamma^\mu\psi A_\mu \quad (5.4)$$

and has the gauge transformation property

$$A^\mu \rightarrow A'^\mu \equiv A^\mu + \frac{1}{e}(\partial^\mu\alpha). \quad (5.5)$$

Gauge invariance immediately dictates many of the properties of the photon field. First, it must be bosonic because  $\alpha(x)$  is just a number, so if  $A^\mu$  was fermionic, the gauge transformation eq. (5.5) would destroy the canonical anticommutation relations. Moreover, the interaction term eq. (5.4) would not be Lorentz invariant. (Three spin 1/2 fields can never couple to a scalar quantity.) More specifically,  $A^\mu$  should be a *real* bosonic field, otherwise the operator  $\bar{\psi}\gamma^\mu\psi A_\mu$  would not be Hermitian. Moreover,  $A^\mu$  must be a vector field, i.e. it carries a Lorentz index and thus has 4 components. Otherwise, a transformation law like eq. (5.5) would not be Lorentz invariant. Finally, the photon must be massless because mass term of the form  $m^2 A^\mu A_\mu$  would not be gauge invariant.<sup>1</sup> What gauge symmetry does not tell us is the value of the coupling constant  $e$ , which will be interpreted as the electric unit charge.

The final missing ingredient is a kinetic term for the photon. The interaction term (5.4) only describes photon interactions, but would not lead to a photon propagator. To find the form of the photon kinetic term, we use our knowledge from classical electrodynamics, where the Lagrangian describing photons is

$$\mathcal{L}_\gamma \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (5.6)$$

Here, the field strength tensor is defined as

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (5.7)$$

It is easy to check that, with the Lagrangian (5.6), the Euler–Lagrange equations lead to Maxwell’s equations:

$$0 = \partial^\mu \frac{\delta\mathcal{L}_\gamma}{\delta(\partial^\mu A^\nu)}$$

---

<sup>1</sup>There are massive gauge bosons in nature—the  $W$  and  $Z$  bosons mediating the weak interaction. Understanding how they obtain their mass through the Higgs mechanism will be one of the climaxes of the second part of this course.

$$\begin{aligned}
 &= \partial^\mu \frac{\delta}{\delta(\partial^\mu A^\nu)} [\partial^\rho A^\sigma - \partial^\sigma A^\rho] [\partial_\rho A_\sigma - \partial_\sigma A_\rho] \\
 &= \partial^\mu \frac{\delta}{\delta(\partial^\mu A^\nu)} [2(\partial^\rho A^\sigma)(\partial_\rho A_\sigma) - 2(\partial^\rho A^\sigma)(\partial_\sigma A_\rho)] \quad (5.8)
 \end{aligned}$$

$$\begin{aligned}
 &= \partial^\mu [4\partial_\mu A_\nu - 4\partial_\nu A_\mu] \\
 &= 4\partial^\mu F_{\mu\nu}. \quad (5.9)
 \end{aligned}$$

This is just Maxwell's first equation in covariant form:  $\partial^\mu F_{\mu\nu} = 0$ .<sup>2</sup> In summary, the QED Lagrangian reads in total

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu. \quad (5.10)$$

As a final remark related to considerations of gauge symmetry, let us apply Noether's theorem. The conserved current is

$$j^\mu = \frac{\delta\mathcal{L}_{\text{QED}}}{\delta(\partial_\mu\psi)} \cdot \underbrace{\Delta\psi}_{=-i\alpha\psi} \propto \bar{\psi}\gamma^\mu\psi. \quad (5.11)$$

## 5.2 The Feynman Rules for QED

The Feynman rules for QED can be derived in the same way as those of Yukawa theory. The only exception is the photon propagator, a rigorous derivation of which we will defer to a later part of this course, when we will have the path integral formalism at our disposal.

The rules for incoming and outgoing (anti)fermions and for the fermion propagator are completely identical to the ones in Yukawa theory. To guesstimate the rules for photons, let us Fourier expand the photon field:

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\mathbf{p}}}} \sum_s \left( a_{\mathbf{p}}^s \epsilon_\mu^s(p) e^{-ipx} + a_{\mathbf{p}}^{s\dagger} \epsilon_\mu^{s*}(p) e^{ipx} \right). \quad (5.12)$$

We have here exploited the fact that  $A^\mu$  is a *real* field. Otherwise, the coefficients of the positive and negative frequency components would not be the complex conjugates of each other. Since  $A^\mu$  is bosonic, the creation and annihilation operators  $a_{\mathbf{p}}^{s\dagger}$  and  $a_{\mathbf{p}}^s$  obey canonical commutation relations of the form of eq. (2.45).

The numerical coefficients  $\epsilon_\mu^s$  account for the Lorentz structure of the field and will be interpreted as photon polarization vectors. This is in full analogy to the  $u$  and  $v$  spinors in the expansion of the Dirac field. The  $u$  and  $v$  spinors were solutions to the free Dirac equation, and the field was expanded in these solutions. Here, the  $\epsilon_\mu^s$  are solutions

<sup>2</sup>Note that the second of Maxwell's equations,  $\partial^\mu(\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}) = 0$ , is automatically satisfied because

$$\partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\mu} + \partial^\sigma F^{\mu\rho} = 0.$$

to the free Maxwell equations. The sum runs over a complete basis of such solutions. This is actually where the complications with quantizing the photon field come into play: we know that physical photons are always transversely polarized. Therefore, we must restrict the polarization vectors of external photons to have the form  $\epsilon^\mu = (0, \boldsymbol{\epsilon})$ , with  $\boldsymbol{\epsilon} \cdot \mathbf{p} = 0$ . A suitable choice for  $\mathbf{p}$  along the  $z$  axis is  $\epsilon^\mu = (0, 1, \pm i, 0)/\sqrt{2}$ , where the plus sign corresponds to right-handed polarization and the minus sign corresponds to left-handed polarization. With this restriction, we can guess the Feynman rules for incoming (outgoing) photons: they will contribute just a factor  $\epsilon^\mu$  ( $\epsilon^{\mu*}$ ), just as the appearance of  $u$  and  $v$  spinors in the expansion of the Dirac field lead to the appearance of these spinors in the Feynman rules.

The restriction of transversality does not necessarily need to apply to off-shell photons (photons with  $p^2 \neq 0$ ) which can propagate along an internal photon line in a Feynman diagram. Nevertheless, even off-shell photons do not have 4 degrees of freedom because of gauge invariance. The freedom to choose a gauge removes one degree of freedom. This has to be carefully taken into account when deriving the expression for the photon propagator. Here, we will settle for a naïve guesstimate: the propagator should somehow resemble that of the scalar field. However, since the propagator should again be just a two point correlation function of the form  $\langle 0|A^\mu(x)A^\nu(y)|0\rangle$ , it must have some Lorentz structure. The simplest choice is a factor  $g^{\mu\nu}$ . We will settle for that, but we will add an extra minus sign and define the photon propagator as

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}e^{-ip(x-y)}}{p^2 + i\epsilon}. \quad (5.13)$$

The rationale for the extra minus sign in the numerator is as follows: For  $x \rightarrow y$  and  $\mu = \nu$ , the propagator is just the norm of  $A^\mu(x)$ , and that norm should be positive. Evaluating the integral over  $p^0$  in eq. (5.13) using complex contour integration and assuming  $x^0 > y^0$ , we have

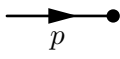
$$\begin{aligned} \Delta^{\mu\nu}(x-y) &\stackrel{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2|\mathbf{p}|} e^{-ip(x-y)} \cdot (-g^{\mu\nu}) \\ &= \frac{-g^{\mu\nu}}{4\pi^2} \int d|\mathbf{p}| \frac{|\mathbf{p}|^2}{2|\mathbf{p}|} \frac{e^{i|\mathbf{p}||\mathbf{x}-\mathbf{y}|} - e^{-i|\mathbf{p}||\mathbf{x}-\mathbf{y}|}}{i|\mathbf{p}||\mathbf{x}-\mathbf{y}|} \\ &= \frac{-g^{\mu\nu}}{8\pi|\mathbf{x}-\mathbf{y}|}. \end{aligned} \quad (5.14)$$

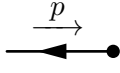
Thus, the spatial components of  $A^\mu$  will have positive norm, while the 0-component has negative norm. This means that at least physical (transversely polarized) photons are properly normalized. Regarding the unphysical time-like photons, we do not have the tools yet to deal with them. We will see later that their contributions exactly cancel against contributions from longitudinal photons (polarized along  $\mathbf{p}$ ), so that only the physical polarization states remain.

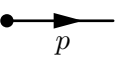
The Feynman rule for the interaction vertex between photons and fermions can be directly read off from the Lagrangian. As in the previously considered theories, all

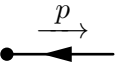
fields appearing in the interaction term end up contracted with other field operators, yielding propagators, and the coupling constant (which here includes a  $\gamma^\mu$  matrix is copied verbatim, but receives an extra factor of  $i$  from the fact that the master formula eq. (4.40) involves an exponential of  $-i \int H_{\text{int}}$  (or equivalently  $+i \int \mathcal{L}_{\text{int}}$ ), not just  $\mathcal{L}_{\text{int}}$ .

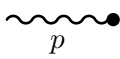
We can thus summarize the QED Feynman rules as follows:

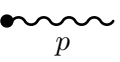
1. Incoming fermion  =  $u^s(p)$  (5.15)

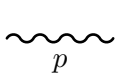
2. Incoming antifermion  =  $\bar{v}^s(p)$  (5.16)

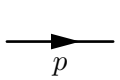
3. Outgoing fermion  =  $\bar{u}^s(p)$  (5.17)

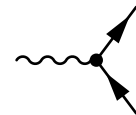
4. Outgoing antifermion  =  $v^s(p)$  (5.18)

5. Incoming photon  =  $\epsilon^\mu$  (5.19)

6. Outgoing photon  =  $\epsilon^{\mu*}$  (5.20)

7. Photon propagator  =  $\frac{-ig^{\mu\nu}}{p^2 + i\epsilon}$  (5.21)

8. Fermion propagator  =  $\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$  (5.22)

9. Vertex  =  $-ie\gamma^\mu$  (5.23)

10. Impose 4-momentum conservation at each vertex.

11. Integrate over momenta not determined by 10.:  $\int \frac{d^4 p}{(2\pi)^4}$  (5.24)

12. Figure out the overall sign of the diagram.

Note that, as in Yukawa theory, QED does not have symmetry factors because the three fields entering the vertex are all distinct.

### 5.3 $e^+e^- \rightarrow \mu^+\mu^-$

We now apply the QED Feynman rules to a simple process: the annihilation of an electron and a positron into a muon and an antimuon,  $e^+e^- \rightarrow \mu^+\mu^-$ . This was for instance one of the most important processes at the Large Electron Positron (LEP) collider at CERN. Similar processes (with the electron and the positron replaced by a quark and an antiquark) are of utmost importance to the LHC. Our first goal is to compute the cross section for  $e^+e^- \rightarrow \mu^+\mu^-$ , developing along the way several important computational techniques that are crucial to the efficient evaluation of Feynman diagrams.

#### 5.3.1 Feynman Diagram and Squared Matrix Element

Let us first apply the Feynman rules, though. Compared to vanilla QED, we have two fermion fields here—the electron and the muon—but their coupling to the photon are identical, therefore the generalization of the Feynman rules to the two-fermion case is trivial. We obtain

$$= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k'). \quad (5.25)$$

Since our goal is to compute a cross section, we ultimately need not the matrix element  $i\mathcal{M}$ , but its modulus square,  $|\mathcal{M}|^2$ . We thus have to multiply eq. (5.25) by its complex conjugate. Along the way, we encounter the complex conjugates of Dirac field bilinearies, e.g.

$$\begin{aligned} (\bar{v}^{s'}(p')\gamma^\mu u^s(p))^* &= [u^s(p)]^\dagger (\gamma^\mu)^\dagger (\gamma^0)^\dagger v^{s'}(p') \\ &= \bar{u}^s(p)\gamma^\mu v^{s'}(p'). \end{aligned} \quad (5.26)$$

In the last step, we have used the identity

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \quad (5.27)$$

which is most easily shown by direct calculation using the explicit form eq. (3.4) of the gamma matrices. The squared matrix element is therefore

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{e^4}{q^4} \left( \bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^{s'}(p') \right) \left( \bar{u}^r(k)\gamma_\mu v^{r'}(k')\bar{v}^{r'}(k')\gamma_\nu u^r(k) \right) \\ &= \frac{e^4}{q^4} \text{tr} \left( v^{s'}(p')\bar{v}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu \right) \text{tr} \left( u^r(k)\bar{u}^r(k)\gamma_\mu v^{r'}(k')\bar{v}^{r'}(k')\gamma_\nu \right). \end{aligned} \quad (5.28)$$

The rearrangements in the second line will come in handy below. In practice, we often have no control over the spin orientation of the initial state particles, and we do not measure the spin orientation of the final state particles. For instance, in a particle collider, it is very difficult to produce polarized beams—it requires that the electrons have some polarizing interaction and to retain their polarization until they collide. One possibility is the so-called Sokolov–Ternov effect: when electrons emit synchrotron radiation in a magnetic field, their spin flips. Since the two spin states have different energies in the magnetic field, transitions from the higher energy state to the lower energy one are slightly more probable than interactions going the opposite way. After the electrons have been in a storage ring for a long time, they can achieve a polarization up to 70%. Measuring the spin of a final state particle can be done by observing the kinematics (in particular angular correlations) between its decay products. In the case of muons, however, this is difficult because they are too long-lived to decay inside a typical detector.

The upshot is that we are often interested in *unpolarized cross sections*, i.e. cross sections averaged over initial state spins and summed over final state spins. We have already seen how to evaluate spin sums in section 3.1.3. Using the relations  $\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \not{p} + m$  and  $\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \not{p} - m$ , we obtain

$$\begin{aligned} \overline{|\mathcal{M}|^2} &\equiv \frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_r \sum_{r'} |\mathcal{M}|^2 \\ &= \frac{e^4}{4q^4} \text{tr}[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu] \text{tr}[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu] \end{aligned} \quad (5.29)$$

Don't let yourself get confused by the fact that  $\mu$  appears both as a Lorentz index here and as an index distinguishing the electron mass  $m_e$  from the muon mass  $m_\mu$ .

### 5.3.2 Trace Technology

The reason an expression like eq. (5.29) is very convenient is that several identities exist that make the evaluation of traces over gamma matrices very efficient. First, note that the trace of a single gamma matrix vanishes:

$$\text{tr} \gamma^\mu = 0. \quad (5.30)$$

In fact the trace of any product of an odd number of gamma matrices vanishes because, using  $(\gamma^5)^2 = 1$  and the cyclic property of the trace, we have

$$\text{tr} \gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}} = \text{tr} (\gamma^5)^2 \gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}} \quad (5.31)$$

$$= -\text{tr} \gamma^5 \gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}} \gamma^5 \quad (5.32)$$

$$= -\text{tr} \gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}}. \quad (5.33)$$

Thus, this trace must be zero.

For the product of two gamma matrices, we have

$$\text{tr} \gamma^\mu \gamma^\nu = \text{tr} [2g^{\mu\nu} \cdot \mathbb{1}_{4 \times 4} - \gamma^\nu \gamma^\mu]$$

$$= 8g^{\mu\nu} - \text{tr } \gamma^\mu \gamma^\nu. \quad (5.34)$$

Therefore,

$$\text{tr } \gamma^\mu \gamma^\nu = 4g^{\mu\nu}. \quad (5.35)$$

For any even number  $2n$  of gamma matrices, we can use the anticommutator  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  to move the leftmost gamma matrix all the way to the right and then use the cyclic property of the trace to bring it back to the left. This leads to the relation

$$\begin{aligned} \text{tr } \gamma^{\mu_1} \dots \gamma^{\mu_{2n}} &= \text{tr} [2g^{\mu_1 \mu_2} \gamma^{\mu_3} \dots \gamma^{\mu_{2n}} - \gamma^{\mu_2} 2g^{\mu_1 \mu_3} \gamma^{\mu_4} \dots \gamma^{\mu_{2n}} \\ &\quad + \dots + \gamma^{\mu_2} \dots \gamma^{\mu_{2n-1}} 2g^{\mu_1 \mu_{2n}} - \gamma^{\mu_2} \dots \gamma^{\mu_{2n}} \gamma^{\mu_1}], \end{aligned} \quad (5.36)$$

which can be used to reduce the trace of  $2n$  gamma matrices to a sum of traces of  $2n-2$  gamma matrices. (In the last term on the right hand side, the cyclic property of the trace needs to be exploited to bring it to the same form as the left hand side.) In particular, for four gamma matrices, this leads to

$$\begin{aligned} \text{tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma &= \text{tr} [g^{\mu\nu} \gamma^\rho \gamma^\sigma - \gamma^\nu g^{\mu\rho} \gamma^\sigma + \gamma^\nu \gamma^\rho g^{\mu\sigma}] \\ &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}). \end{aligned} \quad (5.37)$$

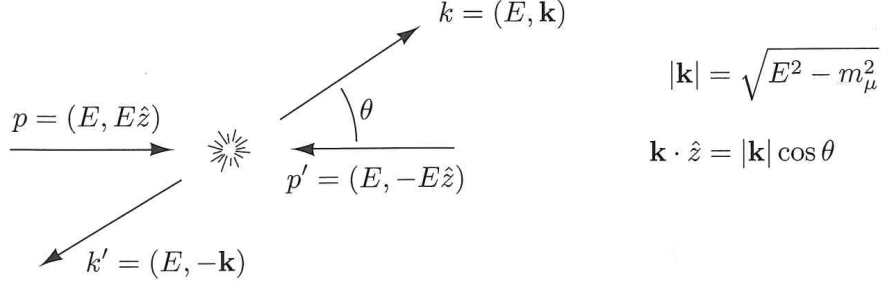
Even though these are all the relations we need to evaluate the squared matrix element for the process  $e^+ e^- \rightarrow \mu^+ \mu^-$ , let us also mention some identities we will encounter later. This concerns in particular traces involving  $\gamma^5$ . Since  $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$ , it counts as an *even* number of gamma matrices. This means in particular that any trace involving one or several  $\gamma^5$  and an *odd* number of  $\gamma^0, \dots, \gamma^3$ , vanishes. Also, the trace of  $\gamma^5$  itself vanishes. This is immediately obvious from the explicit form  $\gamma^5 = \text{diag}(-\mathbb{1}_{2 \times 2}, \mathbb{1}_{2 \times 2})$  in the chiral representation, but can also be shown more abstractly using only the representation invariant properties of the gamma matrices:

$$\text{tr } \gamma^5 = \text{tr } \gamma^0 \gamma^0 \gamma^5 = -\text{tr } \gamma^0 \gamma^5 \gamma^0 = -\text{tr } \gamma^0 \gamma^0 \gamma^5 = -\text{tr } \gamma^5. \quad (5.38)$$

Similarly, for  $\text{tr } \gamma^5 \gamma^\mu \gamma^\nu$ , we can insert the square of a gamma matrix  $\gamma^\rho$  with  $\rho \neq \mu, \nu$  then anticommute one  $\gamma^\rho$  factor all the way to the right to show that  $\text{tr } \gamma^5 \gamma^\mu \gamma^\nu$  is equal to its negative and therefore must be zero. The same procedure works for a combination of  $\gamma^5$  with four other gamma matrices, except in the case where those four matrices are all distinct, i.e. each  $\gamma^0, \dots, \gamma^3$  appears exactly once. Therefore,  $\text{tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$  is proportional to  $\epsilon^{\mu\nu\rho\sigma}$ . By using  $\mu\nu\rho\sigma = 0123$  and calculating explicitly, we find the prefactor to be  $-4i$ . Thus

$$\text{tr } \gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4i \epsilon^{\mu\nu\rho\sigma}. \quad (5.39)$$





**Figure 5.1:** Kinematics of  $e^+e^- \rightarrow \mu^+\mu^-$ . Figure taken from [1].

### 5.3.3 The Squared Matrix Element for $e^+e^- \rightarrow \mu^+\mu^-$ (Part II)

We are now ready to evaluate the spin-averaged matrix element (5.29). The first trace is

$$\begin{aligned} \text{tr}[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu] &= \text{tr}\not{p}'\gamma^\mu\not{p}\gamma^\nu - 4m_e^2g^{\mu\nu} \\ &= 4(p'^\mu p^\nu - (p' \cdot p)g^{\mu\nu} + p'^\nu p^\mu) - 4m_e^2g^{\mu\nu}, \end{aligned} \quad (5.40)$$

and a similar expression holds for the second trace:

$$\text{tr}[(\not{k}' - m_\mu)\gamma_\nu(\not{k} + m_\mu)\gamma_\mu] = 4(k'_\nu k_\mu - (k' \cdot k)g_{\mu\nu} + k'_\mu k_\nu) - 4m_\mu^2g_{\mu\nu}, \quad (5.41)$$

From here on, we will set  $m_e = 0$  to keep our expressions shorter. At a typical high energy collider, where kinetic energies are  $\gg$  GeV, the electron mass of 511 keV is indeed negligible. The squared matrix element is then, after contracting Lorentz indices,

$$|\overline{\mathcal{M}}|^2 = \frac{8e^4}{q^4} \left[ (p' \cdot k')(p \cdot k) + (p' \cdot k)(p \cdot k') + m_\mu^2(p' \cdot p) \right]. \quad (5.42)$$

At this point, it is useful to be a bit more explicit about the kinematics. Let us therefore consider the collision of an  $e^+e^-$  pair in the center of mass frame, where the two particles are moving in opposite directions along the  $z$  axis. (This is the conventional coordinate system chosen for processes at a high energy collider.) This kinematic setup is illustrated in fig. 5.1. With the definitions given in the figure, we can compute the following dot products:

$$\begin{aligned} q^2 &= (p + p')^2 = 4E^2, & p \cdot p' &= 2E^2, \\ p \cdot k &= p' \cdot k' = E^2 - E|\mathbf{k}| \cos \theta, & p \cdot k' &= p' \cdot k = E^2 + E|\mathbf{k}| \cos \theta. \end{aligned} \quad (5.43)$$

With these replacements, the squared matrix element is

$$\begin{aligned} |\overline{\mathcal{M}}|^2 &= \frac{e^4}{2E^4} \left[ E^2(E - |\mathbf{k}| \cos \theta)^2 + E^2(E + |\mathbf{k}| \cos \theta)^2 + 2m_\mu^2E^2 \right] \\ &= e^4 \left[ \left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right]. \end{aligned} \quad (5.44)$$

We have written the last line such that the high energy limit  $E \gg m_\mu$ , relevant for instance for the LEP collider, is easy to take. We have eliminated  $|\mathbf{k}|^2$  in favor of  $E^2$  and  $m_\mu^2$ .

### 5.3.4 The Cross Section — General Results

The procedure for obtaining the cross section from a squared matrix element is the same in QFT as in QM. First, remember that, in our conventions for  $\mathcal{M}$ , we had pulled out a factor  $(2\pi)^4 \delta^{(4)}(p + p' - k - k')$ . We should now put that factor back. When computing the transition probability by squaring the matrix element (including the delta function), we then encounter an awkward square of a delta function. To deal with it, rewrite one of the  $\delta$ -factors by using

$$2\pi\delta(\Delta E) = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} dt e^{i\Delta E t} = \lim_{T \rightarrow \infty} \frac{2}{\Delta E} \sin \frac{\Delta E T}{2}. \quad (5.45)$$

The physical interpretation here is that the process is happening in a finite time interval  $[-T/2, T/2]$ , so the time integral that leads to the delta function only leads to an approximate delta function. The squared  $\delta$  function is thus

$$\begin{aligned} (2\pi)^2 \delta^2(\Delta E) &= (2\pi)\delta(\Delta E) \lim_{T \rightarrow \infty} \lim_{\Delta E \rightarrow 0} \frac{2}{\Delta E} \sin \frac{\Delta E T}{2} \\ &= (2\pi)\delta(\Delta E) \lim_{T \rightarrow \infty} T. \end{aligned} \quad (5.46)$$

Similar expressions hold of course for the delta functions in the other components of the 4-momentum vector. Here, the integration interval  $[-L/2, L/2]$  is interpreted as the boundaries of a large box of volume  $V = L^3$ , to which the process is confined.

We also have to worry about the normalization of the external states: remember that one-particle plane wave states are normalized according to  $\langle \mathbf{p} | \mathbf{q} \rangle = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$ . To derive a physical observable, we would rather like to have them normalized to one. This can be achieved by adding a factor  $[2E_{\mathbf{p}}]^{-1}$  for each particle. For the initial state particles, we should also divide by  $V$  (the height of the  $\delta$ -peak in a finite volume). For the final states, we instead integrate over the 3-momentum (the phase space integral).

Finally, in a collider environment we are not colliding individual electron with individual positrons, but rather beams containing many particles. We should therefore multiply by the total number of electrons,  $N_-$ , and positrons,  $N_+$ .

In total, we thus have for the transition probability:

$$\begin{aligned} \mathcal{P} = \lim_{T, L \rightarrow \infty} TV N_+ N_- \frac{1}{2E_{\mathbf{p}} V} \frac{1}{2E_{\mathbf{p}'} V} \int \frac{d^3 k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{d^3 k'}{(2\pi)^3 2E'_{\mathbf{k}}} \\ \cdot (2\pi)^4 \delta^{(4)}(p + p' - k - k') \cdot |\overline{\mathcal{M}}|^2. \end{aligned} \quad (5.47)$$

The total transition rate is this quantity divided by  $T$ , and the total cross section is

$$\sigma(e^+ e^- \rightarrow \mu^+ \mu^-) = \frac{\text{total transition rate}}{\# \text{ of target } e^- \times \text{flux of incident } e^+}, \quad (5.48)$$

where we have arbitrarily called the  $e^-$  the “target particles” and the  $e^+$  the “incident particles”. The number of target  $e^-$  in the denominator cancels the factor  $N_-$  from eq. (5.47). The  $e^+$  flux is given by the  $e^+$  number density,  $N_+/V$  times the relative velocity  $v_{\text{rel}}$  of the  $e^+$  and the  $e^-$ . Note that also the factors  $T$  and  $V$  cancel, so that our final expression for the cross section carries no trace of the fact that we temporarily worked with a finite spacetime volume. In other words, the limit for  $T, L \rightarrow \infty$  is trivial to take and we obtain a master formula for  $2 \rightarrow 2$  cross sections:

$$\sigma(e^+e^- \rightarrow \mu^+\mu^-) = \frac{1}{2E_{\mathbf{p}} 2E_{\mathbf{p}'} v_{\text{rel}}} \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{d^3k'}{(2\pi)^3 2E_{\mathbf{k}'}} \cdot (2\pi)^4 \delta^{(4)}(p + p' - k - k') \cdot |\mathcal{M}|^2. \quad (5.49)$$

For processes with more than two particles in the final state, one simply has to add more final state phase space factors  $\int d^3k_j / [(2\pi)^3 E_{\mathbf{k}_j}]$ .

For the  $2 \rightarrow 2$  case, the 4-dimensional delta function can be used to remove for instance the integral over  $d^3k'$  and the integral over  $d|\mathbf{k}|$ , leaving only the integral over the solid angle of the first final state particle. If the masses of the two final state particles are  $m$  and  $m'$ , we find

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}} \frac{d^3k'}{(2\pi)^3 2E_{\mathbf{k}'}} (2\pi)^4 \delta^{(4)}(p + p' - k - k') \\ &= \int d\Omega \int d|\mathbf{k}| \frac{|\mathbf{k}|^2}{(2\pi)^2 4E_{\mathbf{k}} E_{\mathbf{k}'}} \delta(E_{\mathbf{p}} + E_{\mathbf{p}'} - \sqrt{|\mathbf{k}|^2 + m^2} - \sqrt{|\mathbf{k}|^2 + m'^2}) \\ &= \int d\Omega \frac{|\mathbf{k}|^2}{(2\pi)^2 4E_{\mathbf{k}} E_{\mathbf{k}'}} \left( \frac{|\mathbf{k}|}{E_{\mathbf{k}}} + \frac{|\mathbf{k}|}{E_{\mathbf{k}'}} \right)^{-1} \\ &= \int d\Omega \frac{|\mathbf{k}|}{(2\pi)^2 4(E_{\mathbf{k}} + E_{\mathbf{k}'})} \end{aligned} \quad (5.50)$$

If we are not interested in the total cross section, but in a differential cross section (for instance  $d\sigma/d\Omega$ ), we simply omit the integral over the corresponding kinematic variable (for instance  $\Omega$ ) in the final state. For instance, for  $e^+e^- \rightarrow \mu^+\mu^-$ , we have in the center of mass frame,

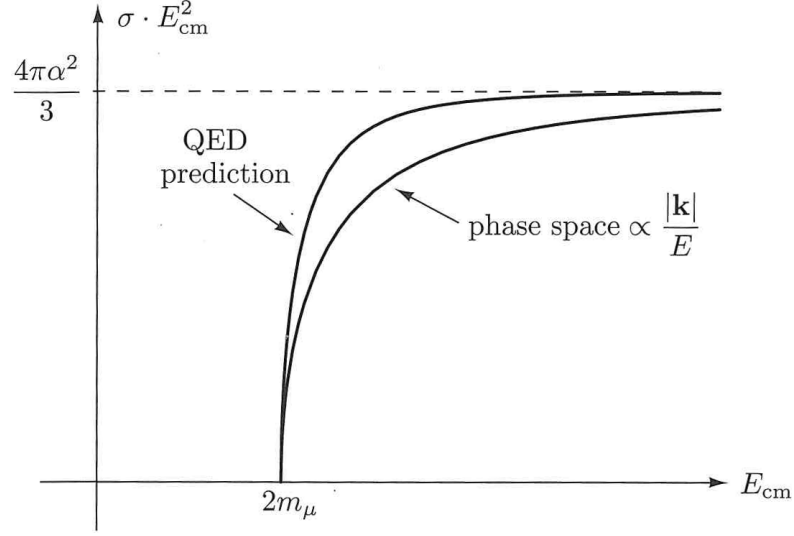
$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{128\pi^2 E^2 v_{\text{rel}}} \sqrt{1 - \frac{m_\mu^2}{E^2}}, \quad (5.51)$$

where  $E$  is the common energy of the electrons and muons.

### 5.3.5 The Cross Section for $e^+e^- \rightarrow \mu^+\mu^-$

Let us plug the squared matrix element eq. (5.44) into the cross section formula eq. (5.51):

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[ \left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right]. \quad (5.52)$$



**Figure 5.2:** The total cross section for  $e^+e^- \rightarrow \mu^+\mu^-$  as a function of the center of mass energy  $E_{\text{cm}} \equiv 2E$ . The vertical axis has been multiplied by  $E_{\text{cm}}^2$  to better expose the asymptotic behavior for  $E_{\text{cm}} \rightarrow \infty$ . Figure taken from [1].

Here we have introduced the electromagnetic fine structure constant  $\alpha \equiv e^2/(4\pi)$ . The total cross section is

$$\sigma_{\text{total}} = \frac{\pi\alpha^2}{3E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left( 1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right). \quad (5.53)$$

This expression is plotted in fig. 5.2

### 5.3.6 $e^+e^- \rightarrow \mu^+\mu^-$ : Summary

Let us briefly summarize how we obtained the cross section for the annihilation of an electron–positron pair to muons:

1. Draw all relevant Feynman diagrams (here, there was only one)
2. Apply the Feynman rules to find the amplitude  $\mathcal{M}$
3. Square it and average over initial state spins/sum over final state spins. Use eqs. (3.25) and (3.29) to rewrite expressions of the form  $\sum_s \bar{u}^s(p) u^s(p)$  as traces over momenta, masses and Dirac matrices.
4. Use the relations derived in section 5.3.2 to evaluate the traces.
5. Pick a suitable reference frame and write the 4-momenta in terms of the measurable kinematic variables in that frame.

6. Integrate  $|\overline{\mathcal{M}}|^2$  over the final state phase space according to the master formula eq. (5.49) to obtain the cross section.

## 5.4 More Technology for Evaluating QED Feynman Diagrams

Now that we have gone through one QED calculation in its full glory, we already have most of the tools we need to evaluate any tree level diagram in QED. We will practice these computational techniques thoroughly in the exercises.

A few important tricks are still missing, though.

### 5.4.1 Scattering of Polarized Particles

In section 5.3.1, we argued that it is difficult though not impossible to collide polarized beams of particles and to analyze the spin of final state particles. It is therefore sometimes important to compute scattering amplitudes between particles of definite spin. We could of course use the explicit expressions for the  $u$  and  $v$  spinors directly after applying the Feynman rules (i.e. in eq. (5.25)), but the resulting expressions can be quite horrible.

In the high energy limit, where all particles are ultrarelativistic, there is a much more elegant approach, based on the observation that chirality and helicity coincide in that limit (see section 3.4.1. Namely, we can simply restrict the amplitude to particular helicities by inserting the chirality projection operators  $P_L$ ,  $P_R$  (see eq. (3.111)) in the fermion currents. For instance, to consider the annihilation of a left-handed electron and a right-handed positron, we would replace

$$\begin{aligned}\bar{v}^{s'}(p')\gamma^\mu u^s(p) &\rightarrow [P_L v^{s'}(p')]^\dagger \gamma^0 \gamma^\mu P_L u^s(p) \\ &= \bar{v}^{s'}(p')\gamma^\mu \frac{1+\gamma^5}{2} u^s(p).\end{aligned}\tag{5.54}$$

(Remember that a *left-handed*  $v$ -spinor annihilates a *right-handed* positron.)

### 5.4.2 External Photons

According to the Feynman rules, an incoming or outgoing photon in a scattering amplitude contributes a polarization vector  $\epsilon^\mu(p)$  or  $\epsilon^{\mu*}(p)$ . If the polarization of photons is unknown, we have to sum over it when computing the squared matrix element, just as we did for fermions in eq. (5.29). In this context an extremely useful identity is

$$\sum_{\text{polarizations}} \epsilon_\mu^*(p) \epsilon_\nu(p) \leftrightarrow -g_{\mu\nu}.\tag{5.55}$$

Here,  $\leftrightarrow$  means that the left hand side and the right hand side are not identical, but when computing squared matrix elements they can be used interchangeably. Here is the

proof: consider the two physical transverse polarization states  $\epsilon_\mu^\pm(p) \equiv (0, 1, \pm i, 0)/\sqrt{2}$ , assuming that  $p$  is aligned with the  $z$  axis. Then

$$\begin{aligned} \sum_{s=+,-} \epsilon_\mu^*(p) \epsilon_\nu(p) &= \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \\ &= -g_{\mu\nu} + \hat{e}_{t\mu} \hat{e}_{t\nu} - \hat{e}_{z\mu} \hat{e}_{z\nu}, \end{aligned} \quad (5.56)$$

where  $\hat{e}_t^\mu = (1, 0, 0, 0)$  and  $\hat{e}_z^\mu = (0, 0, 0, 1)$ . Next, note that

$$\hat{e}_z^\mu = \frac{p^\mu - (\hat{e}_t \cdot p) \hat{e}_t^\mu}{\hat{e}_t \cdot p}. \quad (5.57)$$

We now argue that the term proportional to  $p^\mu$  on the right hand side does not contribute to a Feynman amplitude. The argument is gauge invariance: the matrix element of any physical process involving an external photon field  $A^\mu(x)$  is invariant under  $A^\mu(x) \rightarrow A^\mu(x) + \frac{1}{e} \partial^\mu \alpha(x)$ . In momentum space, this gauge transformation rule reads

$$\tilde{A}^\mu(p) \rightarrow \tilde{A}^\mu(p) - \frac{i}{e} p^\mu \tilde{\alpha}(p). \quad (5.58)$$

For the external photon,  $\tilde{A}^\mu(p) \propto \epsilon^\mu(p)$ . Thus, any contribution  $\propto p^\mu$  to  $\epsilon^\mu(p)$  must vanish. Therefore,

$$\sum_{s=+,-} \epsilon_\mu^*(p) \epsilon_\nu(p) \leftrightarrow -g_{\mu\nu} + \hat{e}_{t\mu} \hat{e}_{t\nu} - \hat{e}_{t\mu} \hat{e}_{t\nu} = -g_{\mu\nu}. \quad (5.59)$$

# 6

## Path Integrals

Now that we have developed and applied the techniques required to compute physical observables in quantum field theory, it is useful to pause a moment and think about the physics hiding behind the algebra. In this chapter, we will therefore discuss an approach to QFT different from the canonical quantization procedure introduced in chapters 2 and 3: the path integral formalism. In that formalism, a scattering amplitude is computed by considering all possible phase space trajectories that particle could take to go from the initial state to the final state. The amplitudes for all these paths are then integrated up, true to the superposition principle of quantum mechanics.

### 6.1 Path Integrals in Quantum Mechanics

To introduce path integrals, let us first take several steps back and consider good old non-relativistic QM again. In particular, we consider the motion of a particle with momentum  $p$  and coordinate  $q$  in one dimension. The Hamilton operator is<sup>1</sup>

$$\hat{H}(p, q) = \frac{\hat{p}^2}{2m} + V(\hat{q}). \quad (6.1)$$

Here,  $V(q)$  is the potential. We wish to compute the amplitude  $\langle q' | e^{-i\hat{H}t} | q \rangle$  for the particle to propagate from  $q$  at time 0 to  $q'$  at time  $t$ . We split the time interval  $[0, t]$  into  $n + 1$  small subintervals of length  $\delta t \equiv t/(n + 1)$ . Between intervals, we insert a complete set of coordinate eigenstates:

$$\langle q' | e^{-i\hat{H}t} | q \rangle = \int \left( \prod_{j=1}^n dq_j \right) \langle q' | e^{-i\hat{H}\delta t} | q_n \rangle \langle q_n | e^{-i\hat{H}\delta t} | q_{n-1} \rangle \cdots \langle q_1 | e^{-i\hat{H}\delta t} | q \rangle. \quad (6.2)$$

---

<sup>1</sup>For the moment, we write operators with a hat, e.g.  $\hat{H}$  and  $c$ -numbers without.

We would like to split each exponential factor into the kinetic part and the potential part using the Campbell-Baker-Hausdorff formula

$$e^{\hat{T}+\hat{V}} = e^{\hat{T}} e^{\hat{V}} e^{-\frac{1}{2}[\hat{T},\hat{V}]} + \dots \quad (6.3)$$

In the limit  $n \rightarrow \infty$ , i.e.  $\delta t \rightarrow 0$ , the commutator term, which is of order  $(\delta t)^2$  and all higher order terms can be neglected. Let us moreover insert a complete set of momentum eigenstates. Then, each factor in eq. (6.2) takes the form

$$\langle q_{j+1} | e^{-i\hat{H}\delta t} | q_j \rangle \simeq \int \frac{dp_j}{2\pi} \langle q_{j+1} | \exp \left( -i \frac{\hat{p}^2}{2m} \delta t \right) | p_j \rangle \langle p_j | \exp \left( -i V(\hat{q}) \delta t \right) | q_j \rangle \quad (6.4)$$

$$\simeq \int \frac{dp_j}{2\pi} \exp \left[ -i \frac{p_j^2}{2m} \delta t - i V(q_j) \delta t \right] e^{ip_j(q_{j+1}-q_j)} \quad (6.5)$$

$$\simeq \int \frac{dp_j}{2\pi} \exp \left[ -i \frac{p_j^2}{2m} \delta t - i V(q_j) \delta t + ip_j \dot{q}_j \delta t \right]. \quad (6.6)$$

In the second line, we have used that  $\langle q | p \rangle = e^{ipq}$ , and in the last line, we have used that  $(q_2 - q_1)/\delta t \simeq \dot{q}$ . To write the factors corresponding to first and the last time step in the form of eq. (6.6), we define  $q_0 \equiv q$ ,  $q_{n+1} \equiv q'$ . Equation (6.2) now becomes

$$\langle q' | e^{-i\hat{H}t} | q \rangle = \int \left( \prod_{j=1}^n \prod_{k=0}^n dq_j \frac{dp_k}{2\pi} \right) \exp \left[ -i \sum_{k=0}^n \left( \frac{p_k^2}{2m} + V(q_k) - p_k \dot{q}_k \right) \delta t \right]. \quad (6.7)$$

We now take the continuum limit  $n \rightarrow \infty$  and introduce the abbreviations  $\mathcal{D}q \equiv \lim_{n \rightarrow \infty} \prod_{j=1}^n dq_j$  and  $\mathcal{D}p \equiv \lim_{n \rightarrow \infty} \prod_{k=0}^n \frac{dp_k}{2\pi}$  for the integration measures. The sum in the exponent becomes an integral in the continuum limit:

$$\langle q' | e^{-i\hat{H}t} | q \rangle = \int \mathcal{D}q \mathcal{D}p \exp \left[ i \int dt \left( p \dot{q} - \frac{p^2}{2m} - V(q) \right) \right]. \quad (6.8)$$

Note that the integrand in the exponent is just the Legendre transform of the Hamiltonian, i.e. the Lagrangian. Thus,

$$\langle q' | e^{-i\hat{H}t} | q \rangle = \int \mathcal{D}q \mathcal{D}p \exp \left( i \int dt L(q, p) \right). \quad (6.9)$$

The physical interpretation of eq. (6.9) is the following: each set  $\{q_1, \dots, q_n, p_0, \dots, p_n\}$  corresponds to one possible phase space trajectory leading from  $q$  to  $q'$ . The amplitude for each path is given by the action  $\exp(i \int dt L)$ . The path integral (integral over  $\mathcal{D}q \mathcal{D}p$ ) sums up the amplitudes for all the paths. In that sense, any quantum mechanical transition process can be viewed as a continuum generalization of the two-slit experiment.

Already at this stage, we can glimpse why the path integral formalism may be useful in QFT. Imagine we want to compute the expectation value of the particle's position at a time  $t_1$ , somewhere in the interval  $[0, t]$ . It is given by

$$\langle q' | \hat{q}(t_1) | q \rangle \equiv \langle q' | e^{-iH(t-t_1)} \hat{q} e^{-iH(t_1-0)} | q \rangle$$



$$= \int \mathcal{D}q \mathcal{D}p q(t_1) \exp \left( i \int dt L(q, p) \right). \quad (6.10)$$

So far, so good. Now, let's insert one more  $q$  factor. If  $t_1 > t_2$ , we have

$$\begin{aligned} \langle q' | \hat{q}(t_1) \hat{q}(t_2) | q \rangle &\equiv \langle q' | e^{-iH(t-t_1)} \hat{q} e^{-iH(t_1-t_2)} \hat{q} e^{-iH(t_2-0)} | q \rangle \\ &= \int \mathcal{D}q \mathcal{D}p q(t_1) q(t_2) \exp \left( i \int dt L(q, p) \right), \end{aligned} \quad (6.11)$$

while for  $t_1 < t_2$ , it is

$$\begin{aligned} \langle q' | \hat{q}(t_2) \hat{q}(t_1) | q \rangle &\equiv \langle q' | e^{-iH(t-t_2)} \hat{q} e^{-iH(t_2-t_1)} \hat{q} e^{-iH(t_1-0)} | q \rangle \\ &= \int \mathcal{D}q \mathcal{D}p q(t_1) q(t_2) \exp \left( i \int dt L(q, p) \right). \end{aligned} \quad (6.12)$$

Thus, overall,

$$\int \mathcal{D}q \mathcal{D}p q(t_1) q(t_2) \exp \left( i \int dt L(q, p) \right) = \begin{cases} \langle q' | \hat{q}(t_1) \hat{q}(t_2) | q \rangle & \text{if } t_1 > t_2 \\ \langle q' | \hat{q}(t_2) \hat{q}(t_1) | q \rangle & \text{if } t_2 > t_1 \end{cases} \quad (6.13)$$

$$= \langle q' | T \hat{q}(t_1) \hat{q}(t_2) | q \rangle. \quad (6.14)$$

In other words, if we had an efficient way of evaluating path integrals, we'd have a new way of computing time-ordered correlation functions. And we have seen in our discussion of the LSZ reduction formula (section 4.4) that these are of paramount importance for computing scattering amplitudes.

We will now introduce the path integral formalism in QFT, we will argue that it leads to the same Feynman rules as our previous approach involving creation and annihilation operators (though in an arguably more elegant way), and we will use path integrals to proof the Feynman rule for the photon propagator.

## 6.2 The Path Integral for a Free Scalar Field

Generalizing the results of the previous section to a field theory is straightforward. First, the initial and final states, which were coordinate points in QM, are now scalar field configuration  $\phi(\mathbf{x}, t)$ . Also the Hamilton operator changes and becomes the one for a free scalar field (see eq. (2.46)):

$$\hat{H} = \int d^3x \left( \frac{1}{2} [\hat{\pi}(\mathbf{x}, t)]^2 + \frac{1}{2} [\nabla \hat{\phi}(\mathbf{x}, t)]^2 + \frac{1}{2} m^2 [\hat{\phi}(\mathbf{x}, t)]^2 \right), \quad (6.15)$$

with the canonical momentum  $\pi(\mathbf{x}, t) = \dot{\phi}(\mathbf{x}, t)$ . Finally, instead of discretizing the trajectory only in time, we discretize all four spacetime directions and integrate over the field value at each spacetime point. The corresponding integration measure is

$$\mathcal{D}\phi \equiv \prod_{i,j,k,l} d\phi(x_i^1, x_j^2, x_k^3, t_l). \quad (6.16)$$

Similarly, also the path integral over the canonical momentum  $\int \mathcal{D}\pi$ , is now understood as

$$\mathcal{D}\pi \equiv \prod_{i,j,k,l} \frac{d\pi(x_i^1, x_j^2, x_k^3, t_l)}{2\pi}. \quad (6.17)$$

The transition amplitude between an initial field configuration  $\phi(\mathbf{x}, 0)$  and a final field configuration  $\phi'(\mathbf{x}, t)$  is then, in complete analogy to eq. (6.2),

$$\begin{aligned} \langle \phi'(\mathbf{x}, t) | e^{-i\hat{H}t} | \phi(\mathbf{x}, 0) \rangle &= \int \mathcal{D}\phi \langle \phi'(\mathbf{x}, t) | e^{-i\hat{H}\delta t} | \phi(\mathbf{x}, t_n) \rangle \\ &\cdot \langle \phi(\mathbf{x}, t_n) | e^{-i\hat{H}\delta t} | \phi(\mathbf{x}, t_{n-1}) \rangle \cdots \langle \phi(\mathbf{x}, t_1) | e^{-i\hat{H}\delta t} | \phi(\mathbf{x}, 0) \rangle. \end{aligned} \quad (6.18)$$

Inserting complete sets of canonical momentum eigenstates, each factor in this product becomes

$$\begin{aligned} &\langle \phi'(\mathbf{x}, t_{m+1}) | e^{-i\hat{H}\delta t} | \phi(\mathbf{x}, t_m) \rangle \\ &\simeq \int \prod_{i,j,k} \frac{d\pi(x_i^1, x_j^2, x_k^3, t_m)}{2\pi} \left\langle \phi(\mathbf{x}, t_{m+1}) \left| \exp \left( -i \int d^3x \frac{1}{2} [\hat{\pi}(\mathbf{x}, t)]^2 \delta t \right) \right| \pi(\mathbf{x}, t_m) \right\rangle \\ &\cdot \left\langle \pi(\mathbf{x}, t_m) \left| \exp \left( -i \int d^3x \left\{ \frac{1}{2} [\nabla \hat{\phi}(\mathbf{x}, t)]^2 + \frac{1}{2} m^2 [\hat{\phi}(\mathbf{x}, t)]^2 \right\} \delta t \right) \right| \phi(\mathbf{x}, t_m) \right\rangle \end{aligned} \quad (6.19)$$

$$\begin{aligned} &\simeq \int \prod_{i,j,k} \frac{d\pi(x_i^1, x_j^2, x_k^3, t_m)}{2\pi} \exp \left[ -i \int d^3x \left\{ \frac{1}{2} [\pi(\mathbf{x}, t_m)]^2 + \frac{1}{2} [\nabla \phi(\mathbf{x}, t_m)]^2 \right. \right. \\ &\quad \left. \left. + \frac{1}{2} m^2 [\phi(\mathbf{x}, t_m)]^2 - \pi(\mathbf{x}, t_m) \dot{\phi}(\mathbf{x}, t_m) \right\} \delta t \right]. \end{aligned} \quad (6.20)$$

Overall, we thus find in analogy to eq. (6.7),

$$\begin{aligned} &\langle \phi'(\mathbf{x}, t) | e^{-i\hat{H}t} | \phi(\mathbf{x}, 0) \rangle \\ &= \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[ i \int_0^t dt \int d^3x \left( \pi \dot{\phi} - \frac{1}{2} \pi^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 \right) \right] \end{aligned} \quad (6.21)$$

The path integral over  $\mathcal{D}\pi$  can actually be evaluated directly. We have for each spacetime point  $x = (x_i^1, x_j^2, x_k^3, t_m)$  an expression proportional to

$$\int \frac{d\pi(x_i^1, x_j^2, x_k^3, t_m)}{2\pi} \exp \left[ i\pi(x) \dot{\phi}(x) \delta^4 x - \frac{i}{2} \pi^2(x) \delta^4 x \right]. \quad (6.22)$$

This Gaussian integral can be evaluated after regularizing it by multiplying the exponent by  $(1 - i\epsilon)$ , where  $\epsilon$  is an infinitesimal positive real number. The integral yields

$$\left( \frac{2\pi i}{\delta^4 x} \right)^{1/2} \exp \left[ \frac{1}{2} i \delta^4 x \dot{\phi}^2(x) \right]. \quad (6.23)$$

The prefactor can be absorbed into a redefinition of the integration measure  $\mathcal{D}\phi$ . Then, the matrix element eq. (6.21) is

$$\begin{aligned}\langle \phi'(\mathbf{x}, t) | e^{-i\hat{H}t} | \phi(\mathbf{x}, 0) \rangle &= \int \mathcal{D}\phi \exp \left[ i \int_0^t dt \int d^3x \left( \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right) \right] \\ &= \int \mathcal{D}\phi \exp \left[ i \int_0^t dt \int d^3x \mathcal{L}(\dot{\phi}, \phi) \right].\end{aligned}\quad (6.24)$$

In QFT, an object of central importance will be the transition amplitude from the vacuum at  $t = -\infty$  to the vacuum at  $t = +\infty$ . It is called the *partition function* and is given by

$$Z_0 \equiv \langle 0(t = +\infty) | e^{-i\hat{H}t} | 0(t = -\infty) \rangle = \int \mathcal{D}\phi \exp \left[ i \int d^4x \mathcal{L}(\dot{\phi}, \phi) \right]. \quad (6.25)$$

Note the allusion to statistical mechanics implied by the term “partition function”. In statistical mechanics, the partition function  $Z \equiv \sum_n \langle n | e^{-\hat{H}/T} | n \rangle$  sums up the Boltzmann factors for all microscopic states  $|n\rangle$  of a system. In the path integral formalism of QFT, the states become trajectories in configuration space, the temperature is set to 1, and there is an extra factor of  $i$  in the exponent. Since a non-interacting system starting out in its ground state will remain there forever,  $Z_0 = 1$  in the free theory. Defining the partition function still makes sense because it will differ from unity as soon as we add an interaction.

Note that, as in QM, the path integral formalism provides a way of writing time-ordered correlation functions:

$$\langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle = \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp \left[ i \int d^4x \mathcal{L}(\dot{\phi}, \phi) \right]. \quad (6.26)$$

Before concluding this section, we will add a very particular interaction to the Lagrangian: a *source term*

$$\mathcal{L}_{\text{source}} \equiv \phi \cdot J. \quad (6.27)$$

This term can be interpreted as a vertex that generates a particle out of nowhere, or annihilates a particle into nothingness, without requiring the presence of another particle. The real coefficient function  $J(x)$  gives the strength of such processes and can be interpreted as a classical (non-quantized) field. The reason for adding the source term will become clear shortly. But first, let us define

$$Z_0[J] \equiv \int \mathcal{D}\phi \exp \left[ i \int d^4x (\mathcal{L}(\dot{\phi}, \phi) + \phi J) \right]. \quad (6.28)$$

Using the *functional derivative*  $\delta/\delta f(x)$ , defined via

$$\frac{\delta}{\delta f(x_1)} f(x_2) \equiv \delta^{(4)}(x_1 - x_2), \quad (6.29)$$

we can then write time-ordered correlation functions very elegantly:

$$\langle 0|\phi(x_1)|0\rangle = \frac{1}{i} \frac{\delta}{\delta J(x_1)} Z_0[J] \Big|_{J=0}, \quad (6.30)$$

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \left( \frac{1}{i} \frac{\delta}{\delta J(x_1)} \right) \left( \frac{1}{i} \frac{\delta}{\delta J(x_2)} \right) Z_0[J] \Big|_{J=0}, \quad (6.31)$$

and so on. Because of these relations,  $Z[J]$  is also called the *generating functional* of the theory.

### 6.3 The Feynman Propagator from the Path Integral

Let us rewrite the action for a free real scalar field

$$S_0 = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \phi J \right] \quad (6.32)$$

in Fourier space, using

$$\phi(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\phi}(k). \quad (6.33)$$

We obtain

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k) \right]. \quad (6.34)$$

Note that we have split the Fourier transform of the source term into two (identical) pieces and added a factor 1/2. Since the Fourier transform is a linear unitary transformation, we can replace the path integral over  $\mathcal{D}\phi$  in the definition of  $Z_0[J]$  by a path integral over  $\mathcal{D}\tilde{\phi}$ . The Jacobian corresponding to this substitution is 1.

Our goal is to simplify eq. (6.34). To this end, let us complete the square (in  $\tilde{\phi}$ ) in the exponent by defining

$$\tilde{\chi}(k) \equiv \tilde{\phi}(k) + \frac{\tilde{J}(k)}{k^2 - m^2 + i\epsilon}. \quad (6.35)$$

This substitution, being a constant shift, does not change the integration measure either, i.e. we can simply replace  $\mathcal{D}\tilde{\phi}$  by  $\mathcal{D}\tilde{\chi}$ . The action now becomes

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k) - \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} \right] \quad (6.36)$$

and the partition function is consequently

$$Z_0[J] = {}_{+\infty} \langle 0|0 \rangle_{-\infty} = \int \mathcal{D}\tilde{\chi} \exp \left[ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left[ \tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k) \right] \right]$$

$$\cdot \exp \left[ -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\epsilon} \right]. \quad (6.37)$$

The first line is just  $Z_0[J=0] = 1$ . The second line does not depend on  $\tilde{\chi}$  any more—we have gotten rid of the path integral. We transform the second line back to coordinate space using the obvious definition

$$\tilde{J}(k) \equiv \int d^4 x e^{ikx} J(x). \quad (6.38)$$

Note that the  $k$ -dependent term obtained that way is just the Feynman propagator  $D_F(x-y)$ . We thus obtain

$$Z_0[J] = \exp \left[ -\frac{1}{2} \int d^4 x d^4 x' J(x') D_F(x-x') J(x) \right]. \quad (6.39)$$

## 6.4 Wick's Theorem from the Path Integral

We have already seen in section 6.2 how time-ordered correlation functions can be constructed by repeatedly applying the functional derivative with respect to  $J(x)$  to the partition function  $Z_0[J]$ . Let us now apply this procedure to the path integral-less form of  $Z_0[J]$ , eq. (6.39).

For the two-point function, this yields

$$\begin{aligned} \langle 0|T\phi(x_1)\phi(x_2)|0\rangle &= \left( \frac{1}{i} \frac{\delta}{\delta J(x_1)} \right) \left( \frac{1}{i} \frac{\delta}{\delta J(x_2)} \right) Z_0[J] \Big|_{J=0} \\ &= -\frac{1}{i} \left( \frac{1}{i} \frac{\delta}{\delta J(x_1)} \right) \int d^4 x D_F(x_2-x) J(x) Z_0[J] \Big|_{J=0} \\ &= [D_F(x_2-x_1) + (\text{terms proportional to } J)] \cdot Z_0[J] \Big|_{J=0} \\ &= D_F(x_2-x_1). \end{aligned} \quad (6.40)$$

We knew from section 2.4.3 that this had to be the outcome, but it is reassuring to see that it can be obtained in the path integral formalism as well.

Consider next the 4-point function. By applying the functional derivative four times, we obtain

$$\begin{aligned} \langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3) \\ &= \begin{array}{c} \begin{array}{ccc} 1 & \text{---} & 2 \\ 3 & \text{---} & 4 \end{array} \\ + \end{array} \begin{array}{c} \begin{array}{ccc} 1 & & 2 \\ | & & | \\ 3 & & 4 \end{array} \\ + \end{array} \begin{array}{c} \begin{array}{ccc} 1 & & 2 \\ & \diagdown & \diagup \\ & 3 & 4 \end{array} \end{array}. \end{aligned} \quad (6.41)$$

This generalizes to arbitrary (even) numbers of functional derivative: each functional derivative can act either on the factor  $Z_0[J]$ , bringing an additional propagator (multiplied by  $J(x)$ ) down, or it can act on one of the  $J(x)$  factors brought down by one of the previous functional derivatives. In total, we obtain one term for each way of picking out pairs of points from the set  $\{x_1, \dots, x_n\}$  and connecting them with Feynman propagators  $D_F(x_j - x_k)$ . This is just Wick's theorem, now proven in a completely different way than in section 4.2.

Correlation functions of an odd number of fields will always vanish because there will always be at least one factor of  $J$  in every term.

## 6.5 Interacting Field Theories in the Path Integral Formalism

The step from a free theory to an interacting theory in the path integral formalism is quite straightforward. We split the Lagrangian into the free piece and the non-interacting piece. For instance, in  $\phi^4$  theory, we have

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^2 \quad (6.42)$$

and

$$\mathcal{L}_1 = -\frac{\lambda}{4!} \phi^4. \quad (6.43)$$

We then define the partition function for the interacting theory,

$$Z_1[J] \propto \int \mathcal{D}\phi \exp \left[ i \int d^4x \mathcal{L}_1 \right] \exp \left[ i \int d^4x (\mathcal{L}_0 + \phi J) \right]. \quad (6.44)$$

By expanding the first exponential (but not the second), we recover a series of time-ordered correlation function that coincides precisely with the perturbation series in the numerator of our master formula, eq. (4.40). Wick's theorem then tells us that, to evaluate the correlation function, we have to consider all possible ways of connecting the  $\phi$  factors by propagators. We thus immediately recover the Feynman rules from section 4.3.

We have thus rederived the formalism for evaluating correlation functions in terms of Feynman diagrams without ever having to invoke creation and annihilation operators. From here on, the remaining step towards the computation of physical observables are the same as before: the LSZ formula relates correlation functions to matrix elements, which are then integrated over phase space.

Note that all correlation functions we encountered in the path integral formalism so far were correlation functions of the *free* theory, as they must be for our master formula to be applicable. As we have seen in eq. (4.40), going from the free theory to the full theory leads to an additional normalization factor, and this factor is the reason we have written “ $\propto$ ” instead of “ $=$ ” in eq. (6.44).

## 6.6 Quantization of the Photon Field

With the path integral formalism at our disposal, we can now justify the expression for the photon propagator in QED (eq. (5.13)),

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}e^{-ip(x-y)}}{p^2 + i\epsilon}. \quad (6.45)$$

Consider the path integral for the free photon field,

$$Z_0[J] = \int \mathcal{D}A \exp \left[ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right) \right]. \quad (6.46)$$

Here,  $\int \mathcal{D}A = \int \mathcal{D}A_0 \mathcal{D}A_1 \mathcal{D}A_2 \mathcal{D}A_3$  implies the path integral over all four components of  $A^\mu$ . Note that also the source  $J^\mu$  is now a Lorentz vector. In other words, there is a separate source for each component of  $A^\mu$ .  $J^\mu$  must satisfy  $\partial^\mu J_\mu = 0$  in order to preserve gauge invariance. (To see this, consider how the term  $J^\mu A_\mu$  transforms under a gauge transformation, and integrate by parts.)

For the following discussion, it will be useful to rewrite

$$\begin{aligned} -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} &= -\frac{1}{2} \int d^4x [(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu)] \\ &= \frac{1}{2} \int d^4x [A^\nu \partial^\mu \partial_\mu A_\nu - A^\nu \partial^\mu \partial_\nu A_\mu]. \end{aligned} \quad (6.47)$$

In the second step, we have integrated by parts. As we did for the scalar field in eq. (6.34), we now transform the path integral to Fourier space:

$$\begin{aligned} Z_0[J] = \int \mathcal{D}\tilde{A} \exp \left[ -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left( \tilde{A}_\mu(k)(k^2 g^{\mu\nu} - k^\mu k^\nu) \tilde{A}_\nu(-k) \right. \right. \\ \left. \left. - \tilde{J}^\mu(k) \tilde{A}_\mu(-k) - \tilde{J}^\mu(-k) \tilde{A}_\mu(k) \right) \right]. \end{aligned} \quad (6.48)$$

We have again split up the source term into two identical pieces and included a factor 1/2 to compensate for the implied double counting.

As in section 6.3, we would like to complete the square in the exponent to rewrite  $Z_0[J]$  as  $Z_0[0] = 1$ , multiplied by a term not involving a path integral any more, but containing  $J$  and the photon propagator. Doing so would require us to invert the matrix  $k^2 g^{\mu\nu} - k^\mu k^\nu \equiv k^2 P^{\mu\nu}$ , so that we could define  $\tilde{B}^\mu(k) \equiv \tilde{A}^\mu(k) - [(k^2 P)^{-1}]^{\mu\nu} \tilde{J}_\nu(k)$ . Unfortunately, the matrix  $P$  is singular because  $P^{\mu\nu} k_\nu = 0$ .

The reason for the singular nature of  $P$  is gauge invariance.  $\mathcal{L}$  is invariant under  $A^\mu(x) \rightarrow A^\mu(x) + (1/e) \partial^\mu \alpha(x)$ , which in momentum space translates to  $\tilde{A}^\mu(k) \rightarrow \tilde{A}^\mu(k) - (i/e) k^\mu \tilde{\alpha}(k)$ . This implies that the component of  $A^\mu(k)$  proportional to  $k^\mu$  is unphysical, but it also suggests a solution to our difficulties defining the photon propagator: we simply drop the unphysical component of the photon field and redefine  $\mathcal{D}\tilde{A}$  to mean

path integration only over components of  $\tilde{A}$  orthogonal to  $k$ , i.e.  $k^\mu \tilde{A}_\mu(k) = 0$ . In other words, we fix the gauge to the Lorenz gauge. Within this subspace,  $P^{\mu\nu}$  is just the identity matrix since

$$\begin{aligned} P^{\mu\nu}(k) P_\nu{}^\lambda(k) &= \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \left( g_\nu{}^\lambda - \frac{k_\nu k^\lambda}{k^2} \right) \\ &= g^{\mu\lambda} - \frac{k^\mu k^\lambda}{k^2} \\ &= P^{\mu\lambda}(k) \end{aligned} \tag{6.49}$$

and

$$g^{\mu\nu} P_{\mu\nu}(k) = 3. \tag{6.50}$$

The first relation implies that  $P^{\mu\nu}(k)$  is a projection operator, i.e. its eigenvalues can only be zero or one. The second relation gives just the trace of  $P^{\mu\nu}$  and implies that three of the eigenvalues are 1 and the fourth one is zero. We have already argued that the zero eigenvalue corresponds to the direction of  $k^\mu$ , which we have removed from the path integral. Therefore, in the remaining subspace, all three eigenvalues are 1 and  $P^{\mu\nu}(k)$  must be the identity matrix. Therefore, in this subspace,  $(k^2 P^{\mu\nu})^{-1} = (1/k^2) P^{\mu\nu}$ . We define

$$\tilde{B}^\mu(k) \equiv \tilde{A}^\mu(k) - \frac{P^{\mu\nu}(k) \tilde{J}_{\mu\nu}(k)}{k^2} \tag{6.51}$$

and turn the partition function into

$$\begin{aligned} Z_0[J] &= \int \mathcal{D}\tilde{B} \exp \left[ -\frac{i}{2} \int \frac{d^4 k}{(2\pi)^4} \left( \tilde{B}^\mu(k) k^2 P_{\mu\nu} \tilde{B}^\nu(-k) - \tilde{J}^\mu(k) \frac{P_{\mu\nu}}{k^2} \tilde{J}^\nu(-k) \right) \right] \\ &= \exp \left[ -\frac{1}{2} \int d^4 x d^4 x' \tilde{J}^\mu(x') \Delta_{\mu\nu}(x-x') \tilde{J}^\nu(x) \right]. \end{aligned} \tag{6.52}$$

In the second step, we have used that  $Z_0[0] = 1$  to eliminate the path integral over  $\mathcal{D}\tilde{B}$ . We have also exploited the fact that  $k^\mu J_\mu(x) = 0$ , so that replacing  $P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2$  by  $g^{\mu\nu}$  leaves  $Z_0[J]$  invariant. Finally, we emphasize that the integration measure  $\mathcal{D}\tilde{B}$  implies integration only over the components of  $\tilde{B}$  that are orthogonal to  $k^\mu$ .

## 6.7 Path Integrals for Fermions

We now generalize the path integral formalism to fermions. The problem is that we must somehow reproduce the Pauli exclusion principle—intimately related to the anticommutation relations of fermion fields—in a formalism that does not involve operators, but only numbers. The solution is to invoke so-called *anticommuting number* or *Grassmann numbers*, which we will now introduce.



### 6.7.1 Grassmann Numbers

The set  $G$  of (real) Grassmann numbers is defined by the property that, for  $\theta, \eta \in G$ ,

$$\theta\eta = -\eta\theta. \quad (6.53)$$

For finite sets of Grassmann numbers, one can construct matrix representations of  $G$ . The above definition is, however, more general. One immediate consequence of eq. (6.53) is that

$$\theta^2 = 0. \quad (6.54)$$

Therefore, any real or complex valued function  $f(\theta)$  can be written as

$$f(\theta) = A + B\theta. \quad (6.55)$$

In other words, the Taylor expansion of  $f(\theta)$  terminates after the linear term. Note that  $A$  is an ordinary number and  $B$  is a Grassmann number, so that  $B\theta$  is an ordinary number.

We will also want to integrate over Grassmann numbers, and we therefore define  $\int d\theta f(\theta)$  by requiring the properties

$$\int d\theta c \cdot f(\theta) = c \cdot \int d\theta f(\theta), \quad (6.56)$$

$$\int d\theta [f(\theta) + g(\theta)] = \int d\theta f(\theta) + \int d\theta g(\theta), \quad (6.57)$$

$$\int d\theta f(\theta + \eta) = \int d\theta f(\theta), \quad (6.58)$$

$$\int d\theta 1 = 0. \quad (6.59)$$

for  $\theta, \eta \in G$  and  $c \in \mathbb{C}$ . These properties, in particular the invariance under shifts of the integration variable, can be satisfied only if  $\int d\theta f(\theta)$  is a constant times  $B$ . We therefore define

$$\int d\theta f(\theta) = \int d\theta (A + B\theta) \equiv B. \quad (6.60)$$

For multidimensional Grassmann integrals, the ordering of the integrals and the Grassmann numbers in the integrand is important. We choose the convention

$$\int d\theta d\eta \eta \theta = +1. \quad (6.61)$$

To take derivatives with respect to Grassmann numbers, we use the definition

$$\frac{d}{d\eta} \theta \eta = -\frac{d}{d\eta} \eta \theta = -\theta. \quad (6.62)$$

We can define a complex Grassmann number  $\theta$  according to

$$\theta \equiv \frac{1}{\sqrt{2}}(\theta_r + i\theta_i), \quad (6.63)$$

where  $\theta_r$  and  $\theta_i$  are real Grassmann numbers. The prefactor  $1/\sqrt{2}$  is convention. The complex conjugate of  $\theta$  is

$$\theta^* \equiv \frac{1}{\sqrt{2}}(\theta_r - i\theta_i). \quad (6.64)$$

When taking the complex conjugate of a product of complex Grassmann numbers, we again have to worry about their ordering. We define

$$(\theta\eta)^* \equiv \eta^*\theta^* = -\theta^*\eta^*. \quad (6.65)$$

When integrating over  $\theta$ , we can either treat  $\theta_r, \theta_i$  as independent degrees of freedom, or  $\theta, \theta^*$ . We will choose the latter convention and thus have

$$\int d\theta^* d\theta \theta \theta^* = 1. \quad (6.66)$$

Let us consider a few examples for working with Grassmann numbers:

1. A Gaussian integral: for complex Grassmann variables  $\theta, \theta^*$  and a real number  $b$ , we compute

$$\int d\theta^* d\theta e^{-\theta^* b \theta} = \int d\theta^* d\theta (1 - \theta^* b \theta) = b. \quad (6.67)$$

Compare this to the corresponding integral with only real numbers,  $\int dx dx^* e^{-x^* b x} = 2\pi/b$ .

2. Another Gaussian integral:

$$\int d\theta^* d\theta \theta \theta^* e^{-\theta^* b \theta} = 1. \quad (6.68)$$

3. A multidimensional Gaussian integral. Let  $B$  be a Hermitian matrix with elements  $B_{ij} \in \mathbb{C}$ . We would like to evaluate

$$I = \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n e^{-\theta_i^* B_{ij} \theta_j}. \quad (6.69)$$

We first note that, in the Taylor expansion of the exponential, the only term that contributed is the one in which all  $\theta_i, \theta_i^*$  appear exactly once:  $\prod_i \theta_i^* \theta_i$ . The strategy is to transform to an eigenbasis of  $B$  by applying a unitary transformation  $\theta_i \rightarrow \theta'_i =$

$U_{ij}\theta_j$ . We will denote the eigenvalues of  $B$  by  $b_k$ . The product of all  $\theta_i$  transforms into

$$\prod_i \theta'_i = \frac{1}{n!} \epsilon^{j_1 j_2 \dots} \theta'_{j_2} \theta'_{j_2} \dots \quad (6.70)$$

$$= \frac{1}{n!} \epsilon^{j_1 j_2 \dots} U_{j_1 k_1} \theta_{k_1} U_{j_2 k_2} \theta_{k_2} \dots \quad (6.71)$$

$$= \frac{1}{n!} \epsilon^{j_1 j_2 \dots} \epsilon^{k_1 k_2 \dots} U_{j_1 k_1} U_{j_2 k_2} \dots \left( \prod_i \theta_i \right) \quad (6.72)$$

$$= (\det U) \left( \prod_i \theta_i \right) \quad (6.73)$$

$$= \prod_i \theta_i. \quad (6.74)$$

In the last step we have used the unitarity of  $U$ . We can now write

$$I = \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n e^{-\sum_k \theta_k'^* b_k \theta_k'} \quad (6.75)$$

$$= (-1)^n \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n \prod_k b_k \theta_k'^* \theta_k \quad (6.76)$$

$$= \prod_k b_k \quad (6.77)$$

$$= \det B. \quad (6.78)$$

This should again be compared to the corresponding integral over ordinary numbers,  $\int dx_1^* dx_1 \dots dx_n^* dx_n e^{-x_i^* B_{ij} x_j} = (2\pi)^n / \det B$ .

### 6.7.2 Partition Function, Functional Derivative and Correlation Functions for Fermions

Consider the free Dirac field with the Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta. \quad (6.79)$$

Note that we have again added source terms. The crucial point in the following will be that we treat  $\psi(x)$ ,  $\bar{\psi}(x)$ ,  $\eta(x)$  and  $\bar{\eta}(x)$  as spinor-valued functions, with the entries of the spinors being Grassmann numbers. We define the partition function for the free fermion field as

$$Z_0[\eta, \bar{\eta}] \equiv \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[ i \int d^4x \mathcal{L}_0 \right]. \quad (6.80)$$

Remember that for the free scalar field, we were able to rewrite the partition function in a form without the path integral as

$$Z_0[J] = \exp \left[ -\frac{1}{2} \int d^4x d^4x' J(x') D_F(x - x') J(x) \right], \quad (6.81)$$

(see eq. (6.39)). To do the same for  $Z_0[\eta, \bar{\eta}]$ , we follow completely analogous arguments. We transform the action  $S_0 = \int d^4x \mathcal{L}_0$  into Fourier space, using

$$\psi(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\psi}(k), \quad (6.82)$$

$$\eta(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\eta}(k). \quad (6.83)$$

This yields

$$S_0 = \int \frac{d^4k}{(2\pi)^4} \left[ \tilde{\psi}(k)(\not{k} - m)\tilde{\psi}(k) + \tilde{\eta}(k)\tilde{\psi}(k) + \tilde{\psi}(k)\tilde{\eta}(k) \right]. \quad (6.84)$$

We then shift the field  $\tilde{\psi}(k)$  by defining

$$\tilde{\chi}(k) \equiv \tilde{\psi}(k) + \frac{\tilde{\eta}(k)}{\not{k} - m + i\epsilon}. \quad (6.85)$$

This leads to

$$S_0 = \int \frac{d^4k}{(2\pi)^4} \left[ \tilde{\chi}(k)(\not{k} - m)\tilde{\chi}(-k) - \tilde{\eta}(k) \frac{1}{\not{k} - m + i\epsilon} \tilde{\eta}(k) \right] \quad (6.86)$$

$$= \int d^4x \bar{\chi}(x)(i\not{\partial} - m)\chi(x) - \int d^4x d^4x' \frac{d^4k}{(2\pi)^4} \bar{\eta}(x) \frac{e^{ik(x'-x)}}{\not{k} - m + i\epsilon} \eta(x'). \quad (6.87)$$

from which it follows that

$$Z_0[\eta, \bar{\eta}] \equiv Z_0[0, 0] \exp \left[ - \int d^4x d^4x' \bar{\eta}(x) S_F(x - x') \eta(x') \right]. \quad (6.88)$$

Here  $S_F(x - x')$  is the Feynman propagator for fermions,

$$S_F(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-x')}}{\not{k} - m + i\epsilon}. \quad (6.89)$$

Note that  $Z_0[0, 0] = 1$  since a field that is initially in the vacuum state will remain there forever in the absence of external sources.

We can again obtain time-ordered correlation functions by taking functional derivatives of  $Z_0[\eta, \bar{\eta}]$ . For instance

$$\langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle = \left( \frac{1}{i} \frac{\delta}{\delta \bar{\eta}(x_1)} \right) \left( - \frac{1}{i} \frac{\delta}{\delta \eta(x_2)} \right) Z_0[\eta, \bar{\eta}] \Big|_{\eta=\bar{\eta}=0}. \quad (6.90)$$

Note the extra minus sign in the functional derivative with respect to  $\eta(x_2)$ . Its origin is the fact that, in the last term in the Lagrangian eq. (6.79),  $\eta$  appears to the right of  $\bar{\psi}$ , so that the derivative operator  $\delta/\delta\eta(x_2)$  needs to be anticommutated past  $\bar{\psi}$  before evaluating it.

## 6.8 The Quantum Equations of Motion: Schwinger–Dyson Equations

Consider an  $n$ -point correlation function in a scalar field theory,

$$\langle \Omega | \phi(x_1) \cdots \phi(x_n) | \Omega \rangle \propto \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{i \int d^4x \mathcal{L}[\phi]}. \quad (6.91)$$

In quantum mechanics, we are used to deriving equations of motion by demanding that the action  $S = \int dt L$  is invariant under infinitesimal variations of the system's phase space trajectory. Let us consider now infinitesimal variations of the field  $\phi(x)$ :

$$\phi(x) \rightarrow \phi'(x) \equiv \phi(x) + \epsilon(x). \quad (6.92)$$

Since the path integral means that the field value  $\phi(x)$  is integrated over at each spacetime point, this shift is nothing but a simple substitution of variables in the path integral. At each spacetime point, eq. (6.92) is just a constant shift, which leaves the integration measure  $\mathcal{D}\phi$  invariant. Therefore,

$$\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) e^{i \int d^4x \mathcal{L}[\phi]} = \int \mathcal{D}\phi \phi'(x_1) \cdots \phi'(x_n) e^{i \int d^4x \mathcal{L}[\phi']}. \quad (6.93)$$

Expanding in  $\epsilon(x)$ , this leads to

$$0 = \int \mathcal{D}\phi e^{i \int d^4x \mathcal{L}[\phi]} \left[ i \int d^4x \epsilon(x) \phi(x_1) \cdots \phi(x_n) \frac{\delta}{\delta \phi(x)} \left( \int d^4x' \mathcal{L}[\phi(x')] \right) + \sum_{j=1}^n \phi(x_1) \cdots \phi(x_{j-1}) \epsilon(x_j) \phi(x_{j+1}) \cdots \phi(x_n) \right] \quad (6.94)$$

The functional derivative in this expression is

$$\frac{\delta}{\delta \phi(x)} \left( \int d^4x' \mathcal{L}[\phi(x')] \right) = \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))}. \quad (6.95)$$

This is just the expression that vanishes in the classical theory according to the Euler–Lagrange equations of motion. In the terms in the second line of eq. (6.94), we can write

$$\epsilon(x_j) = \int d^4x \epsilon(x) \delta^{(4)}(x - x_j). \quad (6.96)$$

Noting now that eq. (6.94) has to hold for any  $\epsilon(x)$ , and writing the path integral in terms of correlation functions, we find

$$\left\langle \Omega \left| T \phi(x_1) \cdots \phi(x_n) \frac{\delta}{\delta \phi(x)} \left( \int d^4x' \mathcal{L}[\phi(x')] \right) \right| \Omega \right\rangle$$

$$= \sum_{j=1}^n \langle \Omega | T \phi(x_1) \cdots \phi(x_{j-1}) i\delta^{(4)}(x - x_j) \phi(x_{j+1}) \cdots \phi(x_n) | \Omega \rangle. \quad (6.97)$$

These equations are called the *Schwinger–Dyson equations*. The terms involving  $\delta$ -functions on the right hand side are called *contact terms* because they are non-zero only when  $x$  equals one of the  $x_j$ . The Schwinger–Dyson equations can be interpreted as the equations of motion of the quantum theory: they tell us that the classical equations of motion (which would dictate that eq. (6.95) vanishes) still hold inside a correlation function, provided the coordinate  $x$  does not equal any of the  $x_j$  appearing in that correlation function.

## 6.9 The Ward–Takahashi Identity

We now carry out a similar derivation as in the previous section for QED. Instead of the general shifts considered above, we focus in particular on the transformation

$$\psi(x) \rightarrow \psi'(x) \equiv [1 + ie\alpha(x)]\psi(x), \quad (6.98)$$

with  $\alpha(x)$  infinitesimal. This is just an infinitesimal gauge transformation on  $\psi(x)$ . Note, however, that we do *not* apply the corresponding gauge transformation to  $A^\mu(x)$ , i.e. the Lagrangian is *not* invariant under eq. (6.98). Instead, it transforms as

$$\mathcal{L}_{\text{QED}}[\psi, \bar{\psi}, A] \rightarrow \mathcal{L}_{\text{QED}}[\psi', \bar{\psi}', A] \equiv \mathcal{L}_{\text{QED}}[\psi, \bar{\psi}, A] - e [\partial_\mu \alpha(x)] \bar{\psi} \gamma^\mu \psi. \quad (6.99)$$

The crucial point is that the Lagrangian changes only by a term containing a derivative of  $\alpha(x)$ . The reason is that  $\mathcal{L}_{\text{QED}}$  is invariant under global (i.e.  $x$ -independent) transformations  $\psi(x) \rightarrow (1 + ie\alpha)\psi(x)$ .

Consider now the transformation of the path integral

$$\begin{aligned} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{i \int d^4x \mathcal{L}_{\text{QED}}[\psi, \bar{\psi}, A]} \psi(x_1) \bar{\psi}(x_2) \\ = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{i \int d^4x \mathcal{L}_{\text{QED}}[\psi', \bar{\psi}', A]} \psi'(x_1) \bar{\psi}'(x_2). \end{aligned} \quad (6.100)$$

The identity follows from the fact that  $\mathcal{D}\psi$  and  $\mathcal{D}\bar{\psi}$  are invariant under eq. (6.98). Expanding the right hand side of eq. (6.100) in  $\alpha(x)$ , and introducing the notation

$$j^\mu(x) \equiv e \bar{\psi}(x) \gamma^\mu \psi(x), \quad (6.101)$$

we obtain

$$\begin{aligned} 0 = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{i \int d^4x \mathcal{L}_{\text{QED}}[\psi, \bar{\psi}, A]} \left\{ -i \int d^4x [\partial_\mu \alpha(x)] j^\mu(x) \psi(x_1) \bar{\psi}(x_2) \right. \\ \left. + [ie\alpha(x_1)\psi(x_1)] \bar{\psi}(x_2) + \psi(x_1) [-ie\alpha(x_2)\bar{\psi}(x_2)] \right\} \end{aligned}$$

$$\begin{aligned}
 &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A e^{i \int d^4x \mathcal{L}_{\text{QED}}[\psi, \bar{\psi}, A]} \int d^4x \left\{ i\alpha(x) [\partial_\mu j^\mu(x)] \psi(x_1) \bar{\psi}(x_2) \right. \\
 &\quad \left. + ie\delta^{(4)}(x - x_1) \alpha(x_1) \psi(x_1) \bar{\psi}(x_2) - ie\delta^{(4)}(x - x_2) \psi(x_1) \alpha(x_2) \bar{\psi}(x_2) \right\}.
 \end{aligned} \tag{6.102}$$

In the last step, we have integrated by parts in the first term and pulled the last two terms under the integral by introducing appropriate  $\delta$  functions. Since eq. (6.102) has to hold for any infinitesimal  $\alpha(x)$ , it must hold also when the integral  $\int d^4x$  is omitted. Written in terms of correlation functions, this implies

$$\begin{aligned}
 i\partial_\mu \langle 0 | T j^\mu(x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle &= -ie\delta^{(4)}(x - x_1) \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle \\
 &\quad + ie\delta^{(4)}(x - x_2) \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle.
 \end{aligned} \tag{6.103}$$

This identity, which is called the *Ward–Takahashi identity*, can be expressed diagrammatically as

$$i\partial_\mu \cdot x \quad = \quad ie \left( \delta^{(4)}(x - x_1) \quad - \delta^{(4)}(x - x_2) \right). \tag{6.104}$$

Note that the left hand side is just the full QED vertex (including all quantum corrections), but with the photon field  $A_\mu(x)$  replaced by a derivative  $\partial_\mu$ . The contact terms on the right hand side involve, besides the  $\delta$ -functions, the full propagator.

Let us elaborate more on the connection between the left hand side of the Ward–Takahashi identity and the correlation function  $\langle 0 | T A^\mu(x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle$  (the full QED vertex), and also on its connection to a corresponding scattering amplitude  $\langle f | i \rangle$  obtained by applying the LSZ reduction formula to it. We use the equations of motion for the photon field. Working in Lorenz gauge  $\partial_\mu A^\mu = 0$ , they read

$$\partial^\nu \partial_\nu A^\mu(x) = j^\mu(x). \tag{6.105}$$

We know from the previous section that these equations of motion still hold inside a correlation function, up to contact terms. Therefore, eq. (6.103) is equivalent to

$$i\partial_\mu \partial^2 \langle 0 | T A^\mu(x) \psi(x_1) \bar{\psi}(x_2) | 0 \rangle = \text{contact terms}. \tag{6.106}$$

The double derivative  $\partial^2$  on the left hand side appears also in the LSZ reduction formula. We now put in all the remaining pieces of that formula except the polarization vector  $\epsilon^\mu(pk)$ . (Since  $A^\mu(x)$  is already contracted with  $\partial_\mu$ , we would not have anything to

contract it with.) On the left-hand side of the LSZ formula, we write the scattering amplitude as

$$\langle f|i \rangle = \epsilon_\mu(k) \cdot i\mathcal{M}^\mu(p, k, q). \quad (6.107)$$

The Feynman rules tell us that this is always possible. We obtain

$$\begin{aligned} -ik_\mu \mathcal{M}^\mu(p, k, q) &= \int d^4x \int d^4x_1 \int d^4x_2 e^{-ikx} \partial^2 e^{iqx_1} [\bar{u}(q)(i\cancel{\partial}_{x_1} - m)] \\ &\quad \times \langle 0|T\partial_\mu A^\mu(x)\psi(x_1)\bar{\psi}(x_2)|0 \rangle [(-i\overleftarrow{\cancel{\partial}}_{x_2} - m)u(p)] e^{-ipx_2}. \end{aligned} \quad (6.108)$$

$$\begin{aligned} &= -i \int d^4x \int d^4x_1 \int d^4x_2 e^{-ikx} e^{iqx_1} [\bar{u}(q)(i\cancel{\partial}_{x_1} - m)] \\ &\quad \times (\text{contact terms}) [(-i\overleftarrow{\cancel{\partial}}_{x_2} - m)u(p)] e^{-ipx_2}. \end{aligned} \quad (6.109)$$

Let us first consider eq. (6.108). Integrating by parts over  $x_1$ , it turns into

$$\begin{aligned} -ik_\mu \mathcal{M}^\mu(p, k, q) &= \lim_{q^2 \rightarrow m^2} (-i) \int d^4x_1 e^{iqx_1} [\bar{u}^s(q)(\not{q} - m)] \dots \langle 0|T\psi(x_1) \dots |0 \rangle \\ &= \lim_{q^2 \rightarrow m^2} (-i) [\bar{u}^s(q)(\not{q} - m)] \dots \langle 0|T\tilde{\psi}(q) \dots |0 \rangle. \end{aligned} \quad (6.110)$$

We have written out only the pieces corresponding to  $\psi(x_1)$  here. The dots indicate the other pieces. The limit is introduced to avoid the otherwise infinite expression  $1/(q^2 - m^2)$ , which would appear in the correlation function according to the Feynman rules. In the second line of eq. (6.110), we have introduced the Fourier-transformed field  $\tilde{\psi}(q) = \int d^4x e^{iqx} \psi(x)$ . Since the scattering matrix element  $\langle f|i \rangle$  must be finite (up to the overall energy-momentum conserving  $\delta$  function), the correlation function  $\langle 0|T\tilde{\psi}(q) \dots |0 \rangle$  must include a pole proportional to  $1/(\not{q} - m)$  to cancel the factor  $\not{q} - m$  from the LSZ formula, and similar poles also for all other external particles. For the specific case of the full QED vertex, the requisite pole structure is

$$\frac{1}{\not{q} - m} \frac{1}{\not{p} - m} \frac{1}{k^2}. \quad (6.111)$$

When we discussed the Feynman rules for correlation functions, we saw that these pole corresponded to the propagators attached to the external vertices. It is the coefficient of all these poles that gives  $\langle f|i \rangle$ . Terms that are lacking one or several of the poles do not contribute since they go to zero in the limit  $q^2 \rightarrow m^2$ ,  $p^2 \rightarrow m^2$ ,  $k^2 \rightarrow 0$ .

Consider now the contact terms (right hand side of eq. (6.106) and last expression in eq. (6.109)). The  $\delta$ -function in a contact term eliminates one of the coordinate integrals and simplifies the momentum dependence. For instance, consider the term containing the delta function  $\delta^{(4)}(x - x_1)$ . After eliminating the  $x$  or  $x_1$  integral using this delta function, the resulting term depends only on  $q - k$ , but not on the orthogonal combination



$q + k$ . Therefore, it cannot have separate  $1/k^2$  and  $1/(q - m)$  poles. It thus does not have the structure of eq. (6.111) and therefore does not contribute to  $\langle f|i \rangle$ .

These arguments can be generalized to arbitrary correlation functions and also to theories other than QED. Therefore, we conclude that *contact terms never contribute to scattering matrix elements*.

Coming back to the electron vertex function, we can now conclude from eqs. (6.108) and (6.109) that

$$k_\mu \mathcal{M}^\mu = 0. \quad (6.112)$$

The implications of this become more clear when expressed diagrammatically:



$$= 0. \quad (6.113)$$

In other words, when we take the matrix element describing the interaction of a photon with two fermions and replace the photon polarization vector  $\epsilon^\mu(k)$  by  $k^\mu$ , the result is zero.

Of course, we could have carried out a fully analogous derivation also for QED diagrams with more external vertices. The result would be the same: *any* QED scattering matrix element  $\epsilon^\mu(k) \mathcal{M}^\mu$  satisfies  $k_\mu \mathcal{M}^\mu = 0$ .

Remember that our starting point in this section was the invariance of  $\mathcal{L}_{\text{QED}}$  under global gauge transformations (the statement that under a *local* gauge transformation  $\mathcal{L}_{\text{QED}}$  changes only by a term containing a derivative of  $\alpha(x)$ ). The Ward–Takahashi identity and the relation  $k_\mu \mathcal{M}^\mu = 0$  are direct consequences of this. Another way of phrasing eq. (6.112) is to say that the amplitude for the production of a (hypothetical) longitudinal photon vanishes. It is often useful when computing complicated Feynman diagrams to check the calculation by verifying that eq. (6.112) is satisfied.



# 7

## Weyl and Majorana Fermions

In section 3.4.1, we have already argued that the left-chiral and right-chiral components of a 4-component spinor belong to different representations of the Lorentz group. (We have argued that the transformation matrices are block diagonal, thus not mixing the upper and lower components of a 4-component spinor.) In the Lagrangians we have encountered so far, the only term that mixes the left-handed and right-handed pieces is the mass term. For instance, consider the QED Lagrangian. Using

$$\begin{aligned}\psi(x) &= \frac{1 - \gamma^5}{2}\psi(x) + \frac{1 + \gamma^5}{2}\psi(x) \\ &= P_L\psi(x) + P_R\psi(x) \\ &\equiv \psi_L(x) + \psi_R(x),\end{aligned}\tag{7.1}$$

it can be written as

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \bar{\psi}_L(i\not{\partial})\psi_L - e\bar{\psi}_L\gamma^\mu\psi_L A_\mu + \bar{\psi}_R(i\not{\partial})\psi_R - e\bar{\psi}_R\gamma^\mu\psi_R A_\mu \\ &\quad - (m\bar{\psi}_L\psi_R + h.c.) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}\tag{7.2}$$

We see that, indeed, apart from the mass term, this looks like the Lagrangian of two independent fields, each coupled to the photon. Group theory allows one to classify all possible representations of the Lorentz group  $SO^+(1,3)$  and give them labels.<sup>1</sup> By convention, the representation corresponding to  $\psi_L$  is called  $(\frac{1}{2}, 0)$ , and the representation

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<sup>1</sup>When we write “Lorentz group” in the following, we consider for simplicity only the proper orthochronous Lorentz group, not the more general Lorentz group  $O(1,3)$ , which includes also time reversal and parity transformations.

corresponding to  $\psi_R$  is called  $(0, \frac{1}{2})$ . Other representations include  $(0, 0)$  (Lorentz scalar) and  $(\frac{1}{2}, \frac{1}{2})$  (Lorentz vector).

In the following, we will exploit the possibility to separate left-handed from right-handed spinors a little further. This is a preparation for a discussion of the Standard Model of particle physics in the second part of this course, where the left-handed and right-handed fermions in fact have different interactions. It is moreover useful when studying massless or approximately massless fermions, for instance in collider physics (where the quark masses are negligible compared to the typical collision energies) and in neutrino physics (where we're dealing with particles whose mass is  $\ll$  eV).

## 7.1 Spinor Indices

We first introduce some useful notation, recapitulating along the way the Lorentz transformation properties of spinors.

### 7.1.1 Left-handed spinors

Consider a left-handed spinor  $\Psi(x)$ . It can be understood as a 4-component object with the lower two components set to zero,  $\Psi(x) = (\psi(x), 0)^T$ . However, it is cumbersome to always carry the zero components along, therefore we will in the following only consider two-component spinors, in this case  $\psi(x)$ . We will write out their spinor indices explicitly, e.g.  $\psi_\alpha(x)$ . The position of the spinor index (subscript or superscript) will be meaningful, as it is for Lorentz vector indices. We will see below how to raise or lower spinor indices.

Remember from eq. (3.68) that a (Dirac) spinor  $\Psi$  transforms under a Lorentz transformation  $\Lambda^\mu{}_\nu$  according to

$$\Psi(x) \rightarrow \Psi'(x) = S(\Lambda) \Psi(\Lambda^{-1}x), \quad (7.3)$$

where  $S(\Lambda)$  is the  $4 \times 4$  block-diagonal matrix given in eq. (3.74) (for infinitesimal transformations). We write

$$S(\Lambda) = \begin{pmatrix} s(\Lambda) & \\ & (\bar{s}^T)^{-1}(\Lambda) \end{pmatrix} \quad (7.4)$$

(The transposition and the inverse in the second row are conventions that will turn out to be useful later in this chapter. For now, this is just how we name the lower  $2 \times 2$  block of  $S(\Lambda)$ .) The upper  $2 \times 2$  block, which is relevant for the transformation of left-handed spinors, has the form

$$s(\Lambda)_\alpha{}^\beta = \delta_\alpha{}^\beta - \frac{i}{4} (\sigma_{\mu\nu})_\alpha{}^\beta \omega^{\mu\nu}, \quad (7.5)$$

where in the 2-component world,  $\sigma^{\mu\nu}$  is understood to mean

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta. \quad (7.6)$$

The tensor  $\omega$  is, as before, defined via  $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ .

Thus, in 2-component notation, the transformation property of a left-handed spinor is

$$\psi_\alpha(x) \rightarrow \psi'_\alpha(x) = s(\Lambda)_\alpha{}^\beta \psi_\beta(\Lambda^{-1}x), \quad (7.7)$$

As we hinted at above, the position of the spinor indices used here is not arbitrary. The convention is that an upper index is always contracted with a lower index and vice-versa.

### 7.1.2 Raising and lowering spinor indices

There is also a method for raising and lowering spinor indices, in analogy to the raising and lowering of Lorentz vector indices using the metric  $g^{\mu\nu}$ . To develop this method, let us consider the antisymmetric tensor in two dimensions,  $\epsilon_{\alpha\beta}$ . Its diagonal components vanish, while

$$\epsilon_{12} = -\epsilon_{21} = -1. \quad (7.8)$$

(The normalization is a mere convention.) The salient feature of  $\epsilon_{\alpha\beta}$  is its invariance under Lorentz transformations. To see this, note first that  $\det s(\Lambda) = 1$ , as befits a symmetry transformation of a quantum field (after all the “ $O$ ” in  $SO^+(1,3)$  stands for “orthogonal”). According to the definition of the determinant, this means in component notation

$$1 = s(\Lambda)_1{}^\gamma s(\Lambda)_2{}^\delta (-\epsilon_{\gamma\delta}), \quad (7.9)$$

or, more generally,

$$\epsilon_{\alpha\beta} = s(\Lambda)_\alpha{}^\gamma s(\Lambda)_\beta{}^\delta \epsilon_{\gamma\delta}. \quad (7.10)$$

This proves the invariance of  $\epsilon_{\alpha\beta}$  under Lorentz transformations, making it the equivalent of  $g_{\mu\nu}$  for spinor indices. We define  $\epsilon^{\alpha\beta}$  (with upper indices) as the inverse of  $\epsilon_{\alpha\beta}$  (with lower indices). By defining

$$\epsilon^{12} = \epsilon_{21} = 1, \quad \epsilon^{21} = \epsilon_{12} = -1, \quad (7.11)$$

we see that

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma. \quad (7.12)$$

Now, we can also define a spinor with an upper index:

$$\psi^\alpha \equiv \epsilon^{\alpha\beta} \psi_\beta. \quad (7.13)$$

Its transformation property is

$$\psi^\alpha(x) \rightarrow \psi'^\alpha(x) = \epsilon^{\alpha\beta} s(\Lambda)_\beta{}^\gamma \psi_\gamma(\Lambda^{-1}x) \quad (7.14)$$

This can be modified further by using the following relation, which follows from eq. (7.10):

$$\begin{aligned}
 \epsilon^{\theta\beta} s(\Lambda)_\beta^\delta \epsilon_{\alpha\delta} \epsilon^{\alpha\kappa} &= \epsilon^{\theta\beta} s^{-1}(\Lambda)_\alpha^\eta \epsilon_{\eta\beta} \epsilon^{\alpha\kappa} \\
 \Leftrightarrow \epsilon^{\theta\beta} s(\Lambda)_\beta^\kappa &= s^{-1}(\Lambda)_\alpha^\theta \epsilon^{\alpha\kappa} \\
 \Leftrightarrow \epsilon^{\alpha\beta} s(\Lambda)_\beta^\gamma &= s^{-1}(\Lambda)_\beta^\alpha \epsilon^{\beta\gamma}.
 \end{aligned} \tag{7.15}$$

This relation leads to

$$\psi'^\alpha(x) = \psi^\beta(x) s^{-1}(\Lambda)_\beta^\alpha \tag{7.16}$$

Lowering indices works the same way as raising them:

$$\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta = \epsilon_{\alpha\beta} \epsilon^{\beta\gamma} \psi_\gamma = \delta_\alpha^\gamma \psi_\gamma. \tag{7.17}$$

However, the antisymmetry of  $\epsilon^{\alpha\beta}$  introduces complications in the form of extra minus signs sometimes:

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta = -\psi_\beta \epsilon^{\beta\alpha}. \tag{7.18}$$

Also, when contracting several spinors, care must be taken:

$$\psi^\alpha \chi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \chi_\alpha = -\psi_\beta \epsilon^{\beta\alpha} \chi_\alpha = -\psi_\beta \chi^\beta = \chi^\beta \psi_\beta. \tag{7.19}$$

The last equality follows from the anticommutation property of spinor fields. In the following we will adopt the following convention for contractions of two spinors written *without spinor indices*:

$$\psi\chi \equiv \psi^\alpha \chi_\alpha. \tag{7.20}$$

Then, eq. (7.19) shows for instance that

$$\psi\chi = \chi\psi. \tag{7.21}$$

### 7.1.3 Right-handed spinors

Let us now consider a right-handed spinor  $\bar{\psi}$ , i.e. a two-component object transforming in  $(0, \frac{1}{2})$  of  $SO^+(1, 3)$ . (For now, the bar in  $\bar{\psi}$  merely helps us to distinguish left-handed from right-handed spinors. It has nothing to do yet with hermitian conjugation or so. We will, however, make this connection shortly.) To distinguish left-handed from right-handed spinors, we introduce the convention that spinor indices transforming in the right-handed representation  $(0, \frac{1}{2})$  are dotted, e.g.  $\bar{\psi}^{\dot{\alpha}}$ , while those transforming in the left-handed representation  $(\frac{1}{2}, 0)$  are undotted as before, e.g.  $\psi_\alpha$ . The transformation property of  $\bar{\psi}^{\dot{\alpha}}$  is

$$\bar{\psi}^{\dot{\alpha}}(x) \rightarrow \bar{\psi}'^{\dot{\alpha}}(x) = (\bar{s}^T)^{-1}(\Lambda)^{\dot{\alpha}}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}}(\Lambda^{-1}x) = \bar{\psi}^{\dot{\beta}}(\Lambda^{-1}x) \bar{s}^{-1}(\Lambda)_{\dot{\beta}}^{\dot{\alpha}}, \tag{7.22}$$

where

$$\bar{s}^{-1}(\Lambda)_{\dot{\beta}}^{\dot{\alpha}} = \delta_{\dot{\beta}}^{\dot{\alpha}} - \frac{i}{4}(\bar{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} \omega^{\mu\nu}, \quad (7.23)$$

with

$$(\bar{\sigma}^{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} = \frac{i}{2}(\bar{\sigma}^{\mu}\sigma^{\nu} - \bar{\sigma}^{\nu}\sigma^{\mu})_{\dot{\beta}}^{\dot{\alpha}}. \quad (7.24)$$

Also in the  $(0, \frac{1}{2})$  representation, we can define an invariant tensor  $\epsilon^{\dot{\alpha}\dot{\beta}}$  and its inverse  $\epsilon_{\dot{\alpha}\dot{\beta}}$  by exactly the same arguments as for the  $(\frac{1}{2}, 0)$  representation. The relations defining these tensors are identical to eq. (7.11). Lowering and raising dotted indices works in exactly the same way as for undotted indices, i.e. there are relations identical to eqs. (7.12), (7.13) and (7.17) to (7.19), but with all undotted indices replaced by dotted ones. The conventions for contractions without explicitly written indices is, however, different: for right-handed spinors, we define

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \quad (7.25)$$

#### 7.1.4 Conjugate spinors

The next question to answer is what happens when we take the hermitian conjugate of a spinor field. We know already from section 3.4.1 that, if  $\psi$  is a left-handed spinor field, it describes left-handed particles and *right-handed* antiparticles. Therefore,  $\psi^{\dagger}$  must describe right-handed particles and left-handed antiparticles, and should therefore transform under  $(0, \frac{1}{2})$  of  $SO^+(1, 3)$ . (An alternative way of seeing this is to note that the transformation of a RH spinor, eq. (7.23), is just the Hermitian conjugate of the transformation of a LH spinor, eq. (7.5).) Thus,  $\psi^{\dagger}$  should come with a dotted index:  $\psi_{\dot{\alpha}}^{\dagger}$ . Note that this notation means  $(\psi^{\dagger})_{\dot{\alpha}}$ , *not*  $(\psi_{\dot{\alpha}})^{\dagger}$ ! The fact that, for a left-handed field  $\psi$ , its hermitian conjugate  $\psi^{\dagger}$  is always a right-handed field, justifies a posteriori the notation  $\bar{\psi}$  that we have used above. When working with Weyl (2-component) spinors, one typically writes all right-handed spinors as hermitian conjugates of left-handed spinors.

#### 7.1.5 Lorentz invariance of the Pauli matrices

There is one more object invariant under Lorentz transformations: the Pauli matrices, provided we write them with one undotted and one dotted index:  $\sigma_{\alpha\dot{\alpha}}^{\mu}$ . This follows directly from eq. (3.72), which read

$$S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda) = \Lambda^{\mu}_{\nu}\gamma^{\nu}. \quad (7.26)$$

Remembering that the  $\gamma$  matrices are composed of Pauli matrices, we can conclude immediately that, in two-component notation

$$s^{-1}(\Lambda)\sigma^{\mu}(\bar{s}^T)^{-1}(\Lambda) = \Lambda^{\mu}_{\nu}\sigma^{\nu}$$

$$\sigma^\mu = \Lambda^\mu{}_\nu s(\Lambda) \sigma^\nu \bar{s}^T(\Lambda) \quad (7.27)$$

Since the first index on  $\sigma^\nu$  transforms with  $s(\Lambda)$ , it must be an undotted index. Since the second one transforms with  $\bar{s}(\Lambda)$ , it must be a dotted index. Putting in spinor indices, eq. (7.27) can be rewritten as

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = \Lambda^\mu{}_\nu s(\Lambda)_\alpha{}^\beta \bar{s}(\Lambda)_{\dot{\alpha}}{}^{\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}}. \quad (7.28)$$

Here, the index structure of  $s(\Lambda)$  can be read off from eq. (7.5). That of  $\bar{s}^T(\Lambda)$  must be the same as that of  $(\bar{s}^T)^{-1}(\Lambda)$  from eqs. (7.22) and (7.23).

In the same way, we can also show that  $\bar{\sigma}^{\mu\dot{\alpha}\alpha}$  is invariant under Lorentz transformations. Alternatively, we can also show this by using that

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}}. \quad (7.29)$$

### 7.1.6 One more example: the vector current

To get some practice with the formalism of dotted and undotted spinor indices, consider a vector current:

$$\psi^\dagger \bar{\sigma}^\mu \chi = \psi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \chi_\beta. \quad (7.30)$$

Here, according to our conventions  $\chi$  and  $\psi$  are left-handed and thus have undotted indices, while  $\psi^\dagger \equiv \bar{\psi}$  is right handed and has a dotted index.

Since the spinor indices in eq. (7.30) are all contracted, the vector current transforms like a Lorentz vector:

$$\psi^\dagger(x) \bar{\sigma}^\mu \chi(x) \rightarrow \Lambda^\mu{}_\nu \psi^\dagger(\Lambda^{-1}x) \bar{\sigma}^\nu \chi(\Lambda^{-1}x). \quad (7.31)$$

Let us compute the hermitian conjugate of this vector current. This is something we need to do for instance when computing the squared matrix element  $|\overline{\mathcal{M}}|^2$  for a scattering process. We find

$$\begin{aligned} [\psi^\dagger \bar{\sigma}^\mu \chi]^\dagger &= [\psi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \chi_\beta]^\dagger \\ &= \chi^\dagger_{\dot{\beta}} (\bar{\sigma}^{\mu*})^{\dot{\beta}\alpha} \psi_\alpha \\ &= \chi^\dagger_{\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\alpha} \psi_\alpha \\ &= \chi^\dagger \bar{\sigma}^\mu \psi. \end{aligned} \quad (7.32)$$

The third equality follows from the hermiticity of  $\bar{\sigma}^\mu$ . Remember that, for 4-component spinors we had  $[\bar{\psi} \gamma^\mu \chi]^\dagger = \bar{\chi} \gamma^\mu \psi$ . We have now derived the 2-component version of this identity.

Another useful relation is

$$\psi^\dagger \bar{\sigma}^\mu \chi = \psi^\dagger_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\beta} \chi_\beta$$



$$\begin{aligned}
 &= \psi^{\dagger\dot{\alpha}} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\sigma}^{\mu\dot{\beta}\gamma} \epsilon_{\gamma\delta} \chi^{\delta} \\
 &= -\chi^{\delta} \sigma_{\delta\dot{\alpha}}^{\mu} \psi^{\dagger\dot{\alpha}} \\
 &= -\chi \sigma^{\mu} \psi^{\dagger},
 \end{aligned} \tag{7.33}$$

where in the third step we have used eq. (7.29) as well as the anticommutation property of fermion fields.

## 7.2 The QED Lagrangian in 2-Component Notation

As a preparation for the discussion of the Standard Model of particle physics and its extensions in the second part of this course, let us rewrite the QED Lagrangian using 2-component spinor fields. We write the 4-component electron field  $\Psi$  as

$$\Psi = \begin{pmatrix} \psi_{\alpha} \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{7.34}$$

The QED Lagrangian is then

$$\begin{aligned}
 \mathcal{L}_{\text{QED}} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i\not{\partial} - m) \Psi - e \bar{\Psi} \gamma^{\mu} \Psi A_{\mu} \\
 &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \bar{\sigma}^{\mu} \partial_{\mu} \psi + i\chi \sigma^{\mu} \partial_{\mu} \bar{\chi} - m\bar{\psi} \bar{\chi} - m\chi \psi - e\bar{\psi} \bar{\sigma}^{\mu} \psi - e\chi \sigma^{\mu} \bar{\chi} \\
 &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \bar{\sigma}^{\mu} \partial_{\mu} \psi + i\bar{\chi} \bar{\sigma}^{\mu} \partial_{\mu} \chi - m\bar{\psi} \bar{\chi} - m\chi \psi - e\bar{\psi} \bar{\sigma}^{\mu} \psi + e\bar{\chi} \bar{\sigma}^{\mu} \chi.
 \end{aligned} \tag{7.35}$$

In the last line, we have used eq. (7.33) followed by an integration by parts on the kinetic term of  $\chi$ . We have also applied eq. (7.33) to the gauge interaction term of  $\chi$ , where it introduces a minus sign compared to the corresponding term for  $\psi$ . This makes sense: remember that  $\bar{\chi}$  is the right-handed component of  $\Psi$ , i.e. it has the same charge as  $\psi$ . Consequently,  $\chi$  must have opposite charge. As expected, we see that, apart from the mass terms,  $\psi$  and  $\chi$  are decoupled.

Note that eq. (7.35) is invariant under the transformation

$$\begin{aligned}
 \psi_{\alpha}(x) &\rightarrow C^{-1} \psi_{\alpha}(x) C \equiv \chi_{\alpha}(x), \\
 \chi_{\alpha}(x) &\rightarrow C^{-1} \chi_{\alpha}(x) C \equiv \psi_{\alpha}(x), \\
 e &\rightarrow -e,
 \end{aligned} \tag{7.36}$$

which corresponds to *charge conjugation*. To see that it is equivalent to the  $C$  transformation from section 3.4.4, note that eq. (7.36) is implemented on the 4-component spinor  $\Psi = (\psi_{\alpha}, \bar{\chi}^{\dagger\dot{\alpha}})^T$  by the operation

$$\Psi(x) \rightarrow \Psi^c(x) \equiv \mathcal{C} \bar{\Psi}^T(x) \tag{7.37}$$

with the *charge conjugation matrix*

$$\mathcal{C} \equiv -i\gamma^2\gamma^0. \quad (7.38)$$

(This is identical to eq. (3.149).) In fact, when written in components, we have

$$\mathcal{C} = -i \begin{pmatrix} & \sigma^2 \\ \bar{\sigma}^2 & \end{pmatrix} \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \quad (7.39)$$

$$= \begin{pmatrix} -i\sigma^2 & \\ & -i\bar{\sigma}^2 \end{pmatrix} \quad (7.40)$$

$$= \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}, \quad (7.41)$$

and therefore

$$\mathcal{C}\bar{\Psi}^T(x) = \mathcal{C}\gamma^0\Psi^*(x) \quad (7.42)$$

$$= \begin{pmatrix} & -i\sigma^2 \\ -i\bar{\sigma}^2 & \end{pmatrix} \begin{pmatrix} \psi^\dagger_\beta \\ \chi^\beta \end{pmatrix} \quad (7.43)$$

$$= \begin{pmatrix} & \epsilon_{\alpha\beta} \\ \epsilon^{\dot{\alpha}\dot{\beta}} & \end{pmatrix} \begin{pmatrix} \psi^\dagger_{\dot{\beta}} \\ \chi^{\dot{\beta}} \end{pmatrix} \quad (7.44)$$

$$= \begin{pmatrix} \chi_\alpha \\ \psi^{\dagger\dot{\alpha}} \end{pmatrix}. \quad (7.45)$$

One conclusion we can draw from this is that charge conjugation transforms a left-handed 4-spinor  $(\psi_\alpha, 0)^T$  into a right-handed 4-spinor  $(0, \psi^{\dagger\dot{\alpha}})$ .

Note that the matrix  $\mathcal{C}$  has the following useful properties:

$$\mathcal{C}^T = \mathcal{C}^\dagger = \mathcal{C}^{-1} = -\mathcal{C} \quad \text{and} \quad \mathcal{C}^* = \mathcal{C}. \quad (7.46)$$

This can be used for instance to show that applying charge conjugation twice is the identity:

$$(\Psi^c)^c = \mathcal{C}\gamma^0(\Psi^c)^* = \mathcal{C}\gamma^0(\mathcal{C}\gamma^0\Psi^*)^* = -\mathcal{C}^2(\gamma^0)^2\Psi = \Psi. \quad (7.47)$$

Other relations that are sometimes useful are

$$\overline{\Psi^c} = (\mathcal{C}\gamma^0\Psi^*)^\dagger\gamma^0 = (\gamma^0\mathcal{C}\gamma^0\Psi)^T = (-\mathcal{C}\Psi)^T = \mathcal{C}\Psi^T \quad (7.48)$$

and

$$\bar{\Psi}\Theta^c = \Psi^\dagger\gamma^0\mathcal{C}\gamma^0\Theta^*$$

$$\begin{aligned}
 &= -\Theta^\dagger (\gamma^0)^T \mathcal{C}^T (\gamma^0)^T \Psi^* \\
 &= \bar{\Theta} \Psi^c.
 \end{aligned} \tag{7.49}$$

In the second equality, we have used the fact that all spinor indices are contracted, i.e. the expression is just a number and we can without loss of generality replace it with its transpose. We have to introduce a minus sign, however, since the fermion fields  $\Theta$  and  $\Psi$  change places. (Write out the indices to see this!) In analogy to eq. (7.49), we also have

$$\bar{\Psi}^c \Theta = \bar{\chi}^c \Psi. \tag{7.50}$$

### 7.3 Majorana Fermions

We could also have rewritten the Lagrangian eq. (7.35) by substituting

$$\psi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \tag{7.51}$$

$$\chi = \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2). \tag{7.52}$$

Plugging this into eq. (7.35) and using eqs. (7.21) and (7.32) leads to

$$\begin{aligned}
 \mathcal{L}_{\text{QED}} = & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\xi}_1\bar{\sigma}^\mu\partial_\mu\xi_1 + i\bar{\xi}_2\bar{\sigma}^\mu\partial_\mu\xi_2 - \frac{1}{2}(m\xi_1\xi_1 + m\xi_2\xi_2 + h.c.) \\
 & - e\bar{\xi}_1\bar{\sigma}^\mu\xi_2 - e\bar{\xi}_2\bar{\sigma}^\mu\xi_1.
 \end{aligned} \tag{7.53}$$

The kinetic and mass terms now describe two completely independent 2-component spinor fields. The gauge interaction term is instead off-diagonal.

For *uncharged* particles (i.e. particles without electromagnetic gauge interaction terms, for instance neutrinos), the decoupling of  $\xi_1$  and  $\xi_2$  is complete. Removing one of them from the theory should therefore still lead to a valid model, with a Lagrangian of the form

$$\mathcal{L}_{\text{Majorana}} = i\bar{\xi}\bar{\sigma}^\mu\partial_\mu\xi - \frac{1}{2}m\xi\xi - \frac{1}{2}m\xi^\dagger\xi^\dagger. \tag{7.54}$$

A field of this type is called a *Majorana fermion*. It is a *massive* fermion with only *two* degrees of freedom. This is possible because the right-handed state of a Majorana fermion is identical to the antiparticle of the left-handed state. To see this, define

$$\Psi(x) \equiv \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}, \tag{7.55}$$

which implies that  $\Psi^c = \Psi$ . We can also rewrite eq. (7.54) in 4-component notation:

$$\mathcal{L}_{\text{Majorana}} = \frac{1}{2}\bar{\Psi}\gamma^\mu i\partial_\mu\Psi - \frac{1}{2}m\bar{\Psi}\Psi. \tag{7.56}$$

## 7.4 Application: Majorana Neutrinos and the Seesaw Mechanism

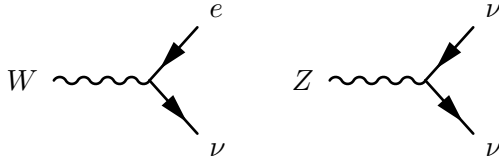
As alluded to before, neutrinos have all the properties required to be Majorana fermions—in particular they are electrically neutral. In fact, we do not know whether they are Dirac or Majorana particles. The interactions of the Standard Model (SM) couple only to left-handed neutrinos (and their antiparticles, the right-handed anti-neutrinos). The only interactions that couple right-handed neutrinos to left-handed neutrinos are the mass terms, which are tiny for neutrinos. Thus, if right-handed neutrinos exist as distinct particles (i.e. if neutrinos are Dirac), the only way to produce or detect them is through interactions that are suppressed by the tiny neutrino masses. With present experiments, it is impossible to detect such tiny interactions. Our best hope is the search for *neutrinoless double beta decay*, a process in which two (virtual) neutrinos annihilate. This is only possible if neutrinos are their own antiparticles, i.e. for Majorana neutrinos.

### 7.4.1 Neutrino mass terms

Let us now elaborate further on the physics of Majorana neutrinos. For phenomenological purposes, we could simply write down a mass term

$$\mathcal{L} \supset -\frac{1}{2} m^{ij} \nu_L^i \nu_L^j + h.c., \quad (7.57)$$

where  $\nu_L^j$  is the 2-component neutrino spinor, and  $j$  is a flavor index. (There are three species, or “flavors” of neutrinos:  $\nu_e$ ,  $\nu_\mu$ ,  $\nu_\tau$ , just like there are three flavors of charged leptons: the electron, the muon and the tau lepton.) Note that we have kept an index  $L$  in eq. (7.57) to remind ourselves that this is a left-handed spinor. In the context of the Standard Model, this index also implies that  $\nu_L^j$  couples to the weak interaction, which affects only left-handed particles. The interaction vertices are



where  $W$  and  $Z$  are the gauge bosons mediating weak interactions, and  $\ell^j$  are the charged lepton fields. But let's come back to the mass term in eq. (7.57). One may ask the question whether the “charge” of the neutrinos under the weak interaction does not forbid this mass term, just like the electric charge forbids Majorana mass terms for the other SM fermions. The truth is, eq. (7.57) *is* forbidden in the SM. In fact, *all* fermion mass terms (Dirac and Majorana) are forbidden. Then, why are fermions obviously massive? The solution is the Higgs mechanism, which we will discuss in detail in the second part of this course, the essentials of which we can however already understand now. The SM symmetries allow for the following Yukawa couplings:

$$\mathcal{L} \supset y^{ij} L^i \tilde{H} N^j + \dots + h.c., \quad (7.58)$$

where  $L^i = (\nu_L^i, \ell_L^i)^T$  consists of the left-handed neutrino and charged lepton fields of the  $i$ -th generating, arranged here in a 2-vector, and  $\tilde{H}$  is the Higgs field, which is also a 2-vector. (The reason for arranging these particles in 2-vectors is that they form a 2-dimensional representation of  $SU(2)$ , one of the symmetry groups of the SM.) The field  $N$  in eq. (7.58) describes the right-handed neutrinos, but as before, we write a right handed field as the hermitian conjugate of a left-handed one, i.e.  $N$  are the left-handed antiparticles of the RH neutrinos, and  $\bar{N}$  are the RH neutrinos themselves. Note that in the original form of the SM, right-handed neutrinos are not included. Therefore, the original SM predicted neutrinos to be exactly massless. With the discovery of neutrino oscillations, which prove that neutrinos have nonzero masses, the model had to be amended. The dots in eq. (7.58) represent similar couplings for the charged leptons and the quarks. The crucial point is now that  $\langle \Omega | \tilde{H} | \Omega \rangle = (v, 0)^T \neq 0$ , i.e. the Higgs field has a non-vanishing vacuum expectation value. In other words, even when no Higgs particles are present in a given process, the field is non-zero. Expanding  $\tilde{H}$  around this vacuum expectation value turns the Yukawa couplings in eq. (7.58) into

$$\mathcal{L} \supset \underbrace{y^{ij} v}_{\equiv m_D^{ij}} \nu_L^i N^j + \dots + h.c., \quad (7.59)$$

a Dirac mass term for neutrinos. (The notation  $m_D^{ij}$  indicates that this parameter describes a Dirac mass.) The dots again indicate many terms we have omitted because they are not relevant to our present discussion.

### 7.4.2 The seesaw mechanism

But the story does not end here: the right-handed singlet neutrinos  $\bar{N}^j$  are uncharged under all SM symmetries, i.e. they can have a Majorana mass term

$$\mathcal{L} \supset -\frac{1}{2} M^{ij} N^i N^j + h.c.. \quad (7.60)$$

The total Lagrangian describing neutrino masses (composed of eq. (7.59) and eq. (7.60)) can be written in matrix form:

$$\mathcal{L} \supset \frac{1}{2} (\nu_L, N) \begin{pmatrix} 0 & m_D \\ m_D^T & M \end{pmatrix} \begin{pmatrix} \nu_L \\ N \end{pmatrix} + h.c.. \quad (7.61)$$

Here, it is implied that  $\nu_L = (\nu_L^e, \nu_L^\mu, \nu_L^\tau)^T$  contains the three left-handed SM neutrino flavors, and  $N = (N^1 \dots N^n)$  contains the singlet neutrinos.  $m_D$  is thus a  $3 \times n$  matrix and  $M$  is an  $n \times n$  matrix.

We see that the neutrino mass matrix is off-diagonal, i.e. there is *mixing* between the  $\nu_L$  and  $N$  fields. For instance, an electron neutrino does not have a definite mass, but is a superposition of several mass eigenstates. Similarly, a neutrino of definite mass is a mixture of different flavor eigenstates. To transform the mass matrix in eq. (7.61) to the mass basis, we diagonalize it by applying a unitary transformation to the neutrino state.

To obtain the form of this transformation, let us consider for simplicity a simplified model with only one  $\nu_L$  flavor and one  $N$  flavor. The desired transformation can be written as

$$\begin{pmatrix} \nu_{mL} \\ N_m \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_L \\ N \end{pmatrix}. \quad (7.62)$$

Here, the subscript  $m$  denotes mass eigenstates. By computing

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (7.63)$$

and requiring that the off-diagonal elements vanish, we find for the mixing angle

$$\tan 2\theta = -\frac{2m_D}{M}. \quad (7.64)$$

The eigenvalues of the mass matrix are

$$m_{1,2} = \frac{1}{2}(M \mp \sqrt{M^2 + 4m_D^2}), \quad (7.65)$$

and the corresponding mass terms read

$$\mathcal{L}_{2\text{-flavor}} \supset -\frac{1}{2}m_1\nu_{mL}\nu_{mL} - \frac{1}{2}m_2N_mN_m + h.c.. \quad (7.66)$$

There are theoretical reasons to believe that  $M \gg m_D$ . Namely, while  $m_D$  is determined by the vacuum expectation value (vev) of the Higgs field  $v \sim 174$  GeV according to eq. (7.59), the scale of  $M$  is completely free. (Note that  $y^{ij}$  cannot be much larger than  $\mathcal{O}(1)$  in a consistent QFT, and values much smaller than  $\mathcal{O}(1)$  are considered unnatural by many theorists.) If  $M$  is determined by some underlying, more complete, theory, the typical mass scales of that theory should be  $\gg v$ , otherwise we would have discovered it already. If  $M \gg m_D$ , we see from eq. (7.65) that one neutrino mass eigenstate is of order  $M$ , while the other is

$$m_1 \simeq \frac{m_D^2}{M}. \quad (7.67)$$

This provides a natural explanation of the smallness of neutrino masses. For instance, for  $m_D \sim 100$  GeV,  $M \sim 10^{13}$  GeV, we obtain neutrino masses of order 0.1 eV, consistent with experimental constraints.

The above mechanism is called the *type-I seesaw mechanism* because the fact that the  $N$  have a very large mass term reduces the effective mass of the light neutrinos, which are mostly  $\nu_L$ , with a small admixture of  $N$ .

Lifting the restrictions of the 2-flavor toy model and treating  $m_D$  and  $M$  as matrices again, the seesaw formula eq. (7.67) generalizes to

$$m_\nu \simeq -m_DM^{-1}m_D^T. \quad (7.68)$$

Note that the matrices  $m_D$  and  $M$  can be off-diagonal.  $m_D$  can in fact be an arbitrary complex matrix, while  $M$  has to be complex symmetric. This follows from the form of the  $N$  mass term,  $M^{ij}N^iN^j + h.c.$  (see eq. (7.60)), which shows that any antisymmetric contribution would cancel and thus be unphysical. The off-diagonal nature of the mass matrices implies that also the SM neutrinos can mix among each other. This means, for instance, that the electron neutrino does not have a definite mass, but is a mixture of the three light mass eigenstate (plus a tiny, usually negligible admixture of order  $m_D/M$  of the heavy mass eigenstates). Conversely, the light neutrino mass eigenstates are mixtures of  $\nu_e$ ,  $\nu_\mu$  and  $\nu_\tau$  (and a tiny admixture of the heavy flavor eigenstates). This mixing is at the core of the so-called *neutrino oscillations*.

### 7.4.3 Interlude: measuring neutrino masses

#### Direct neutrino mass measurements

Let us pause for a moment to discuss what we know experimentally about neutrino masses. Obviously, decay processes involving neutrinos depend on their masses. The most direct way of measuring neutrino masses is by studying the kinematics of  $\beta$  decay processes like

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_e. \quad (7.69)$$

The maximum energy of the electron is

$$E_e^{\max} = Q - \min_j(m_j), \quad (7.70)$$

where  $Q = m_{\text{H}} - m_{\text{He}}$  is the decay energy and  $m_j$  are the neutrino mass eigenstates.

The spectrum of electron energies from  $\beta$  decay is given by

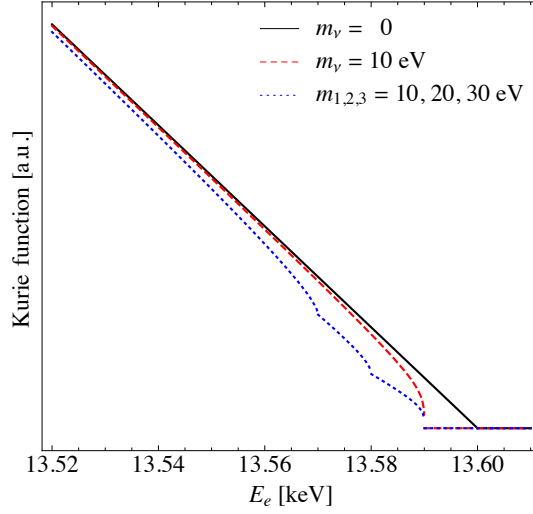
$$\frac{dN_e(E_e)}{dE_e} \propto F(Z, E_e) \sqrt{E_e^2 - m_e^2} E_e (Q - E_e) \sqrt{(Q - E_e)^2 - m_{\bar{\nu}_e}^2} \theta(Q - E_e - m_{\bar{\nu}_e}). \quad (7.71)$$

Here, the *Fermi function*  $F(Z, E_e)$  accounts for the interaction of the produced electron with the Coulomb field of the nucleus. By  $\theta(x)$  we denote the Heaviside function.

To make the dependence on the neutrino mass more visible, one defines the *Kurie functions*

$$K(E_e) \equiv \sqrt{\frac{dN_e(E_e)/dE_e}{F(Z, E_e)p_e E_e}}. \quad (7.72)$$

For  $m_{\bar{\nu}_e} = 0$ , it should be a straight line  $\propto Q - E_e$ , while for  $m_{\bar{\nu}_e} \neq 0$ , it has a cutoff, see fig. 7.1.



**Figure 7.1:** The Kurie plot.

It is important that  $\bar{\nu}_e$  does not have a definite mass, i.e. when we write  $m_{\bar{\nu}_e}$  above, this is actually ill-defined. We should therefore regard the process (7.69) as the combination of the processes

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_1, \quad (7.73)$$

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_2, \quad (7.74)$$

$${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \bar{\nu}_3. \quad (7.75)$$

Since we can only detect the electron, we do not know on an event-by-event basis which of these processes has occurred, hence we always have to take all three of them into account. Thus, in particular, the maximum energy of the electron  $E_e^{\text{max}}$  is given by the maximum of the kinematic endpoints of the three processes. For neutrino mass  $m_j \gg \max_k |m_j - m_k|$ , this subtlety is negligible.

For smaller neutrino masses, the Kurie function is

$$K(E_e) = \sqrt{|U_{e1}|^2[K_1(E_e)]^2 + |U_{e2}|^2[K_2(E_e)]^2 + |U_{e3}|^2[K_3(E_e)]^2}, \quad (7.76)$$

where  $U_{ej}$  are elements of the leptonic mixing matrix that determine the admixture of the mass eigenstate  $\nu_j$  to the flavor eigenstate  $\nu_e$ , and

$$K_j(E_e) \equiv \mathcal{C} \sqrt{(Q - E_e) \sqrt{(Q - E_e)^2 - m_j^2}} \theta(Q - E_e - m_j). \quad (7.77)$$

Here,  $\mathcal{C}$  is a  $j$ -independent normalization constant. For  $E_e$  not too close to the endpoint, we can expand

$$\sqrt{(Q - E_e)^2 - m_j^2} \simeq Q - E_e - \frac{m_j^2}{2(Q - E_e)}. \quad (7.78)$$



Plugging this into eq. (7.77) and (7.76), we get

$$K(E_e) \simeq \mathcal{C} \sqrt{(Q - E_e) \left[ Q - E_e - \sum_j |U_{ej}^2| \frac{m_j^2}{2(Q - E_e)} \theta(Q - E_e - m_j) \right]}. \quad (7.79)$$

At energies below  $Q - \max_j(m_j)$  (where all the  $\theta$  functions are 1), this can be written as

$$K(E_e) \simeq \mathcal{C} \sqrt{(Q - E_e) \sqrt{(Q - E_e)^2 - m_{\bar{\nu}_e}^2}}, \quad (7.80)$$

with the *definition* of the *effective mass*

$$m_{\bar{\nu}_e} \equiv \sqrt{\sum_j |U_{ej}^2| m_j^2}. \quad (7.81)$$

Very close to the endpoint, this description breaks down. Instead, several kinks are expected in the spectrum.

### Neutrinoless double beta decay

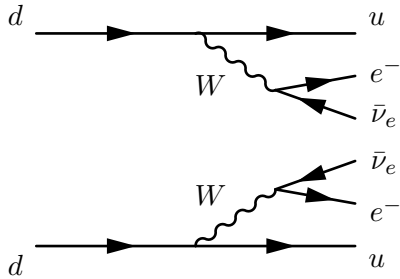
A second way of obtaining information on the masses of Majorana neutrinos is *neutrinoless double beta decay*. To introduce this process, consider the level scheme in fig. 7.2, taken from a table of isotopes [3]. Note that the isotopes in general follow two parabolas: a higher energy one for the odd–odd nuclei (i.e. nuclei with an odd number of protons and an odd number of neutrons), and a lower energy one for the even–even nuclei. This leads to the situation that the energy of  $^{136}\text{Xe}$  is lower than that of the isotope directly to its right,  $^{136}\text{Cs}$ , so that the direct  $\beta^-$  decay of  $^{136}\text{Xe}$  to  $^{136}\text{Cs}$  is energetically forbidden. On the other hand, two simultaneous  $\beta^-$  decays are allowed:

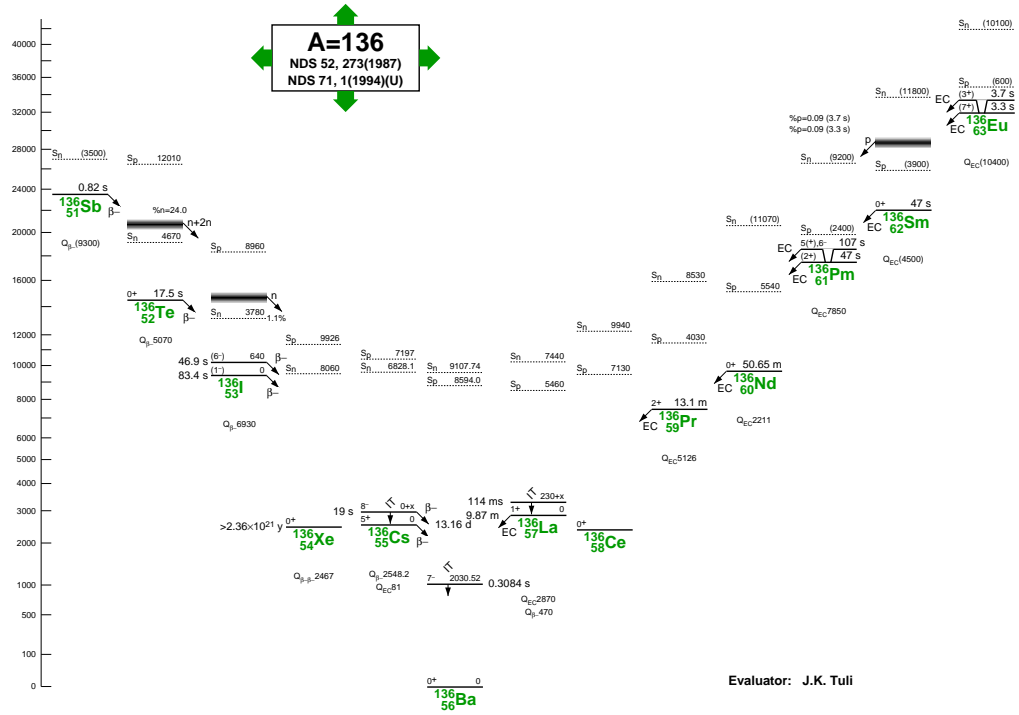
$$^{136}\text{Xe} \rightarrow ^{136}\text{Ba} + 2e^- + 2\bar{\nu}_e. \quad (7.82)$$

The Feynman diagram for such a process of the form

$$(A, Z) \rightarrow (A, Z - 2) + 2e^- + 2\bar{\nu}_e \quad (7.83)$$

is





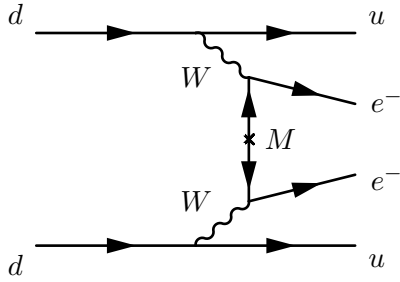
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**Figure 7.2:** Level scheme for  $A = 136$  nuclei. Note that  $^{136}\text{Xe}$  cannot undergo direct  $\beta^-$  decay to  $^{136}\text{Cs}$ , but  $0\nu 2\beta$  decay to  $^{136}\text{Ba}$  is allowed. Similarly,  $^{136}\text{Ce}$  cannot decay via  $\beta^+$  decay to  $^{136}\text{La}$ , but  $0\nu 2\beta^+$  decay to  $^{136}\text{Ba}$  is allowed.

Remember that replacing an outgoing antiparticle in a Feynman diagram by an ingoing particle yields a valid Feynman diagram as well (“crossing symmetry”). Therefore, if the neutrino is its own antiparticle (Majorana neutrinos), the two neutrino lines in the above diagram can be connected. In other words, the neutrino emitted in the decay of the first down quark can be absorbed by the second down quark. This leads to *neutrinoless double beta decay*

$$(A, Z) \rightarrow (A, Z - 2) + 2e^{-}, \quad (7.84)$$

the Feynman diagram for which is



This diagram is sometimes called “lobster diagram”. Since my artistic skills are not sufficient to explain why, I have to appeal to the reader’s imagination . . .

The crucial point is that the rate of neutrinoless double beta decay is proportional to the neutrino masses. To see this, we must know that the  $W$  boson couples only to left-handed particles and right-handed antiparticles. For instance, the upper vertex in the lobster diagram can be viewed as emitting a LH electron and a RH antineutrino. The lower vertex absorbs a LH neutrino and emits a LH electron. This is only possible if the neutrino experiences a chirality flip along the vertical propagator, i.e. if there is a mass insertion (denoted by a little cross in the Feynman diagram).

While in two-neutrino double beta decay, part of the decay energy is carried away by the neutrinos, in neutrinoless double beta decay it is all carried by the electrons and thus visible to a detector. The telltale signature of neutrinoless double beta decay is thus a monoenergetic peak at the  $Q$  value of the decay.

## 7.5 Twistors

### 7.5.1 Unifying spinors and momentum 4-vectors

In high energy physics, one often deals with particles that are approximately massless. For instance, at LHC energies of multiple TeV, the up, down and strange quark masses of order MeV are completely negligible. We now introduce a formalism that allows scattering amplitudes involving massless particles to be written in a very compact form. Consider a massless spinor  $u^s(p)$ , with  $s = \pm$ . Using the spin sum  $\sum_r u^r(p)\bar{u}^r(p) = \not{p}$

(see eq. (3.25)), we can write its outer product with itself as

$$u^s(p)\bar{u}^s(p) = \frac{1+s\gamma^5}{2} \left( \sum_r u^r(p)\bar{u}^r(p) \right) \frac{1-s\gamma^5}{2} = \frac{1+s\gamma^5}{2} \not{p}, \quad (7.85)$$

where the projection operators  $(1 \pm s\gamma^5)/2$  make sure that only the term with  $r = s$  contributes in the spin sum. Similarly, for  $v$  spinors:

$$v^s(p)\bar{v}^s(p) = \frac{1-s\gamma^5}{2} \left( \sum_r v^r(p)\bar{v}^r(p) \right) \frac{1+s\gamma^5}{2} = \frac{1-s\gamma^5}{2} \not{p}, \quad (7.86)$$

Note that for massless spinors we have  $v^s(p) = u^{-s}(p)$ . This follows from the explicit expressions for the  $u$  and  $v$  spinors when taking the limit  $m \rightarrow 0$ , see eqs. (3.107) and (3.108). Therefore, in the following, it is sufficient to focus on  $u$  spinors only.

According to the Feynman rules, scattering amplitudes for massless particles depend on spinors and on 4-momenta. We will now write the momenta in terms of spinor-like objects as well. This will lead to some simplifications. We define

$$p_{\alpha\dot{\alpha}} \equiv p_\mu \sigma^\mu_{\alpha\dot{\alpha}}. \quad (7.87)$$

We have already mentioned in eq. (7.29) that  $\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} \sigma^\mu_{\beta\dot{\beta}}$ . Therefore, we also have

$$p^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} p_{\beta\dot{\beta}} = p_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha}. \quad (7.88)$$

Remembering that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \frac{1}{2}(1 - \gamma^5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (7.89)$$

we can write

$$u^-(p)\bar{u}^-(p) = \begin{pmatrix} 0 & p_{\alpha\dot{\alpha}} \\ 0 & 0 \end{pmatrix}. \quad (7.90)$$

To avoid all the zeros in this matrix, define

$$u^-(p) \equiv \begin{pmatrix} \phi_\alpha(p) \\ 0 \end{pmatrix}, \quad \text{and thus} \quad \bar{u}^-(p) \equiv (\phi_\alpha^\dagger(p), 0). \quad (7.91)$$

Then, we have simply

$$p_{\alpha\dot{\alpha}} = \phi_\alpha(p) \phi_{\dot{\alpha}}^\dagger(p). \quad (7.92)$$

The 2-component object  $\phi_\alpha(p)$  is called a “twistor”. It carries a spinor index and transforms accordingly under Lorentz transformation. Since twistors are not fields, but just a numerical quantity, they are *not* anticommuting, though.

We can repeat the above arguments also for positive helicity spinors

$$u^+(p) \equiv \begin{pmatrix} 0 \\ \phi^{*\dot{\alpha}}(p) \end{pmatrix}, \quad \text{and} \quad \bar{u}^+(p) \equiv (\phi^{\alpha T}(p), 0), \quad (7.93)$$

with  $\phi^\alpha(p) = \epsilon^{\alpha\beta} \phi_\beta(p)$ . Note that the  $\phi(p)$  appearing in  $u^+(p)$  is the same as the one appearing in  $u^-(p)$  (up to the raising and lowering of indices). This follows again from the explicit expressions eq. (3.107). We can also write  $p$  in terms of  $u^+(p)$ :

$$u^+(p) \bar{u}^+(p) = \begin{pmatrix} 0 & 0 \\ p^{\dot{\alpha}\alpha} & 0 \end{pmatrix}. \quad (7.94)$$

or

$$p^{\dot{\alpha}\alpha} = \phi^{\dagger\dot{\alpha}}(p) \phi^\alpha(p). \quad (7.95)$$

We have thus seen that both the particles spinor and its momentum can be expressed in terms of the twistor. It is therefore possible to express a Feynman amplitude for massless particles entirely in terms of twistors.

### 7.5.2 Twistor notation

At this point, we again introduce some notation:

$$|p] \equiv u^-(p) = v^+(p) \quad (7.96)$$

$$|p\rangle \equiv u^+(p) = v^-(p) \quad (7.97)$$

$$[p| \equiv \bar{u}^+(p) = \bar{v}^-(p) \quad (7.98)$$

$$\langle p| \equiv \bar{u}^-(p) = \bar{v}^+(p) \quad (7.99)$$

Given two 4-momentum vectors  $p$  and  $k$ , and the corresponding twistors  $\phi_\alpha(p)$  and  $\kappa_\alpha(k)$ , we thus have

$$\begin{aligned} [p k] &\equiv \phi^\alpha(p) \kappa_\alpha(k) \\ \langle p k \rangle &\equiv \phi_\alpha^* \kappa^{*\dot{\alpha}}. \end{aligned} \quad (7.100)$$

Since  $\phi^\alpha \kappa_\alpha = \epsilon^{\alpha\beta} \phi_\beta \kappa_\alpha = -\epsilon^{\beta\alpha} \kappa_\alpha \phi_\beta = \kappa^\alpha \phi_\alpha$ , we find the identity

$$[p k] = -[k p]. \quad (7.101)$$

Similarly,

$$\langle k p \rangle = -\langle p k \rangle \quad (7.102)$$

Note that these identities appear to be at odds with eq. (7.21), which stated that  $\psi\chi = \chi\psi$ . Recall, however, that eq. (7.21) holds for anticommuting spinor fields, while we

are now dealing with commuting twistors. Hence the extra minus sign. The definitions eq. (7.100) imply

$$\langle p k \rangle = [k p]^* . \quad (7.103)$$

Moreover, eqs. (7.91) and (7.93) show that

$$\bar{u}^+(p) u^-(k) = [p k] \quad (7.104)$$

$$\bar{u}^-(k) u^+(p) = \langle p k \rangle \quad (7.105)$$

$$\bar{u}^+(p) u^+(k) = \bar{u}^-(p) u^-(k) = 0 . \quad (7.106)$$

We can also rewrite the dot products of 4-momenta in Minkowski space in terms of the twistor products  $\langle p k \rangle$  and  $[p k]$  by using the identity

$$\begin{aligned} \langle p k \rangle [p k] &= (\phi_\alpha^* \kappa^{*\dot{\alpha}})(\kappa^\alpha \phi_\alpha) \\ &= -\phi_\alpha \phi_\alpha^* (\kappa^{*\dot{\alpha}} \kappa^\alpha) \\ &= -p_{\alpha\dot{\alpha}} k^{\dot{\alpha}\alpha} \\ &= -p_\mu k_\nu \sigma_{\alpha\dot{\alpha}}^\mu \bar{\sigma}^{\nu\dot{\alpha}\alpha} \\ &= -2p^\mu k_\mu . \end{aligned} \quad (7.107)$$

The last identity follows from  $\text{tr}(\sigma^\mu \bar{\sigma}^\nu) = 2g^{\mu\nu}$ .

When we did QED calculations in 4-component notation, we often encountered gamma matrices sandwiched between spinors. In twistor notation, we write this as

$$\bar{u}^+(p) \gamma^\mu u^-(k) = [p | \gamma^\mu | k] , \quad (7.108)$$

$$\bar{u}^-(p) \gamma^\mu u^+(k) = [p | \gamma^\mu | k] , \quad (7.109)$$

etc. If the gamma matrix is contracted with a momentum  $P$ , we abbreviate this by

$$\bar{u}^+(p) \not{P} u^+(k) = [p | P | k] , \quad (7.110)$$

$$\bar{u}^-(p) \not{P} u^-(k) = \langle p | P | k \rangle , \quad (7.111)$$

If  $P$  is a light-like momentum (i.e. the on-shell momentum of a massless particles), we can write  $P$  in terms of twistors, too. In particular, according to eqs. (7.92) and (7.95):

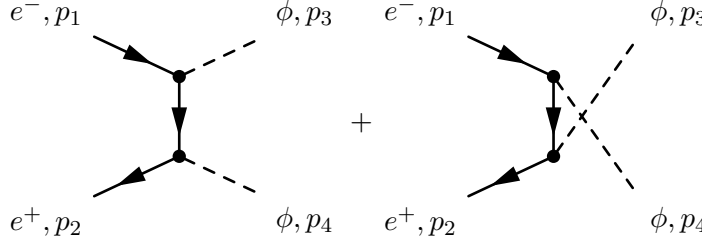
$$\begin{aligned} P_\mu \sigma_{\alpha\dot{\alpha}}^\mu &= |P\rangle \langle P| \\ P_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha} &= |P\rangle [P| , \end{aligned} \quad (7.112)$$

and thus

$$\begin{aligned} [p | P | k] &= [p |^{\dot{\alpha}} P_{\dot{\alpha}\alpha} | k]^\alpha = [p P] \langle P k \rangle , \\ \langle p | P | k \rangle &= \langle p |_\alpha P^{\alpha\dot{\alpha}} | k]_{\dot{\alpha}} = \langle p P \rangle [P k] . \end{aligned} \quad (7.113)$$

### 7.5.3 Examples

As a first example, consider the annihilation of two massless fermions into two scalars in Yukawa theory:



$$\begin{aligned}
 &= g^2 \bar{v}^{s'}(p_2) \left( \frac{i(\not{p}_1 - \not{p}_3)}{(p_1 - p_3)^2} + \frac{i(\not{p}_1 - \not{p}_4)}{(p_1 - p_4)^2} \right) u^s(p_1) \\
 &= -g^2 \bar{v}^{s'}(p_2) \left( \frac{i\not{p}_3}{2p_1 p_3} + \frac{i\not{p}_4}{2p_1 p_4} \right) u^s(p_1). \tag{7.114}
 \end{aligned}$$

In the last line, we have used the Dirac equation  $\not{p}_1 u^s(p_1) = 0$ . If, for instance,  $s = s' = +$ , we obtain

$$\begin{aligned}
 \mathcal{A}^{++} &\equiv g^2 \left( \frac{i[23] \langle 31 \rangle}{\langle 13 \rangle [13]} + \frac{i[24] \langle 41 \rangle}{\langle 14 \rangle [14]} \right) \\
 &= -ig^2 \left( \frac{[23]}{[13]} + \frac{[24]}{[14]} \right). \tag{7.115}
 \end{aligned}$$

Note that, for brevity, we have replaced  $p_j \rightarrow j$  here. Moreover, we have used eq. (7.107) to rewrite the contraction of the 4-momenta in the denominators in terms of twistors, eq. (7.113) to rewrite the momenta contracted with gamma matrices, and eqs. (7.101) and (7.102) to reorder the momenta inside twistor products.

The amplitude for  $s = s' = -$  is obtained in a similar way by replacing square brackets by angle brackets and vice-versa. Using this observation, we can directly read it off from eq. (7.115):

$$\mathcal{A}^{--} = -ig^2 \left( \frac{\langle 23 \rangle}{\langle 13 \rangle} + \frac{\langle 24 \rangle}{\langle 14 \rangle} \right). \tag{7.116}$$

If, on the other hand,  $s = s'$ , we would obtain mixed products of left-chiral and right-chiral spinors like  $\langle pk \rangle$ , which vanish.





# 8

## Radiative Corrections

Our goal in the chapter is to compute the interaction amplitude for an electron with a photon to higher orders in perturbation theory, including in particular loop diagrams.

We start from the tree level interaction between electrons and photons in QED:

$$\mathcal{L} \supset -e\bar{\psi}\gamma^\mu\psi A_\mu. \quad (8.1)$$

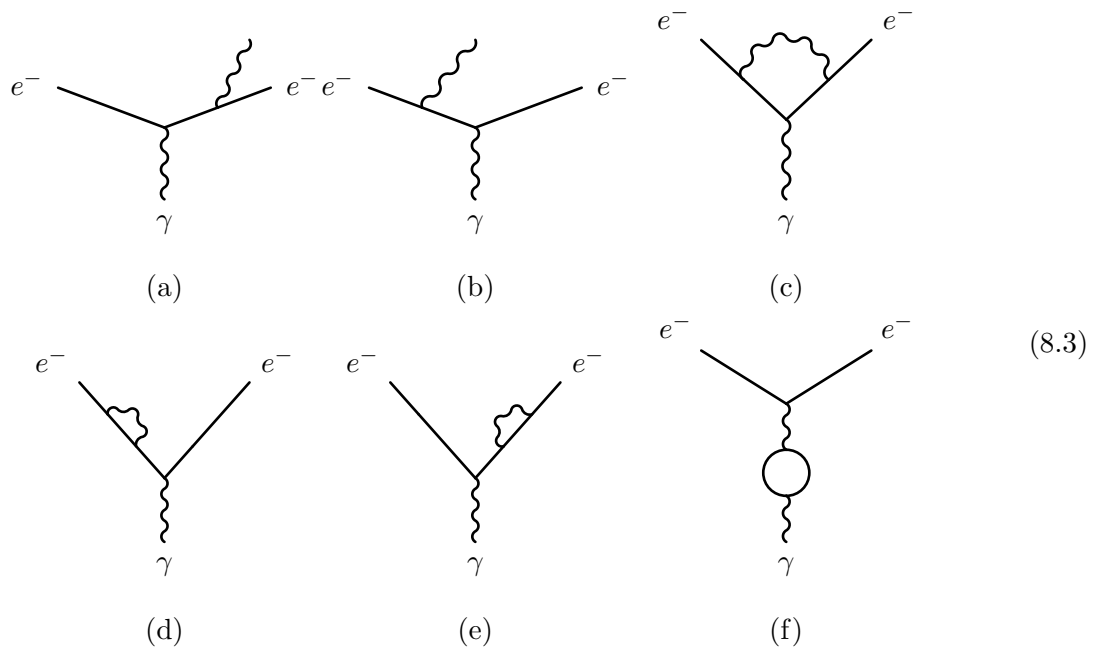
The corresponding diagram



appears as a substructure in every QED Feynman diagram, but can also stand on its own. Namely, if  $A_\mu$  is treated not as a quantum field, but as a classical (external) field, it describes the scattering of electrons of this external field. The latter situation is realized, for instance, when electrons scatter off a very heavy charged target such as a heavy ion.

We will in the following be interested in the Feynman diagrams shown in fig. 8.1. Diagrams (a) and (b) in this figure, corresponding to the emission of an extra photon, are called *real corrections*. An example for such a process is synchrotron radiation: electron interact with a strong magnetic field and emit radiation. Diagrams (c), (d), (e) and (f) are called *virtual corrections* or *loop corrections*.

We will first learn how to compute 1-loop diagrams, using (8.3) (c) as an example. We will then consider also real corrections and their significance.



**Figure 8.1:** Next-to-leading order corrections to the electron–photon vertex.

## 8.1 The Electron Vertex Function

Let us write down the Feynman amplitude of diagram (8.3) (c), which we repeat here, with momenta assigned to the external and internal particles:



$$(8.4)$$

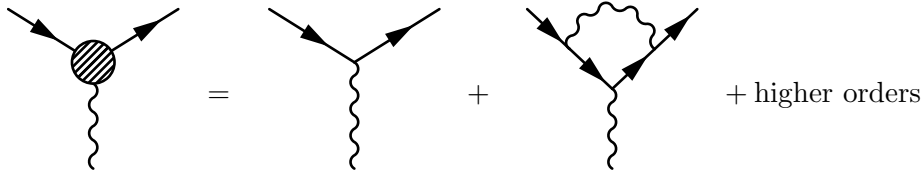
The corresponding Feynman amplitude is, according to the QED Feynman rules,

$$\begin{aligned} i\mathcal{M}^\mu &= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-ie\gamma^\nu) \frac{i(\not{k} + \not{q} + m)}{(k+q)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \\ &\quad \cdot \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} (-ie\gamma^\rho) u(p) \frac{-ig_{\nu\rho}}{(p-k)^2 + i\epsilon} \\ &\equiv \bar{u}(p') (-ie \delta\Gamma^\mu) u(p). \end{aligned} \quad (8.5)$$

Note that, by not including a polarization vector for the external photon and retaining the Lorentz index  $\mu$ , we keep the expression general. It holds if the photon connects to more fermion lines in a larger diagram, but also if it is treated as a non-quantized external field. In the former case, the index  $\mu$  is contracted with a photon propagator, in the latter case it is contracted with the Fourier transformation of the external field,  $\tilde{A}_\mu^{\text{ext}}(q)$ . In the last line of eq. (8.5), we introduce the notation  $\delta\Gamma^\mu$ , which can be interpreted as the 1-loop correction to the  $\gamma^\mu$  factor in the tree level electron-photon vertex.

### 8.1.1 Preliminary Considerations

Before we embark on the actual computation of the Feynman amplitude eq. (8.5), let us investigate its general (Lorentz) structure to get an idea of the result to expect. We follow here the motto that one should never do a complicated calculation before knowing the result. In fact, the arguments we are going to give in the following apply not only to the 1-loop correction to the electron-photon vertex, but to the whole perturbation series



$$\equiv \bar{u}(p') [-ie\Gamma^\mu(p, p')] u(p). \quad (8.6)$$

In the expression in the second line, we have again pulled the spinors and a factor  $-ie$  out. The remaining term

$$\Gamma^\mu(p, p') = \gamma^\mu + \delta\Gamma^\mu + \dots \quad (8.7)$$

is called the *electron vertex function*.

We note the following

1.  $\Gamma^\mu$  can only depend on  $p, p', \gamma^\mu, m$  and  $e$ , as these are the only quantities appearing in the QED Feynman rules. Of these, only  $p^\mu, p'^\mu$  and  $\gamma^\mu$  transform as Lorentz vectors, therefore we can write

$$\Gamma^\mu = \gamma^\mu \cdot A + (p'^\mu + p^\mu) \cdot B + (p'^\mu - p^\mu) \cdot C, \quad (8.8)$$

where  $A, B$  and  $C$  are matrices in spinor space.

2.  $A, B$  and  $C$ , not carrying Lorentz indices, can only contain Dirac matrices  $\gamma^\nu$  when contracted with momenta:  $\not{p}, \not{p}'$ . But since the Dirac equation tells us that  $\not{p}u(p) = mu(p)$  and  $\bar{u}(p')\not{p}' = m\bar{u}(p')$ , we can eliminate any such contraction in favor of  $m$ . Therefore, we can treat  $A, B$  and  $C$  as complex numbers, not matrices in spinor space.
3. The only Lorentz-scalar combinations of  $p$  and  $p'$  are  $p^2 = p'^2 = m^2$  and  $p \cdot p' = \frac{1}{2}q^2 - m^2$ . Therefore,

$$A = A(q^2), \quad B = B(q^2), \quad C = C(q^2). \quad (8.9)$$

In other words,  $A, B$  and  $C$  can depend only on  $q^2, m$  and  $e$ .

4. Gauge invariance, expressed through the Ward identity, implies  $\partial_\mu j^\mu = 0$ , or, in momentum space

$$q_\mu [\bar{u}(p') \Gamma^\mu u(p)] = 0. \quad (8.10)$$

(Note that the arguments put forth in section 6.9 to prove this identity hold even when  $q^2 \neq 0$ . As long as  $p'$  and  $p$  are on-shell momenta, the contact terms on the right hand side of eq. (6.103) preclude these terms from supplying the requisite  $1/(\not{p} - m)$  and  $1/(\not{p}' - m)$  poles to contribute to a scattering amplitude.) We now plug the general form of  $\Gamma^\mu$  from eq. (8.8) into eq. (8.10) and use  $q = p' - p$ . We observe that

$$q_\mu [\bar{u}(p') \gamma^\mu u(p)] = \bar{u}(p')(\not{p}' - \not{p}) u(p) = 0 \quad (8.11)$$

by virtue of the Dirac equation, and

$$q_\mu [\bar{u}(p') (p'^\mu + p^\mu) u(p)] = \bar{u}(p') (p'^2 - p^2) u(p) = 0. \quad (8.12)$$

On the other hand, the term  $q_\mu \bar{u}(p') (p'^\mu - p^\mu) u(p)$  does not vanish automatically. Therefore, to satisfy the Ward identity, its coefficient  $C$  must vanish.

Using now the *Gordon identity*

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[ \frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p) \quad (8.13)$$

(Proof: exercise!), we thus arrive at the following general expression for  $\Gamma^\mu$ :

$$\Gamma^\mu(q^2) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2), \quad (8.14)$$

where  $F_1(q^2)$  and  $F_2(q^2)$  are as yet unknown functions, combinations of the coefficients  $A$  and  $B$  from eq. (8.8).  $F_1(q^2)$  and  $F_2(q^2)$  are often called *form factors*.

### 8.1.2 Physical Interpretation

Before we proceed to evaluate  $F_1(q^2)$  and  $F_2(q^2)$ , let us pause a moment to ponder on their physical meaning. The coupling  $\gamma^\mu F_1(q^2)$  has the same Lorentz structure as the tree level QED coupling. It thus corresponds to a momentum-dependent correction to the electromagnetic coupling strength. At lowest order, without loop corrections, we have  $F_1(q^2) = 1$ . Even including loop corrections, we can without loss of generality take  $F_1(q^2 = 0) = 1$ , since any deviation from unity could be absorbed into a redefinition of the electric charge. At  $q^2 \neq 0$ , of course,  $F_1$  will still differ from unity.

Already at this stage, we can give a qualitative argument in favor of the existence of  $q^2$ -dependent corrections to the electric charge. Remember that  $e^+e^-$  pairs are constantly popping in and out of existence in the QED vacuum. In the presence of a negatively charged particle at the origin, these virtual  $e^+e^-$  pairs will become polarized—the  $e^+$  tends to be closer to the origin than the  $e^-$ . This phenomenon, called *vacuum polarization*, leads to partial screening of the electric charge at the origin because it makes the vacuum a dielectric medium. In other words, an observer inside the cloud of polarized  $e^+e^-$  pairs will always see some dipoles whose negative charge is further away from the origin than the observer's position, while the positive charge is closer to the origin. This reduces the apparent net charge at the origin. A scattering process at very large  $q^2$ , however, can probe very small length scales—much shorter than the typical volume occupied by one of the dipoles. The two scattering partners can then get so close to each other that the screening is reduced.

The form factor  $F_2(q^2)$  is relevant only at  $q \neq 0$ , i.e. it must have to do with magnetic (rather than electrostatic) interactions. Assume that the vertex  $\bar{u}(p')(-ie\Gamma^\mu)u(p)$  is coupled to an external (non-quantized) static magnetic field described by a vector potential

$$A_\mu^{\text{ext}} = (0, \mathbf{A}^{\text{ext}}):$$

$$i\mathcal{M} = +ieA^{\text{ext},i}\bar{u}(p')\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p). \quad (8.15)$$

(Note the extra minus sign from raising the spacelike index on  $A^{\text{ext},i}$ .) It then describes the scattering of a charged particle off a magnetic field. To relate this coupling to familiar quantities from classical electrodynamics, we take the non-relativistic limit. Since the term in square brackets in the above expression vanishes at  $q = 0$ , we go to linear order in the momenta. To do so, we expand

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} \simeq \sqrt{m} \begin{pmatrix} (1 - \mathbf{p} \cdot \boldsymbol{\sigma}/(2m))\xi \\ (1 + \mathbf{p} \cdot \boldsymbol{\sigma}/(2m))\xi \end{pmatrix}, \quad (8.16)$$

where  $\xi = (1, 0)^T$  or  $(0, 1)^T$  is a Pauli spinor. We will assume that this Pauli spinor is the same in  $u(p)$  and  $\bar{u}(p')$ , i.e. that the electron spin does not change in the scattering off the magnetic field. This assumption makes sense especially when the scattering is so soft that a spin flip is energetically not possible. The term containing  $\gamma^i$  becomes

$$\begin{aligned} \bar{u}(p')\gamma^i u(p) &= 2m\xi^\dagger \left( \frac{\mathbf{p}' \cdot \boldsymbol{\sigma}}{2m} \sigma^i + \sigma^i \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2m} \right) \xi \\ &= \xi^\dagger [p'^i + p^i - i\epsilon^{ijk}(p'^j - p^j)\sigma^k] \xi. \end{aligned} \quad (8.17)$$

In the second line, we have used the identity  $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k$ . When plugging eq. (8.17) back into the matrix element, eq. (8.15), we see that we obtain a term proportional to  $(\mathbf{p} + \mathbf{p}') \cdot \mathbf{A}^{\text{ext}}$ , and a term containing  $\mathbf{q} \times \mathbf{A}^{\text{ext}}$ . The former is familiar from non-relativistic quantum mechanics and describes the motion of a charged particle in a magnetic field. The second term appears only in the relativistic theory, and as we will demonstrate shortly, corresponds to a magnetic moment interactions.

Before doing so, however, let us also evaluate the term containing  $\sigma^{i\nu}$  in eq. (8.15). We again expand to first order in  $\mathbf{q}$ , but since there is an explicit factor  $q_\nu$  in the matrix element, we need to expand the spinors  $u(p)$  and  $\bar{u}(p')$  only to zeroth order in the momenta. Using moreover that

$$\sigma^{i0} = \frac{i}{2}[\gamma^i, \gamma^0] = i \begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix} \quad (8.18)$$

and

$$\sigma^{ij} = \epsilon^{ijk} \begin{pmatrix} \sigma^k & \\ & \sigma^k \end{pmatrix} \quad (8.19)$$

(see eq. (3.97)), we obtain

$$\bar{u}(p') \frac{i\sigma^{i\nu}q_\nu}{2m} u(p) = -i\xi^\dagger \epsilon^{ijk} q^j \sigma^k \xi. \quad (8.20)$$

The minus sign comes again from raising the index on  $q_j$ . Overall, by plugging eqs. (8.17) and (8.20) into eq. (8.15) and remembering that we are expanding in  $q$ , so that  $F_1(q^2)$  and  $F_2(q^2)$  need to be evaluated only at  $q^2 = 0$ , we thus arrive at

$$\begin{aligned} i\mathcal{M} &\simeq ieA^{\text{ext},i}\xi^\dagger \left[ -i\epsilon^{ijk}q^j\sigma^k F_1(0) - i\epsilon^{ijk}q^j\sigma^k F_2(0) \right] \xi \\ &= -2e\mathbf{S} \cdot (\mathbf{q} \times \mathbf{A}^{\text{ext}}) [F_1(0) + F_2(0)]. \end{aligned} \quad (8.21)$$

In the second line, we have introduced the notation  $\mathbf{S} \equiv \frac{1}{2}\xi^\dagger \boldsymbol{\sigma} \xi$  for the spin of the electron. Note that  $i\mathbf{q} \times \mathbf{A}^{\text{ext}}$  is just the Fourier transform of  $\mathbf{B} \equiv \nabla \times \mathbf{A}^{\text{ext}}$ .

In a similar way as in section 4.6.5, we can argue that in the non-relativistic limit,  $i\mathcal{M}$  describes the scattering amplitude of an electron off a potential  $V(\mathbf{x})$ , with Fourier transform  $\tilde{V}(\mathbf{q}) = -\mathcal{M}/(2m)$  (see eqs. (4.161) and (4.167)). In our case, the potential in Fourier space is thus

$$\tilde{V}(\mathbf{q}) = -\frac{e}{m} [F_1(0) + F_2(0)] \mathbf{S} \cdot (i\mathbf{q} \times \mathbf{A}^{\text{ext}}). \quad (8.22)$$

Going back to coordinate space, this leads to

$$V(\mathbf{x}) = -\boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{x}), \quad (8.23)$$

with the *magnetic moment of the electron*

$$\boldsymbol{\mu} = g \frac{e}{2m} \mathbf{S}, \quad (8.24)$$

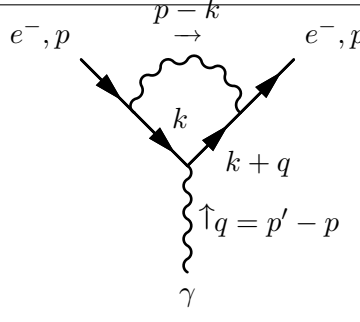
where in turn

$$g = 2[F_1(0) + F_2(0)] = 2 + 2F_2(0) \quad (8.25)$$

is the *Landé  $g$ -factor*. It describes the deviation of the intrinsic magnetic moment of the electron from the classical value  $e/(2m)\mathbf{S}$ . Already at tree level,  $g = 2$  differs from the classical expectation. This is one of the triumphs of the Dirac theory. Loop corrections in QFT modify  $g$  further by the term  $F_2(0)$ , which is why the quantity  $(g-2)/2 = F_2(0)$  is also called the *anomalous magnetic moment* of the electron. It can be calculated up to 10-loop order, and both the calculation and the measurement have reached a relative precision of order  $\text{few} \times 10^{-10}$  (and are in agreement). This is perhaps the best available test of QFT.

### 8.1.3 Simplifying the Integrand

We now proceed to the actual evaluation of  $F_1(q^2)$  and  $F_2(q^2)$  at order  $\alpha$ . For convenience, we repeat here the expression we have to evaluate (see eq. (8.5)):



$$\begin{aligned}
 &= \int \frac{d^4 k}{(2\pi)^4} \bar{u}(p') (-ie\gamma^\nu) \frac{i(\not{k} + \not{q} + m)}{(k+q)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \\
 &\quad \cdot \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} (-ie\gamma^\rho) u(p) \frac{-ig_{\nu\rho}}{(p-k)^2 + i\epsilon}. \quad (8.26)
 \end{aligned}$$

We begin simplifying this expression by exploiting the fact that the Lorentz indices on either side of the internal photon propagator are contracted. Using the identities

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu, \quad (8.27)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}, \quad (8.28)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \quad (8.29)$$

leads to

$$i\mathcal{M} = 2e^3 \int \frac{d^4 k}{(2\pi)^4} \bar{u}(p') \frac{\not{k}\gamma^\mu(\not{k} + \not{q}) - 2m(2k+q)^\mu + m^2\gamma^\mu}{[(k+q)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon][(p-k)^2 + i\epsilon]} u(p). \quad (8.30)$$

We would like to transform this expression such that the loop momentum only appears quadratically. This will ultimately allow us to perform the integration in 4-dimensional spherical coordinates.

### Denominator of eq. (8.30)

We begin with the denominator, and use the following relation, which holds for any  $A_1, \dots, A_n \in \mathbb{R}$ :

**Theorem.** (*Feynman parameters*)

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + \cdots + x_n A_n]^n}. \quad (8.31)$$

The  $x_1, \dots, x_n$  are called Feynman parameters.

*Proof.* The proof is by induction. For  $n = 2$ , the right hand side of eq. (8.31) reads

$$\begin{aligned}
 \int_0^1 dx \frac{1}{[xA_1 + (1-x)A_2]^2} &= -\frac{1}{xA_1 + (1-x)A_2} \frac{1}{A_1 - A_2} \Big|_0^1 \\
 &= -\left(\frac{1}{A_1} - \frac{1}{A_2}\right) \frac{1}{A_1 - A_2} \\
 &= \frac{1}{A_1 A_2}. \quad (8.32)
 \end{aligned}$$



To perform the induction step, we first consider the  $m$ -th derivative of eq. (8.32) with respect to  $A_2$ :

$$\begin{aligned} \frac{(-1)(-2)\cdots(-m)}{A_1 A_2^{m+1}} &= \int_0^1 dx \frac{(-2)(-3)\cdots(-(m+1))(1-x)^m}{[xA_1 + (1-x)A_2]^{2+m}} \\ \Leftrightarrow \frac{1}{A_1 A_2^m} &= \int_0^1 dx \frac{m(1-x)^{m-1}}{[xA_1 + (1-x)A_2]^{1+m}}. \end{aligned} \quad (8.33)$$

We then have for the induction step

$$\begin{aligned} \frac{1}{A_1 \cdots A_n A_{n+1}} &= \int_0^1 dx_1 \cdots dx_n \delta\left(\sum_1^n x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + \cdots + x_n A_n]^n A_{n+1}} \\ &= \int_0^1 dx_1 \cdots dx_n dy \delta\left(\sum_1^n x_i - 1\right) \frac{(n-1)! n(1-y)^{n-1}}{[y A_{n+1} + (1-y)[x_1 A_1 + \cdots + x_n A_n]]^{n+1}}. \end{aligned} \quad (8.34)$$

We now make the transformation  $x_j \rightarrow x_j/(1-y)$ , and we define  $x_{n+1} \equiv y$ . The  $\delta$  function then turns into  $\delta((1-y)^{-1} \sum_{i=1}^n x_i - 1) = (1-y)^n \delta(\sum_{i=1}^{n+1} x_i - 1)$ , and the upper integration boundary of the  $x_1 \cdots x_n$  integrals changes to  $1-x_{n+1}$ . However, since the  $\delta$  function sets the integrand to zero when  $x_j > 1-x_{n+1}$ , we can just as well integrate from 0 to 1 as before. We thus obtain

$$\frac{1}{A_1 \cdots A_n A_{n+1}} = \int_0^1 dx_1 \cdots dx_{n+1} \delta\left(\sum_{i=1}^{n+1} x_i - 1\right) \frac{n!}{[x_1 A_1 + \cdots + x_{n+1} A_{n+1}]^{n+1}}, \quad (8.35)$$

which completes the proof.  $\square$

We now apply the above theorem to the denominators in our matrix element eq. (8.30):

$$\frac{1}{[(k+q)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon][(p-k)^2 + i\epsilon]} = \int dx dy dz \delta(1-x-y-z) \frac{2}{D^3}, \quad (8.36)$$

with

$$\begin{aligned} D &= x[(k+q)^2 - m^2] + y[k^2 - m^2] + z[(p-k)^2 - m^2] + (x+y+z)i\epsilon \\ &= k^2 + 2xkq - 2zpk + xq^2 + zp^2 - (1-z)m^2 + i\epsilon. \end{aligned} \quad (8.37)$$

In the second line, we have used that  $x+y+z=1$ . To complete the square, we define

$$\ell \equiv k + xq - zp, \quad (8.38)$$

which leads to

$$D = \ell^2 - x^2 q^2 - z^2 p^2 + 2xzqp + xq^2 + zp^2 - (1-z)m^2 + i\epsilon$$

$$\begin{aligned}
 &= \ell^2 + q^2(-x^2 + x - xz) - m^2(1 - 2z + z^2) + i\epsilon \\
 &= \ell^2 - \Delta + i\epsilon,
 \end{aligned} \tag{8.39}$$

with

$$\Delta \equiv -xyq^2 + (1 - z)^2 m^2. \tag{8.40}$$

In the second line of eq. (8.39), we have used  $qp = p'p - p^2 = -(p' - p)^2/2 = -q^2/2$ . Note that the shift from  $k$  to  $\ell$  does not affect the integration measure. In other words, we can simply replace  $\int d^4k$  by  $\int d^4\ell$  without further ado.

### Numerator of eq. (8.30)

The next step is to also express  $k$  in terms of  $\ell$  in the numerator of eq. (8.30). In doing so, note that

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0 \tag{8.41}$$

because the denominator depends only on  $\ell^2$  and is thus symmetric in  $\ell^\mu$ , while the numerator is odd. Similarly,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{4} g^{\mu\nu} \ell^2}{D^3}. \tag{8.42}$$

For  $\mu \neq \nu$ , this follows again from the symmetry properties of the integrand. For  $\mu = \nu$ , the integral must be proportional to  $g^{\mu\nu}$  since this is the only available rank-2 Lorentz tensor. (All the momenta in the integrand are contracted into Lorentz-invariant products.) The prefactor  $1/4$  can be checked by contracting both sides of eq. (8.42) with  $g_{\mu\nu}$ . The relevance of eqs. (8.41) and (8.42) lies in the fact that these relations tell us that we can drop any term in the numerator of eq. (8.30) that depends linearly on  $\ell$ , and that terms containing  $\ell^\mu \ell^\nu$  can be greatly simplified. Explicitly, we have

$$\begin{aligned}
 \text{Numerator} &= \bar{u}(p') [\not{k} \gamma^\mu (\not{k} + \not{q}) - 2m(2k + q)^\mu + m^2 \gamma^\mu] u(p) \\
 &= \bar{u}(p') [(\ell - x\not{q} + z\not{p}) \gamma^\mu (\ell + (1-x)\not{q} + z\not{p}) \\
 &\quad - 2m(2\ell - (2x-1)q + 2zp)^\mu + m^2 \gamma^\mu] u(p) \\
 &\rightarrow \bar{u}(p') [\ell \gamma^\mu \ell + (-x\not{q} + z\not{p}) \gamma^\mu ((1-x)\not{q} + z\not{p}) \\
 &\quad + 2m((2x-1)q - 2zp)^\mu + m^2 \gamma^\mu] u(p).
 \end{aligned} \tag{8.43}$$

Note that the last step is not an algebraic identity, but only a replacement allowed in the integrand of a loop integral.

- By virtue of eqs. (8.27) and (8.42), the first term in square brackets in eq. (8.43) simplifies further to

$$\ell \gamma^\mu \ell = \ell_\nu \ell_\rho \gamma^\nu \gamma^\mu \gamma^\rho \rightarrow \frac{1}{4} \ell^2 \gamma^\nu \gamma^\mu \gamma_\nu = -\frac{1}{2} \ell^2 \gamma^\mu. \tag{8.44}$$

- The second term in square brackets can also be simplified significantly. Using the Dirac equation  $\not{p}u(p) = mu(p)$  and  $\bar{u}(p')\not{p}' = \bar{u}(p')m$  and the identities

$$\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p} \quad (8.45)$$

and

$$\begin{aligned} \not{p}\gamma^\mu\not{p}' &= 2p^\mu\not{p}' - \gamma^\mu\not{p}\not{p}' \\ &= 2p^\mu\not{p}' - 2\gamma^\mu p \cdot p' + \gamma^\mu\not{p}'\not{p} \\ &= 2p^\mu\not{p}' - 2\gamma^\mu p \cdot p' + 2p'^\mu\not{p} - \not{p}'\gamma^\mu\not{p}, \end{aligned} \quad (8.46)$$

it becomes

$$\begin{aligned} &\bar{u}(p')(-x\not{p}' + (z+x)\not{p})\gamma^\mu((1-x)\not{p}' + (x+z-1)\not{p})u(p) \\ &= \bar{u}(p')\left[-x(x+z-1)m^2\gamma^\mu \right. \\ &\quad \left. - x(1-x)[2mp'^\mu - m^2\gamma^\mu] + (z+x)(x+z-1)[2mp^\mu - m^2\gamma^\mu] \right. \\ &\quad \left. + (z+x)(1-x)[2mp^\mu - 2\gamma^\mu p \cdot p' + 2mp'^\mu - m^2\gamma^\mu]\right]u(p) \\ &= \bar{u}(p')\left[m^2\gamma^\mu[xy + x(1-x) + y(1-y) - (1-x)(1-y)] \right. \\ &\quad \left. + 2mp'^\mu[-x(1-x) + (z+x)(1-x)] \right. \\ &\quad \left. + 2mp^\mu[(z+x)(x+z-1) + (z+x)(1-x)] \right. \\ &\quad \left. - 2\gamma^\mu p \cdot p'(z+x)(1-x)\right]u(p) \\ &= \bar{u}(p')\left[m^2\gamma^\mu[xy + x(1-x) + y(1-y) - (1-x)(1-y)] \right. \\ &\quad \left. + 2mp'^\mu z(1-x) + 2mp^\mu z(1-y) \right. \\ &\quad \left. - \underbrace{2p \cdot p'}_{=-q^2+2m^2}\gamma^\mu(1-y)(1-x)\right]u(p) \\ &= \bar{u}(p')\left[m^2\gamma^\mu(-z^2 - 2z) + m(p'^\mu + p^\mu)z(1+z) + \gamma^\mu q^2(1-x)(1-y)\right]u(p). \end{aligned} \quad (8.47)$$

Here, we have used multiple times the identity  $x + y + z = 1$ . In the last step we have made the transformation

$$\begin{aligned} &xy + x(1-x) + y(1-y) - 3(1-x)(1-y) \\ &= xy + (1-x)(x-1+y) + (1-y)(y-1+x) - (1-x)(1-y) \\ &= -z(1-x) - z(1-y) - 1 + x + y \\ &= -z(1-x-y) - z - 1 + x + y \\ &= -z^2 - 2z, \end{aligned} \quad (8.48)$$

and we have exploited the fact that the denominator of the loop integral,  $D$ , is symmetric under the interchange  $x \leftrightarrow y$  (see eq. (8.39)). Therefore, we can replace  $x$  by  $y$  (and vice-versa) in any term of the numerator as well if doing so is convenient. This allowed us to rewrite

$$z(1-x) \rightarrow z\left[1 - \frac{1}{2}(x+y)\right] = z\left[1 - \frac{1}{2}(1-z)\right] = \frac{1}{2}z(z+1). \quad (8.49)$$

- In the third term in square brackets in eq. (8.43), we exploit again the symmetry of the denominator  $D$ , which allows us to rewrite

$$2m((2x-1)q - 2zp)^\mu \rightarrow 2m((x+y-1)q - 2zp)^\mu = -2mz(p'^\mu + p^\mu). \quad (8.50)$$

Collecting all the pieces, the numerator eq. (8.43) is then

$$\begin{aligned} & \bar{u}(p') \left[ -\frac{1}{2}\ell^2\gamma^\mu + m^2\gamma^\mu(-z^2 - 2z + 1) \right. \\ & \quad \left. + m(p'^\mu + p^\mu)(z(1+z) - 2z) + q^2\gamma^\mu(1-x)(1-yg) \right] u(p) \\ & = \bar{u}(p') \left[ -\frac{1}{2}\ell^2\gamma^\mu + q^2\gamma^\mu(1-x)(1-y) \right. \\ & \quad \left. - im\sigma^{\mu\nu}q_\nu z(z-1) + m^2\gamma^\mu \underbrace{(-z^2 - 2z + 1 + 2z(z-1))}_{=1-4z+z^2} \right] u(p) \end{aligned} \quad (8.51)$$

The Gordon identity eq. (8.13) came in handy here.

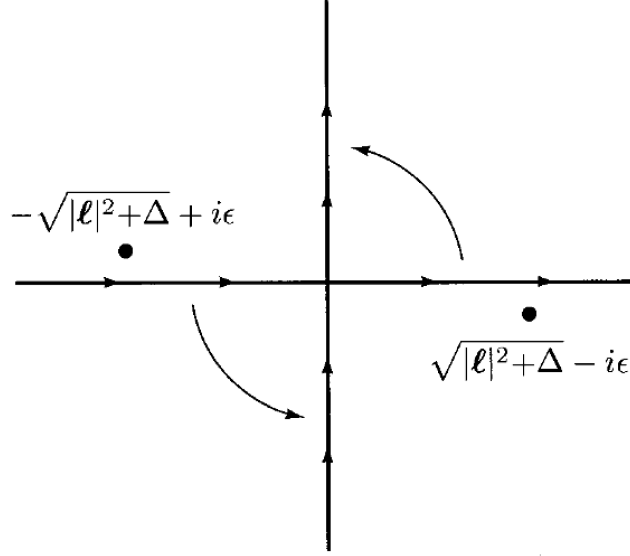
#### 8.1.4 The 4-Momentum Integral

We have now brought the integrand of the loop integral

$$\begin{aligned} \text{Diagram} &= 2e^3 \int dx dy dz \delta(1-x-y-z) \int \frac{d^4\ell}{(2\pi)^4} \frac{2}{D^3} \\ & \quad \cdot \bar{u}(p') \left[ \gamma^\mu \left( -\frac{1}{2}\ell^2 + q^2(1-x)(1-y) + m^2(1-4z+z^2) \right) \right. \\ & \quad \left. - im\sigma^{\mu\nu}q_\nu z(z-1) \right] u(p). \end{aligned} \quad (8.52)$$

to a radially symmetric form. We will now proceed to solving this integral. In particular, we need a way to solve

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta + i\epsilon]^m} \quad \text{and} \quad \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^m} \quad (8.53)$$



**Figure 8.2:** Illustration of the Wick rotation we use to transform the  $\ell^0$  integral. Figure taken from [1].

One complication here is the fact that  $\ell$  is a 4-vector in Minkowski space. This makes it difficult to define spherical coordinates. The brute force method of evaluating eq. (8.52) would involve solving the  $\ell^0$  integral by closing it by an infinite half-circle in the complex plane and applying the residual theorem. The  $\ell^2$  integral over the spatial components of  $\ell$  could then be evaluated in spherical coordinates.

However, there is a more elegant way. Consider the pole structure of the  $\ell^0$  integral in the complex plane:

$$\int d\ell^0 \frac{1}{[\ell^2 - \Delta + i\epsilon]^3} = \int d\ell^0 \frac{1}{(2\ell^0)^3} \left( \frac{1}{\ell^0 - \sqrt{\ell^2 + \Delta} + i\epsilon'} + \frac{1}{\ell^0 + \sqrt{\ell^2 + \Delta} - i\epsilon'} \right)^3. \quad (8.54)$$

The location of the poles is illustrated in fig. 8.2. We can close the integration contour, which runs along the real axis, by an infinite half-circle in either the upper or the lower half-plane. As long as the integrand drops faster than  $\propto 1/\ell^0$  as  $\ell^0 \rightarrow \infty$ , the integral vanishes along this half-circle. We can now rotate the integration contour by 90 degrees counterclockwise, as illustrated in fig. 8.2. (This procedure is called a *Wick rotation*.) Since the contour does not hit any poles during this rotation, the integral over the transformed contour is the same as the original integral. Moreover, the integral over the infinite half-circle, which now lies in the left half-plane, still vanishes. This shows that the integral along the real axis, which we are interested in, is identical to the integral

along the imaginary axis. This suggests the definition

$$\ell_E^0 \equiv -i\ell^0, \quad \ell_E^j \equiv \ell^j \quad (8.55)$$

and the corresponding transformation

$$\int_{-\infty}^{\infty} d\ell^0 \rightarrow \int_{-i\infty}^{i\infty} d\ell^0 = i \int_{-\infty}^{\infty} d\ell_E^0. \quad (8.56)$$

The subscript  $E$  in  $\ell_E^0$  stands for “Euclidean”. It indicates that the metric in the transformed momentum is the Euclidean one (i.e. the unit matrix). In particular  $\ell^2 = (\ell^0)^2 - \ell^2 = -\sum_j (\ell_E^j)^2 \equiv -\ell_E^2$ .

The first of the master integrals in eq. (8.53) is then

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta + i\epsilon]^m} &= i \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{[-\ell_E^2 - \Delta + i\epsilon]^m} \\ &= \frac{(-1)^m i}{(2\pi)^4} \int d\Omega_E \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^m} \\ &= \frac{(-1)^m i}{16\pi^4} 2\pi^2 \left[ \frac{1}{2(m-1)(m-2)} \frac{\ell_E^2(1-m) + \Delta}{[\ell_E^2 + \Delta]^{m-1}} \right]_0^\infty \\ &= \frac{(-1)^m i}{16\pi^2} \frac{\Gamma(m-2)}{\Gamma(m)} \frac{1}{\Delta^{m-2}}. \end{aligned} \quad (8.57)$$

We have used here that the integral over the 4-dimensional angular variables,  $d\Omega_E$ , is  $2\pi^2$ , the surface area of a 4-dimensional unit sphere. The integral over the modulus of the 4-vector  $\ell_E$  (denoted here for simplicity by  $\ell_E$  as well) can be obtained from a table of integrals, from a computer algebra system, or from your own ingenuity. By  $\Gamma(m) = (m-1)!$ , we denote the gamma function. Note also that after the Wick rotation we can safely drop  $i\epsilon$  in the denominator since the integrand contour is now far away from the poles of the integrand.

The second master integral in eq. (8.53) can be computed in a similar way. The result is

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^m} = \frac{(-1)^{m-1} i}{16\pi^2} \frac{2\Gamma(m-3)}{\Gamma(m)} \frac{1}{\Delta^{m-3}}. \quad (8.58)$$

Formulas for additional master integrals are given in appendix A.4 of ref. [1].

Note that eq. (8.57) is well-defined only for  $m > 2$ , while eq. (8.58) holds only for  $m > 3$ . This is a problem for us—we have  $m = 3$ . The fact that eq. (8.58) is ill-defined for  $m = 3$  is easy to see. At very large  $\ell_E$ , the integral has the structure

$$\int d^4\ell_E \frac{\ell_E^2}{[\ell_E^2 + \Delta]^3} \propto \int d\ell_E \frac{\ell_E^5}{\ell_E^6} \simeq \log \ell \Big|_{\ell=0}^\infty, \quad (8.59)$$

which is infinite. We call this type of infinity an *ultraviolet divergence* or *UV divergence* because the divergence comes from the region of large loop momenta. Since the value of

the integral increases logarithmically as the upper integration boundary goes to infinity, we say that the UV divergence is *logarithmic*.

We defer a discussion of this divergence and its implications to section 8.1.6 and first extract some physics from the finite pieces of the loop integral.

### 8.1.5 Lepton magnetic moments

Let us consider in particular the term in the loop integral eq. (8.52) containing  $\sigma^{\mu\nu}$ . Using the master integral eq. (8.57), it becomes

$$\begin{aligned}
 \text{Diagram} &= -4e^3 m \int dx dy dz \delta(1-x-y-z) \int \frac{d^4 \ell}{(2\pi)^4} \frac{z(z-1)}{D^3} [\bar{u}(p') i \sigma^{\mu\nu} q_\nu u(p)] + \dots \\
 &= -4e^3 m \int dx dy dz \delta(1-x-y-z) \frac{-i}{16\pi^2} \frac{\Gamma(1)}{\Gamma(3)} \frac{z(z-1)}{-xyq^2 + (1-z)^2 m^2} \\
 &\quad \times [\bar{u}(p') i \sigma^{\mu\nu} q_\nu u(p)] + \dots
 \end{aligned} \tag{8.60}$$

Comparing to the general expression for  $\Gamma^\mu(q^2)$  in terms of  $F_1(q^2)$  and  $F_2(q^2)$ , eq. (8.14), we can identify

$$F_2(q^2) = -\frac{e^2 m^2}{4\pi^2} \int dx dy dz \delta(1-x-y-z) \frac{z(z-1)}{-xyq^2 + (1-z)^2 m^2}. \tag{8.61}$$

As discussed in eq. (8.25),  $F_2(0)$  gives the anomalous contribution  $(g-2)/2$  to the magnetic moment of the electron. We can now give the  $\mathcal{O}(\alpha)$  contribution to this quantity:

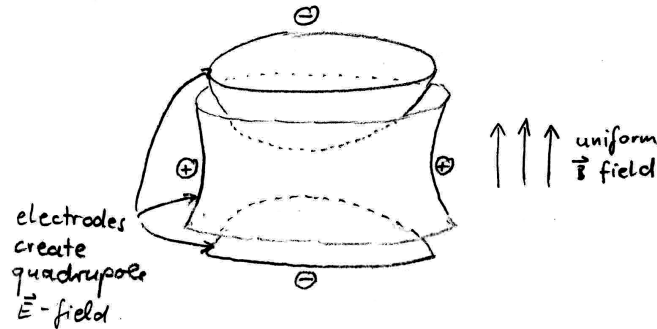
$$\begin{aligned}
 \frac{g-2}{2} = F_2(0) &= -\frac{e^2}{4\pi^2} \int dz (1-z) \frac{z(z-1)}{(1-z)^2} + \mathcal{O}(\alpha^2) \\
 &= \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) \\
 &\simeq 0.001161.
 \end{aligned} \tag{8.62}$$

This is our first precision calculation in quantum field theory!

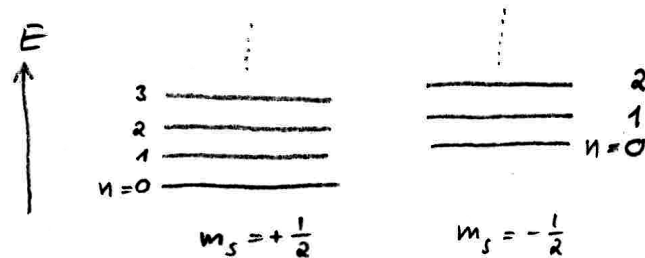
We mentioned before that the agreement between the measured and predicted values of  $(g-2)/2$  is at the level of  $10^{-10}$ . Let us briefly discuss how such a precise measurement of the magnetic moment of a particle is possible.

### Measuring $(g_e - 2)/2$ of the electron

The technique employed is to store an electron in a Penning trap—an ingenious configuration of electric and magnetic fields designed to confine a charged particle to a small volume:



The electric fields prevent the particle from escaping upwards or downwards, while if it attempts to escape sideways, it is instead forced onto a circular trajectory by the magnetic field. Thanks to the magnetic field, an electron in the trap experiences a potential  $V = -\vec{\mu} \cdot \vec{B}$ , so that its energy levels are quantized:



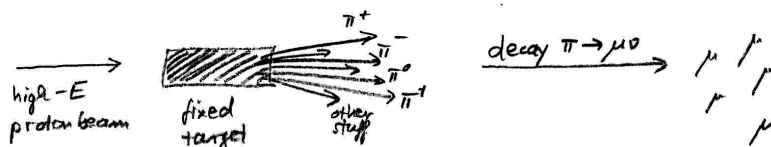
Here, the states on the left correspond to the electron spin oriented along the magnetic field ( $m_s = +1/2$ ), the ones on the right hand side correspond to the spin anti-aligned with  $\vec{B}$  ( $m_s = -1/2$ ). The quantum number  $n$  describes the orbital angular momentum.

One now applies an oscillating electromagnetic field with frequency  $\omega$  to induce transitions between different states. The polarization of the oscillating field determines whether only transitions between different orbital states  $n$  are induced, while keeping  $m_s$  unchanged, or whether both  $n$  and  $m_s$  can change. By monitoring the cyclotron frequency of the electron using the mirror charges it induces on a pickup capacitor, one can determine when  $n$  has changed. By scanning over the frequency  $\omega$  of the external field and determining the values at which transitions between states occur resonantly, one can determine the spectrum of the different energy levels. This allows in particular for a determinations of the splitting between the  $m_s = +1/2$  and  $m_s = -1/2$  states, which in turn depends on  $(g - 2)/2$ .

### Measuring $(g_\mu - 2)/2$ of the muon

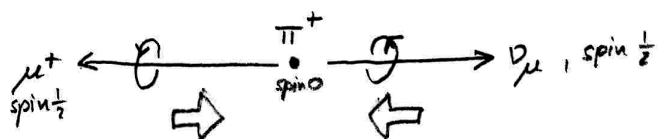
To determine the magnetic moment of the muon, one first needs muons. They can be produced by dumping a high-energy, high-intensity proton beam onto a target. A large fraction of the nuclear debris emerging from the target consists mostly of charged pions  $\pi^\pm$ . These, in turn, decay into muons and neutrinos:  $\pi^+ \rightarrow \mu^+ + \nu_\mu$ ,  $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$ :





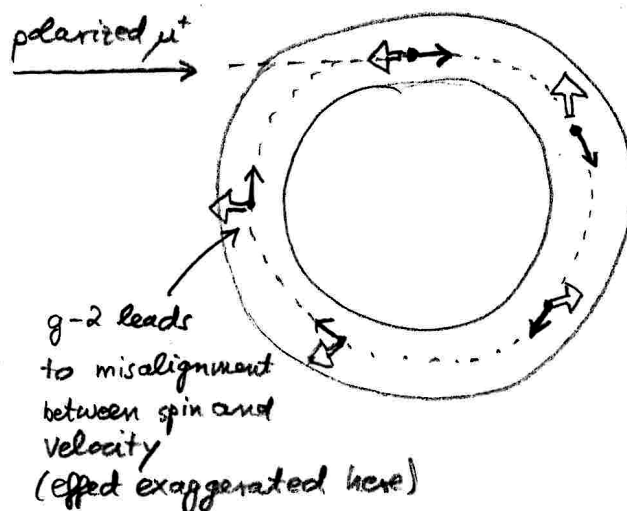
A magnetic filter can be employed to select either  $\pi^+$  or  $\pi^-$ , while discarding all pions with the “wrong” charge, as well as other particles produced in the target.

The muons from pion decay are polarized, as can be understood from angular momentum conservation.



Pions are spin-0 particles, while neutrinos and muons each carry spin  $1/2$ . Moreover, neutrinos ( $\nu_\mu$ ) are always left-handed, while antineutrinos ( $\bar{\nu}_\mu$ ) are always right-handed. Therefore, in the rest frame of a decaying  $\pi^+$  ( $\pi^-$ ), also the  $\mu^+$  ( $\mu^-$ ) must be left-handed (right-handed).

The polarized muons are now injected into a storage ring, where magnetic fields keep them on a circular orbit.



The angular frequency of this cyclotron motion is

$$\omega_c = \frac{eB}{m}, \quad (8.63)$$

as can be easily seen by equating the Lorentz force and the centripetal force. The same magnetic field also leads to a precession of the spin. The torque experienced by the muon

spin is  $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} = ge/(2m) \cdot \mathbf{S} \times \mathbf{B}$ , and since  $\dot{\mathbf{S}} = \boldsymbol{\tau}$ , the spin precession frequency is

$$\omega_s = g \frac{eB}{2m}. \quad (8.64)$$

Hence, the difference between the two frequencies,

$$\omega_a \equiv \omega_s - \omega_c = \frac{g-2}{2} \frac{eB}{m} \quad (8.65)$$

provides a direct measurement of the anomalous magnetic moment.

To measure the spin precession frequency of the muons, one exploits their decay  $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$  (or  $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$ ), which happens after a proper lifetime of  $\sim 2 \times 10^{-6}$  sec, or after a few hundred revolutions in the storage ring. In the decay of a  $\mu^+$ , the direction of the emitted positron carries information on the muon spin:



This can be understood from the fact that the muon has spin 1/2, the neutrino is always left-handed, and the antineutrino is always right-handed. Moreover, the positron is preferentially right-handed due to properties of the weak interaction that mediates muon decay. As a result of these restrictions, the highest energy positrons are preferentially emitted in the direction of the muon spin, while the lowest energy ones are preferentially emitted in the direction opposite to the muon spin. (Exercise: which way are  $e^-$  in the decay of a  $\mu^-$  emitted?) By observing the energy and direction of the positrons from  $\mu^+$  decay, one thus obtains information on the most probable direction of the muon spin at the time of the decay. After a large number of muon decays has been observed, a precision measurement of  $\omega_a$  becomes possible.

The most precise measurements of the anomalous magnetic moment of the electron and the muon are summarized in table 8.1 and compared to theory predictions. We see that our 1-loop prediction is accurate to a few per cent. This is expected since the next term in the perturbation series would be suppressed by an extra power of  $\alpha \sim 0.01$ . More accurate theoretical predictions agree with the measurement at the  $10^{-10}$  level.

Note, however, that for the muon there is a  $\sim 3\sigma$  discrepancy between the prediction and the measurement. It is currently unknown whether this discrepancy is due to a statistical fluctuation, a systematic problem with the measurement, or perhaps due to yet unknown new particles contributing in loop diagrams.

### 8.1.6 Renormalization: a First Glimpse

Let us now come back to the divergent parts of our loop integral eq. (8.52). It seems that our theory is horribly inconsistent. However, note that the only requirement for a

	Data	$\alpha/(2\pi)$	Best theory prediction
$(g-2)/2$ for electron	0.001 159 652 180 73(28)	0.001 161	0.001 159 652 182 79(77)
$(g-2)/2$ for muon	0.001 165 920 80(63)	0.001 161	0.001 165 918 03(49)

**Table 8.1:** Comparison between the best measurements of the anomalous magnetic moment of the electron and the muon to theoretical predictions. Note the  $\sim 3\sigma$  discrepancy for the muon!

consistent theory is that physical observables (cross sections, magnetic moments, etc.) should be finite. The trick to achieve this is called *renormalization*, and amounts to assuming that the coupling constant  $e$  in the Lagrangian is infinite in just the right way to cancel the infinities arising from loop integrals. This makes sense because, after all, the coupling constant is determined by fitting to experimental observations. As long as we had considered only tree level calculations, we found  $e = \sqrt{4\pi\alpha} \sim \sqrt{4\pi/137}$ . Including loop effects, this changes to  $e = \infty$ .

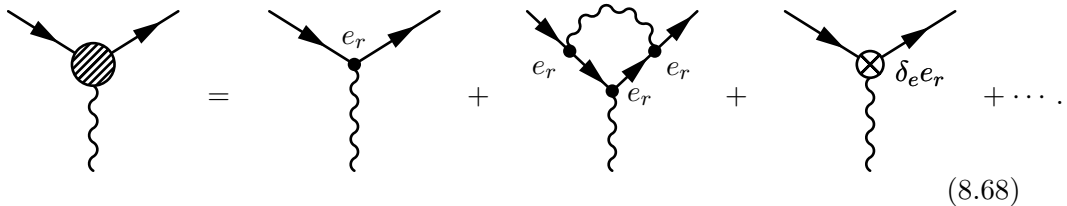
In the following, we will call the infinite coupling constant  $e_0$  instead of  $e$ . Since it is difficult to deal with infinite quantities (even after regularizing the divergence, see section 8.1.7 below), it is convenient to split off the divergent piece of  $e_0$ . Let us write the interaction term in the QED Lagrangian as

$$\begin{aligned}\mathcal{L}_{\text{int,QED}} &= -e_0 \bar{\psi} \gamma^\mu \psi A_\mu \\ &\equiv -e_r \bar{\psi} \gamma^\mu \psi A_\mu - \delta_e e_r \bar{\psi} \gamma^\mu \psi A_\mu.\end{aligned}\tag{8.66}$$

All we've done here is to define a relation

$$e_0 \equiv e_r(1 + \delta_e)\tag{8.67}$$

between the *bare coupling*  $e_0$ , the *renormalized coupling*  $e_r$ , and the *counterterm*  $\delta_e$ .  $e_0$  is split up in such a way that  $e_r = \sqrt{4\pi\alpha} \sim \sqrt{4\pi/137}$  is the physical value of the electric unit charge, and  $\delta_e$  is formally infinite and cancels the infinite terms arising from loop diagrams like the electron vertex correction, eq. (8.4). In renormalized perturbation theory (working with  $e_r$  and  $\delta_e$  as parameters instead of  $e_0$ ), the loop expansion for the electron-photon vertex is



$$\text{Tree-level vertex} = \text{Tree-level vertex with } e_r + \text{Loop diagram with } e_r + \text{Counterterm with } \delta_e e_r + \dots\tag{8.68}$$

The third diagram on the right hand side corresponds to the counterterm. By definition it cancels the infinite piece of the second diagram, the Feynman rule for which is



$$\delta_e e_r = -i \delta_e e_r \gamma^\mu. \quad (8.69)$$

It may seem problematic that we are now treating the infinite quantity  $\delta_e$  perturbatively. However, it is understood that the counterterm vertex is only included in calculations that also include the loop diagrams the infinities of which it cancels. The sum of the loop diagram + counterterm diagram is finite and perturbatively small.

### 8.1.7 Regularization of the Divergence

To give quantitative meaning to this initial discussion of renormalization, and to write down an analytic expression for the counterterm coupling  $\delta_e$ , we first need to regularize the divergence in the loop integral eq. (8.52). There are several ways of doing this

- **UV cutoff.** The most straightforward way of regularizing an improper integral is to replace the infinite integration boundaries by a large but finite value  $\Lambda$ :

$$\int_{-\infty}^{\infty} d^4 \ell \rightarrow \int_{-\Lambda}^{\Lambda} d^4 \ell. \quad (8.70)$$

The physical motivation behind regularization by a cutoff could be the assumption that the theory is valid only up to a certain energy scale, and is embedded into a more complete theory at the scale  $\Lambda$ . An example for such an *effective field theory* is the Fermi theory of weak interactions. It describes the beta decay of a neutron into a proton, an electron, and an electron anti-neutrino through a vertex  $\mathcal{L}_{\text{Fermi}} \supset G_F [\bar{p}(c_V - c_A \gamma^5)n][\bar{e}(1 - \gamma^5)\nu_e]$ . At energy scales  $\gtrsim \text{GeV}$ , the theory breaks down because the quark content of the nucleons is resolved. At even higher scales,  $\gtrsim 80 \text{ GeV}$ , also the  $W$  bosons mediating the interaction are resolved.

The disadvantage of cutoff regularization is that it breaks Lorentz invariance as well as gauge invariance.

- **Pauli–Villars regularization.** In this scheme, a fictitious heavy partner of the photon with mass  $\Lambda$  is added to the theory, with a propagator that differs from the photon propagator by the mass term in the denominator and by an extra minus sign in the numerator. Every diagram containing an internal photon is then supplemented by another diagram containing the heavy photon. For instance, in our loop diagram eq. (8.26), this leads to the replacement

$$\frac{g_{\nu\rho}}{(p-k)^2 + i\epsilon} \rightarrow \frac{g_{\nu\rho}}{(p-k)^2 + i\epsilon} - \frac{g_{\nu\rho}}{(p-k)^2 - \Lambda^2 + i\epsilon}. \quad (8.71)$$

At loop momenta  $k \gg \Lambda$ , the contributions from the real and fictitious photons cancel exactly. At low momenta  $\ll \Lambda$ , the contribution from the fictitious photon is irrelevant.

Pauli–Villars regularization preserves Lorentz invariance, but the mass term for a particle that otherwise behaves like the photon violates gauge invariance. In view of this remaining shortcoming, we will here use a thirs regularization scheme:

- **Dimensional regularization.** In this prescription, the number of spacetime dimensions  $d$  is formally promoted to a variable. The idea is to replace

$$\int d^4\ell \rightarrow \int d^d\ell, \quad (8.72)$$

After promoting  $d$  to a variable, one can exploit the fact that

$$\int d^d\ell \frac{\ell^2}{[\ell^2 - \Delta]^3}, \quad (8.73)$$

which is logarithmically divergent at  $d = 4$ , is finite in  $d = 4 - \epsilon$  dimensions. This is because the number of powers of  $\ell$  in the numerator,  $d + 2$ , is smaller then, while there are always 6 powers of  $\ell$  in the denominator.

Changing the dimensionality of spacetime entails modifications to some of the algebraic relations we have used in simplifying the loop integral. First, eqs. (8.27) to (8.29) now turn into

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(2 - \epsilon) \gamma^\nu, \quad (8.74)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho} - \epsilon \gamma^\nu \gamma^\rho, \quad (8.75)$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + \epsilon \gamma^\nu \gamma^\rho \gamma^\sigma. \quad (8.76)$$

Note that deriving these identities required only the relations  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  and  $g_\mu^\mu = d$ . With these relations, we can formally even allow  $d$  to be non-integer. Another relation that is modified in  $d$  dimensions is eq. (8.42). It now reads

$$\int \frac{d^d\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^d\ell}{(2\pi)^d} \frac{\frac{1}{d} g^{\mu\nu} \ell^2}{D^3}. \quad (8.77)$$

The modified prefactor can be verified by contracting both sides with  $g_{\mu\nu}$ .

We should now carefully examine our derivation of the transformed loop integral eq. (8.52) to check where it needs to be altered in view of eqs. (8.74) to (8.77). The first modification arises in the contractions of gamma matrices that led from eq. (8.26) to eq. (8.30) (the very first step after applying the Feynman rules to the diagram). Equation (8.30) now becomes

$$\begin{aligned} i\mathcal{M} = 2e^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+q)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon][(p-k)^2 + i\epsilon]} \\ \times \bar{u}(p') \left[ \not{k} \gamma^\mu (\not{k} + \not{q}) - \frac{\epsilon}{2} (\not{k} + \not{q}) \gamma^\mu \not{k} \right] \end{aligned}$$

$$-2m[(2k+q)^\mu - \frac{\epsilon}{4}(\not{k} + \not{q})\gamma^\mu - \frac{\epsilon}{4}\gamma^\mu \not{k}] + (1 - \frac{\epsilon}{2})m^2\gamma^\mu]u(p). \quad (8.78)$$

Note, however, that only the terms in the second line contribute to the divergent piece since they are the only ones that contain two powers of the loop momentum. Therefore, in the third line, we can (and will) immediately take the limit  $\epsilon \rightarrow 0$ . After introducing Feynman parameters and substituting  $k = \ell - xq + zp$ , the divergent term (i.e. the one containing  $\ell^\mu \ell^\nu$ ) becomes

$$\begin{aligned} & 2e^3 \int dx dy dz \delta(1-x-y-z) \int \frac{d^d \ell}{(2\pi)^d} \frac{2}{D^3} (1 - \frac{\epsilon}{2}) \bar{u}(p') \not{\ell} \gamma^\mu \not{\ell} u(p) \\ &= \int dx dy dz \delta(1-x-y-z) \int \frac{d^d \ell}{(2\pi)^d} \frac{-\frac{8}{d}e^3(1 - \frac{\epsilon}{2})^2 \ell^2}{D^3} \bar{u}(p') \gamma^\mu u(p). \end{aligned} \quad (8.79)$$

In the second line, we have used eqs. (8.74) and (8.77). We see that instead of the infinite master integral eq. (8.58), we now have to evaluate

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^m}. \quad (8.80)$$

As before, we carry out a Wick rotation by defining  $\ell_E^0 \equiv -i\ell^0$  and  $\ell_E^j \equiv \ell^j$  to obtain a radially symmetric integrand in Euclidean space. The integral can then be evaluated in spherical coordinates. Using the surface area of a  $d$ -dimensional unit sphere,  $\int d\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ ,<sup>1</sup> we obtain

$$\begin{aligned} \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^m} &= \frac{i}{(2\pi)^d} \int d\Omega_d \int_0^\infty d\ell_E \frac{\ell_E^{d+1}}{[-\ell_E^2 - \Delta]^m} \\ &= \frac{(-1)^{m-1}i}{(4\pi)^{d/2}} \frac{2}{\Gamma(\frac{d}{2})} \frac{1}{2} \left(\frac{1}{\Delta}\right)^{m-1-d/2} \frac{\Gamma(1+\frac{d}{2})\Gamma(m-1-\frac{d}{2})}{\Gamma(m)} \\ &= \frac{(-1)^{m-1}i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(m-1-\frac{d}{2})}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-1-d/2}. \end{aligned} \quad (8.81)$$

We have once again dropped  $i\epsilon$  in the denominator since, after the Wick rotation, the integration contour in  $\ell_E^0$  is far away from any poles. It is advantageous to expand this expression around  $d = 4$ . To do so, we use the expansion

$$\Gamma(x) \simeq \frac{1}{x} - \gamma + \mathcal{O}(x) \quad (8.82)$$

<sup>1</sup> This formula is obtained from

$$(\sqrt{\pi})^d = \left( \int dx e^{-x^2} \right)^d = \int d^d x \exp\left(-\sum_{i=1}^d x_i^2\right) = \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} = \left( \int d\Omega_d \right) \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

for the Gamma function at  $x \simeq 0$ . Here,  $\gamma \simeq 0.5772$  is the *Euler-Mascheroni constant*. We moreover use

$$\left(\frac{1}{\Delta}\right)^x \simeq 1 - x \log \Delta + \mathcal{O}(x^2). \quad (8.83)$$

Using these formulas, the  $d = 4 - \epsilon$ -dimensional master formula eq. (8.81) for  $m = 3$  turns into

$$\begin{aligned} & \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 - \Delta + i\epsilon]^3} \\ &= \frac{(-1)^{m-1} i}{(4\pi)^2 \Gamma(3)} \left(1 + \frac{\epsilon}{2} \log 4\pi\right) \left(2 - \frac{\epsilon}{2}\right) \left(\frac{2}{\epsilon} - \gamma\right) \left(1 - \frac{\epsilon}{2} \log \Delta\right) \\ &= \frac{(-1)^{m-1} i}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \frac{1}{2} - \log \Delta + \mathcal{O}(\epsilon)\right). \end{aligned} \quad (8.84)$$

We now put all the pieces together. Using eq. (8.84), the divergent term in the amplitude, eq. (8.79), becomes

$$-2e^3 \int dx dy dz \delta(1 - x - y - z) \frac{i}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - 2 - \log \Delta\right) \bar{u}(p') \gamma^\mu u(p). \quad (8.85)$$

Putting back also the finite pieces from eq. (8.52) and evaluating them using the for  $m = 3$  finite master integral eq. (8.57), the regularized electron vertex function is

$$\begin{aligned} \text{Diagram} &= -ie \frac{\alpha}{2\pi} \int dx dy dz \delta(1 - x - y - z) \bar{u}(p') \left[ \gamma^\mu \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - 2 - \log \Delta\right) \right. \\ &\quad \left. + \gamma^\mu \frac{1}{\Delta} \left(q^2(1-x)(1-y) + m^2(1-4z+z^2)\right) - i\sigma^{\mu\nu} q_\nu \frac{m z(z-1)}{\Delta} \right] u(p). \end{aligned} \quad (8.86)$$

This is the main result of this section. Note that it has the slight blemish of being dimensionally inconsistent: it contains the logarithm of a dimensionful quantity. We will see below that the dimensionality of  $\log \Delta$  is canceled by a similar log term in the counterterm.

### 8.1.8 Renormalization: The Counterterm

If we work in renormalized perturbation theory and split up the bare coupling  $e_0$  into the renormalized coupling  $e_r$  and the counterterm  $\delta_e e_r$ , the quantity  $e$  in eq. (8.86) should be interpreted as  $e_r$ . The counterterm coupling  $\delta_e e_r$  should then be defined such that the combined amplitude

$$\begin{array}{c} \text{triangle with wavy line} \end{array} + \begin{array}{c} \text{triangle with circle} \end{array} + \text{other loop diagrams} \quad (8.87)$$

is finite. This is achieved for instance by defining

$$\delta_e \equiv -\frac{\alpha}{2\pi} \int dx dy dz \delta(1-x-y-z) \left( \frac{2}{\epsilon} - \log \mu^2 \right) + \dots \quad (8.88)$$

Here, we have introduced an auxiliary variable  $\mu$  of mass dimension 1 to compensate for the incorrect dimensionality of eq. (8.86). Of course,  $\mu$  is unphysical, and physical observables should not depend on it. This is achieved by choosing the renormalized coupling  $e_r$  appropriately, by fitting calculated observables (with fixed  $\mu$ ) to experimental data. Of course, any change in  $\mu$  then requires a re-fitting of  $e_r$ , making  $e_r$  a function of  $\mu$ . This is the essence of *renormalization group evolution*, which we will discuss in more detail below in section 8.2.7.

Note that the dots in eq. (8.88) stand for contributions to the counterterm required to cancel other divergent diagrams, e.g.

$$\begin{array}{c} \text{triangle with wavy line} \end{array} + \begin{array}{c} \text{triangle with wavy line} \end{array} + \begin{array}{c} \text{triangle with circle} \end{array} \quad (8.89)$$

At the practical level, adding the counterterm eq. (8.88), means simply removing the divergent term  $2/\epsilon$  from eq. (8.86) and replacing  $\log \Delta$  by  $\log(\Delta/\mu^2)$ . This renormalization scheme is therefore called *minimal subtraction (MS)*.

Since, from the expansion of the Gamma function eq. (8.82), we see that  $\epsilon/2$  always appears in the combination  $2/\epsilon - \gamma + 4\pi$ , it is more convenient to define instead

$$\delta_e \equiv -\frac{\alpha}{2\pi} \int dx dy dz \delta(1-x-y-z) \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log \mu^2 \right) + \dots \quad (8.90)$$

This renormalization scheme is called  $\overline{\text{MS}}$  *renormalization*.

### 8.1.9 Summary of our First 1-Loop Calculation

Let us summarize the recipe we have developed for 1-loop calculation:

1. Write down the amplitude using the Feynman rules.
2. Introduce Feynman parameters to combine the denominators.
3. Complete the square in the denominator to bring it to the form  $D^m = [\ell^2 - \Delta + i\epsilon]$ , where  $\ell$  is the shifted loop momentum and  $\Delta$  is a Lorentz-invariant function of masses and external momenta.
4. Rewrite the numerator in terms of  $\ell$  and simplify it. Drop odd powers of  $\ell$ . Already at this stage, it is useful to go from 4 dimensions to  $d$  dimensions. This is important for instance when contractions of  $\gamma$  matrices (e.g.  $\gamma^\mu \gamma^\nu \dots \gamma^\rho \gamma_\mu$ ) occur.



5. Perform a Wick rotation and carry out the momentum integral in spherical Euclidean coordinates. In practice, it is usually easier to use tabulated expressions for master integrals like eq. (8.84) (see also appendix A.4 of [1]).

Typically, UV divergences appear in the integrals, and we have briefly outlined how splitting a formally infinite bare coupling  $e_0$  into a finite renormalized coupling  $e_r$  and a counterterm  $\delta_e e_r$  can remove these divergences and ensure that physical observables are finite.

## 8.2 Renormalization

### 8.2.1 The renormalized QED Lagrangian

Now that we know how to compute loop diagrams, we are ready for a more comprehensive discussion of renormalization. First, we remark that not only the electromagnetic coupling  $e$  will receive infinite corrections, but also the electron mass parameter  $m$ , and even the normalizations of the field operators  $\psi$  and  $A^\mu$ . We will see below from which diagrams the latter corrections arise. A comprehensive renormalization strategy therefore requires rewriting the bare QED Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\psi}_0(i\cancel{\partial} - m_0)\psi_0 - \frac{1}{4}F_{0,\mu\nu}F_0^{\mu\nu} - e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0,\mu}. \quad (8.91)$$

Here, the subscript “0” is introduced to identify bare quantities. It is understood that  $e_0$ ,  $m_0$ ,  $\psi_0$ ,  $A_0^\mu$  are infinite in such a way as to cancel the infinite terms in loop diagrams, such that all physical observables have finite values. (For the field operators, an infinite value means that the operators as we know them are multiplied by an infinite constant.)

We would now like to split off the infinite terms. This is achieved by splitting the bare terms into renormalized terms and counterterms  $\delta_\psi$ ,  $\delta_A$ ,  $\delta_m$ ,  $\delta_e$  according to

$$\begin{aligned} \mathcal{L}_{\text{QED}} = & (1 + \delta_\psi)\bar{\psi}_r i\cancel{\partial}\psi_r - (m_r + \delta_m)\bar{\psi}_r\psi_r \\ & - \frac{1}{4}(1 + \delta_A)F_{r,\mu\nu}F_r^{\mu\nu} - (1 + \delta_e)e_r\bar{\psi}_r\gamma^\mu\psi_r A_{r,\mu}. \end{aligned} \quad (8.92)$$

It is understood that  $\psi_r$ ,  $A_r$ ,  $e_r$ ,  $m_r$  are finite, while  $\delta_\psi$ ,  $\delta_A$ ,  $\delta_e$ ,  $\delta_m$  as well as the bare quantities  $\psi_0$ ,  $A_0$ ,  $e_0$ ,  $m_0$  are infinite. Working in renormalized perturbation theory, the QED Feynman rules from section 5.2 are understood to involve renormalized quantities only. They are supplemented by the counterterm vertices

$$1. \quad \text{Fermion propagator counterterm} \quad \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad = \quad i(\delta_\psi \not{p} - \delta_m) \quad (8.93)$$

$$2. \quad \text{Photon propagator counterterm} \quad \begin{array}{c} \mu \quad \text{---} \text{---} \text{---} \text{---} \text{---} \quad \nu \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad = \quad -i\delta_A(p^2 g^{\mu\nu} - p^\mu p^\nu) \quad (8.94)$$

$$3. \quad \text{Vertex counterterm} \quad \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \quad = \quad -i\delta_e \gamma^\mu \quad (8.95)$$

When computing a Feynman amplitude, these counterterm vertices should be included along with the bare ones. Since the counterterms receive their lowest order contributions at the 1-loop, or  $\mathcal{O}(e^2)$ , level, they should be counted at this order in the perturbation series. This means that a lowest order calculation should not involve any counterterms. A 1-loop calculation should involve diagrams without counterterms up to 1-loop order and counterterm vertices at the tree level. A 2-loop calculation should involve loop diagrams without counterterms up to 2-loop order, 1-loop diagrams including one counterterm evaluated at  $\mathcal{O}(e^2)$ , and tree level diagrams with two counterterm vertices evaluated at  $\mathcal{O}(e^2)$  or one counterterm evaluated at  $\mathcal{O}(e^4)$ . We will write derive expressions for the counterterms below.

Sometimes it is also convenient to write the renormalized Lagrangian as

$$\mathcal{L}_{\text{QED}} = Z_\psi \bar{\psi}_r i \not{\partial} \psi_r - Z_m m_r \bar{\psi}_r \psi_r - \frac{1}{4} Z_A F_{r,\mu\nu} F_r^{\mu\nu} - Z_e e_r \bar{\psi}_r \gamma^\mu \psi_r A_{r,\mu} \quad (8.96)$$

with the infinite multiplicative renormalization factors  $Z_\psi = 1 + \delta_\psi$ ,  $Z_A = 1 + \delta_A$ ,  $Z_e = 1 + \delta_e$ ,  $Z_m = m_r + \delta_m$ . The renormalized quantities can then be written as

$$\begin{aligned} \psi_r(x) &\equiv Z_\psi^{-1/2} \psi_0(x), \\ A_r^\mu(x) &\equiv Z_A^{-1/2} A_0(x), \\ e_r &\equiv Z_e^{-1} Z_\psi Z_A^{1/2} e_0, \\ m_r &\equiv Z_m^{-1} Z_\psi m_0. \end{aligned} \quad (8.97)$$

In the literature, one often finds also the notation  $Z_1 \equiv Z_e$ ,  $Z_2 \equiv Z_\psi$ , and  $Z_3 = Z_A$ .  $Z_\psi$  and  $Z_2$  are also called the electron wave function renormalization factors, and  $Z_A$ ,  $Z_3$  are the photon wave function renormalization factors.

### 8.2.2 1-Loop Corrections to the Fermion Propagator

We would now like to derive explicit expressions for the 1-loop ( $\mathcal{O}(e^2)$ ) contributions to the counterterms in eq. (8.92). In the following, we drop the index  $r$  from renormalized quantities. Quantities without an index are always implied to be renormalized.

To do so, we need to evaluate all divergences arising in QED at the 1-loop level. Consider first corrections to the electron propagator (often called the *electron self-energy* corrections):

$$\begin{aligned}
 \text{Diagram with shaded circle} &= \text{Diagram with straight line} + \text{Diagram with loop} + \text{Diagram with cross} + \dots \\
 &\equiv \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} [-i\Sigma(p^2)] \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}, \quad (8.98)
 \end{aligned}$$

The 1-loop diagram in this series is

$$e \xrightarrow{p} \text{Diagram with loop} \xrightarrow{p} e = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} [-i\Sigma_2(p^2)] \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (8.99)$$

with

$$-i\Sigma_2(p^2) = i\frac{\alpha}{4\pi} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_e \right) [(2 - \epsilon)x\not{p} - (4 - \epsilon)m] \quad (8.100)$$

and

$$\Delta_e = (1 - x)m^2 - x(1 - x)p^2. \quad (8.101)$$

The notation  $\Sigma_2(p^2)$  indicates that this is the second order (in  $e$ ) contribution to the electron self-energy  $\Sigma(p^2)$ . We defer the derivation of eq. (8.100) using the recipe from section 8.1.9 to the exercises.

### 8.2.3 1-Loop Corrections to the Photon Propagator

Consider moreover the photon self-energy

$$\begin{aligned}
 \text{Diagram with shaded circle} &= \text{Diagram with straight line} + \text{Diagram with loop} + \text{Diagram with cross} + \dots \\
 &\equiv \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} [i(q^2 g_{\nu\rho} - q_\nu q_\rho) \Pi(q^2)] \frac{-ig^{\rho\sigma}}{q^2 + i\epsilon}. \quad (8.102)
 \end{aligned}$$

The 1-loop term is

$$\gamma, \mu \xrightarrow{q} \text{Diagram with loop} \xrightarrow{q} \gamma, \sigma = \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} [i(q^2 g_{\nu\rho} - q_\nu q_\rho) \Pi_2(q^2)] \frac{-ig^{\rho\sigma}}{q^2 + i\epsilon} \quad (8.103)$$

with the  $\mathcal{O}(e^2)$  self-energy

$$\Pi_2(q^2) = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_\gamma \right) \quad (8.104)$$

and

$$\Delta_\gamma = m^2 - x(1-x)q^2. \quad (8.105)$$

### 8.2.4 On-Shell Renormalization

We see that the self-energies eqs. (8.100) and (8.104) are divergent, but the divergent terms have precisely the right structure to be compensated by the counterterms eqs. (8.93) and (8.94). We could now apply  $\overline{\text{MS}}$  renormalization and define the counterterms such that they absorb the terms proportional to  $\frac{2}{\epsilon} - \gamma + \log 4\pi$  in eqs. (8.99) and (8.103). However, note that the splitting of the bare quantities into renormalized quantities and counterterms is not unique. In particular, finite contributions can be traded between them. Physical results, which include the contributions from the finite and the counterterm parts of the Lagrangian, do not depend on how the bare quantities are split. We can use this freedom to define a set of *renormalization conditions* that makes calculations in renormalized perturbation theory particularly easy:

$$\begin{aligned} \Sigma(\not{p} = m_r) &= 0, \\ \frac{d}{d\not{p}} \Sigma(\not{p}) \Big|_{\not{p}=m} &= 0, \\ \Pi(q^2 = 0) &= 0, \\ -ie_r \Gamma^\mu(p' - p = 0) &= -ie_r \gamma^\mu. \end{aligned} \quad (8.106)$$

Since all of these conditions are defined for particles on the mass shell, this scheme is called *on-shell renormalization*. Equation (8.106) means that, for on-shell particles, loop corrections vanish exactly.

We can read off from eq. (8.100), that in on-shell renormalization at 1-loop order,

$$\begin{aligned} m\delta_\psi - \delta_m &= m(Z_\psi - 1) - (Z_m - 1) = \Sigma_2(m) \\ &= -\frac{\alpha m}{4\pi} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log [(1-x)^2 m^2] \right) [-4 + 2x + \epsilon(1-x)] \end{aligned} \quad (8.107)$$

and

$$\begin{aligned} \delta_\psi = Z_\psi - 1 &= -\frac{\alpha}{4\pi} \int_0^1 dx \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log [(1-x)^2 m^2] \right) \\ &\quad \times \left[ (2-\epsilon)x - \frac{\epsilon - 2x(1-x)m^2}{2(1-x)^2 m^2} (-4 + 2x + \epsilon(1-x)) \right] \end{aligned}$$

$$= -\frac{\alpha}{2\pi} \int_0^1 dx x \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log [(1-x)^2 m^2] - 1 + \frac{2(x-2)}{1-x} \right). \quad (8.108)$$

Moreover, eq. (8.103) tells us that

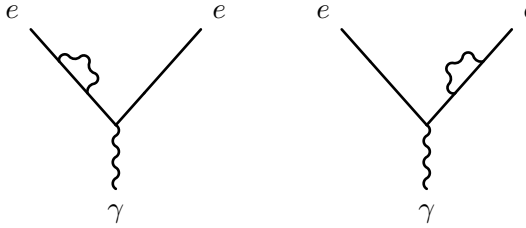
$$\delta_A = Z_A - 1 = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log m^2 \right). \quad (8.109)$$

Finally, the vertex counterterm  $\delta_e$  is obtained from the electron vertex function computed in eq. (8.86):

$$\delta_e = Z_e - 1 = -\frac{\alpha}{2\pi} \int dz (1-z) \left[ \frac{2}{\epsilon} - \gamma + \log 4\pi - 2 - \log [(1-z)^2 m^2] + \frac{1-4z+z^2}{(1-z)^2} \right]. \quad (8.110)$$

This completes the renormalized Lagrangian.

Note that, in on-shell renormalization, loop corrections to external (on-shell) lines in a Feynman diagram vanish. This justifies a posteriori the statement made in section 4.5 that external leg corrections do not contribute to physical quantities. In other renormalization schemes, for instance in  $\overline{\text{MS}}$ , however, diagrams like



$$(8.111)$$




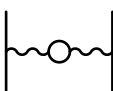
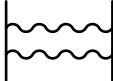
do not vanish and need to be included in the computation of physical quantities.

Note also that the counterterms computed here, when added to the loop amplitudes eqs. (8.86), (8.100) and (8.103), give the overall amplitude the correct mass dimension. They cancel the wrong dimensionality in the log terms of the loop amplitudes without requiring the introduction of an unphysical scale  $\mu$  as in  $\overline{\text{MS}}$  renormalization.

### 8.2.5 Renormalizability

When computing loop diagrams, it is useful to have a simple way of guesstimating a priori whether a diagram will contain ultraviolet divergences. We can achieve this by noting that, at very large loop momentum, an  $L$ -loop integral takes the form

$$\int \frac{d^4 k_1 d^4 k_2 \cdots d^4 k_L}{k_i \cdots (k_j)^2 \cdots}. \quad (8.112)$$

Diagram	Superficial divergence	Actual divergence
	$D = 0$	$\sim \log \Lambda$
	$D = 1$	$\sim \log \Lambda$
	$D = 2$	$\sim \log \Lambda$
	$D = -2$	$\sim \log \Lambda$
	$D = -2$	finite

**Figure 8.3:** The naive degree of divergence and the actual divergence structure in cutoff regularization for several QED diagrams.

Here, the terms  $k_i$  in the denominator arise from internal fermion propagators, while the terms  $(k_j)^2$  arise from internal photon propagators. The integral is usually finite if the integrand contains more powers of the loop momentum in the denominator than in the numerator and diverges otherwise. Here, the differential  $d^4k$  counts towards the powers in the numerator. In formulas, this statements means that a diagram is expected to be finite if the *superficial degree of divergence*,

$$D \equiv 4L - P_e - 2P_\gamma \quad (8.113)$$

is  $< 0$ . Here,  $L$  is the number of loops,  $P_e$  is the number of fermion propagators, and  $P_\gamma$  is the number of photon propagators. For  $D = 0$ , we expect a logarithmic divergence, i.e. a divergence which, in cutoff regularization, behaves as  $\log \Lambda$ . For  $D > 0$ , we expect a divergence of the form  $\Lambda^D$ . The superficial degree of divergence and the actual divergence structure of a few QED diagrams are listed in fig. 8.3. We see that the naive power counting works for the electron vertex function which we computed in section 8.1, but it fails already for the fermion self energy diagram from section 8.2.2, and for the photon self-energy from section 8.2.3. This does not seem like a particularly promising outcome, but the superficial degree of divergence is still useful in practice, in particular if we understand the situations in which it fails.

Consider therefore the third diagram in fig. 8.3, i.e. the photon vertex function. The reason the actual degree of divergence is lower ( $\sim \log \Lambda$ ) than the naive expectation ( $\sim \Lambda^2$ ) is gauge symmetry. In fact, the Ward identity, which demands that the self-energy amplitude vanishes when contracted with the photon momentum  $q_\mu$ , restricts



where  $V$  is the number of vertices in the diagram. To see this, remember that in our derivation of the Feynman rules from the action each propagator contains a 4-momentum integral and each vertex contains a spacetime integral that evaluates to a 4-dimensional delta function in momentum space. One delta function merely enforces overall 4-momentum conservation and is pulled out of the amplitude.

Let us call the number of fermion lines attached to the vertex  $n_\psi$  and the number of boson lines attached to it  $n_\phi$ . For a diagram with  $N_\psi$  ( $N_\phi$ ) external fermion (boson) lines, the number of vertices is

$$V = \frac{1}{n_\phi}(2P_\phi + N_\phi) = \frac{1}{n_\psi}(2P_\psi + N_\psi). \quad (8.118)$$

This follows simply from the fact that each propagator needs to be attached to two vertices and each external line needs to be attached to one vertex. Putting eqs. (8.116) to (8.118) together, we arrive at

$$D = V\left(\frac{3}{2}n_\psi + n_\phi - 4\right) - \frac{3}{2}N_\psi - N_\phi + 4. \quad (8.119)$$

Note that in QED, where  $n_\psi = 2$  and  $n_\phi = 1$ , the superficial degree of divergence is *independent* of the number of vertices and depends only on the number of external lines in the diagram. In other words, all diagrams contributing to a given  $n$ -point amplitude have the same  $D$ . For instance, all higher order contributions to the electron vertex function have  $D = 0$  like the 1-loop contribution. As each additional external line lowers  $D$ , this also means that there are only a finite number of superficially divergent amplitudes in QED. This means that a finite number of counterterms is sufficient to cancel all UV divergences. Such a theory is called *renormalizable*. (Note that, of course, even  $n$ -point amplitudes with  $D < 0$  receive divergent contributions from divergent sub-diagrams. However, to cancel these, the counterterms corresponding to these divergent sub-diagrams are sufficient.)

In other theories, the degree of divergence grows with the number of vertices in a diagram. Consider for example a 4-fermion vertex of the form  $\mathcal{L} \supset (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$ . The coefficient of  $V$  in eq. (8.119) is then 2, so adding more vertices will make an amplitude more and more divergent. In particular, an amplitude with an arbitrary large number of external legs will be divergent at sufficiently high loop orders (sufficiently large number of vertices). To cancel all these divergences, an infinite number of counterterms would be required (one for each combination of  $N_\psi$  and  $N_\phi$ ). Such a theory is called *non-renormalizable*.

Theories for which the coefficient of  $V$  in eq. (8.119) is negative, on the other hand, are called *super-renormalizable*. For such theories, diagrams get more and more convergent at higher loop orders.

As a final, but in practice very important, remark, let us note that the coefficient of  $V$  is simply the mass dimension of the vertex in the Lagrangian, minus 4. Indeed, a fermion field  $\psi(x)$  has mass dimension  $3/2$ , whereas scalar or gauge boson fields have mass dimension 1. Thus, by simple dimensional analysis, one can immediately determine whether a theory is super-renormalizable, renormalizable, or non-renormalizable.



### 8.2.6 Renormalization of the Electric Charge

Let us dwell for a moment on the relation

$$e_r \equiv Z_e^{-1} Z_\psi Z_A^{1/2} e_0 \quad (8.120)$$

between the bare and renormalized values of the electric charge. We are going to argue below that

$$Z_e = Z_\psi, \quad (8.121)$$

i.e. that the renormalization of  $e_r$  depends only on the loop corrections to the photon propagator, not on the vertex corrections and the fermion wave function renormalization. Proving eq. (8.121) is a bit tricky. This relation is certainly not obvious from our formulas for  $\delta_e = Z_e - 1$  in eq. (8.110) and for  $\delta_\psi = Z_\psi - 1$  in eq. (8.108). Therefore, before proving it, let us first discuss its consequences.

#### Consequences of $Z_e = Z_\psi$

**Gauge invariance of the renormalized Lagrangian.** Note first that  $Z_e = Z_\psi$  implies that gauge transformations have the same functional form when written in terms of renormalized and bare quantities. In fact, consider a gauge transformation of the bare fields,

$$\psi_0(x) \rightarrow e^{-i\alpha(x)} \psi_0(x), \quad A_0^\mu(x) \rightarrow A_0^\mu(x) + \frac{1}{e_0} [\partial^\mu \alpha(x)], \quad (8.122)$$

and rewrite it in terms of renormalized quantities using eq. (8.97):

$$\psi_r(x) \rightarrow e^{-i\alpha(x)} \psi_r(x), \quad A_r^\mu(x) \rightarrow A_r^\mu(x) + \frac{1}{\sqrt{Z_A} e_0} [\partial^\mu \alpha(x)]. \quad (8.123)$$

Here, we have used that the renormalization constants  $Z_\psi$ ,  $Z_A$ ,  $Z_m$  and  $Z_e$  are gauge invariant.<sup>2</sup> Since  $\sqrt{Z_A} e_0 = e_r$ , we see that gauge transformations written in terms of renormalized quantities have exactly the same functional form as those written in terms of bare fields.

**Consequences for theories with several fermion species.** Another way of seeing that  $Z_e = Z_\psi$  makes sense is by considering electrodynamics with more than one fermion species, for example by adding muons to the theory. Already from the 1-loop expressions in eqs. (8.86) and (8.100) it is clear that the correction  $Z_\psi^{(e)}$  to the electron propagator differs from the correction  $Z_\psi^{(\mu)}$  to the muon propagator, and that the correction  $Z_e^{(e)}$  to

<sup>2</sup>Gauge invariance of the renormalization constants follows from the gauge invariance of the Feynman rules used to compute them. It holds to all orders in perturbation theory. Remember, though, that we have fixed the Lorenz gauge condition  $\partial^\mu A_\mu = 0$  in deriving the gauge boson propagator in section 6.6. Therefore, gauge invariance here means the residual gauge freedom that is left after fixing this condition.

the electron–photon vertex differs from the correction  $Z_e^{(\mu)}$  to the muon–photon vertex. If the vertex correction and the propagator correction were not identical, the renormalized charge  $e_r = Z_e^{-1} Z_\psi Z_A^{1/2}$  would be different for the electron and the muon. This would be a strange result.

**Physical interpretation of charge renormalization.** Finally, the fact that the bare and renormalized values of the electric charge depend only on the photon self energy correction can also be understood by investigating the physical meaning of the photon self-energy. Remember that we argued in section 8.1.2 that vacuum polarization leads to screening of the (infinite) bare charge  $e_0$  of the electron. A low-momentum photon with  $q^2 \rightarrow 0$  experiences a much lower—renormalized—charge  $e_r$ . Photons with larger momentum  $q^2$  probe deeper and deeper into the cloud of virtual  $e^+e^-$  pairs, experiencing an effective charge somewhere between  $e_r$  and  $e_0$ . This effect is described by the loop corrections to the electron–photon vertex. In fact, since a virtual  $e^+e^-$  pair is nothing but a fermion vacuum bubble, the interactions between photons and virtual  $e^+e^-$  pairs are described, at lowest loop order, by the diagram


(8.124)

This diagram can be viewed as a virtual  $e^+e^-$  pair (the internal fermion loop) exchanging a photon with the electron, thus screening its bare charge. The external photon, in turn, interacts with the virtual  $e^+e^-$  pair, so that overall it experiences only a screened charge. This argument is based entirely on photon self energy diagrams and does not leave room for fermion self energy diagrams or vertex corrections, which could not be interpreted as interactions with vacuum fluctuations. Thus, also from physical intuition, it makes sense that  $e_r = \sqrt{Z_A} e_0$ , without contributions from  $Z_e$  or  $Z_\psi$ .

### Proof of $Z_e = Z_\psi$ at 1-loop order

Having discussed the consequences of the relation  $Z_e = Z_\psi$  at length, we should finally prove it. We will first do this at 1-loop level by showing that our formulas for  $\delta_e = Z_e - 1$  in eq. (8.110) and for  $\delta_\psi = Z_\psi - 1$  in eq. (8.108) are identical. We have

$$\begin{aligned} \delta_e - \delta_\psi &= -\frac{\alpha}{2\pi} \int_0^1 dz (1-z) \left[ \frac{2}{\epsilon} - \gamma + \log 4\pi - 2 - \log [(1-z)^2 m^2] + \frac{1-4z+z^2}{(1-z)^2} \right] \\ &\quad + \frac{\alpha}{2\pi} \int_0^1 dz z \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log [(1-z)^2 m^2] - 1 + \frac{2(z-2)}{1-z} \right). \end{aligned}$$

Integrating the log-term by parts and further simplifying the non-log terms yields

This derivation shows that  $Z_e = Z_\psi$  at 1-loop, but does not reveal anything about the deeper origin of this relation.

Let us now prove the relation  $Z_e = Z_\psi$  to all orders in perturbation theory. In fact, the identity follows from gauge invariance, as expressed through the Ward–Takahashi identity eq. (6.103). By repeating the steps eqs. (6.98) to (6.102) leading to this identity using the renormalized Lagrangian eq. (8.96), we find

Note in particular the factor  $Z_\psi$  on the left hand side, which can be understood from the fact that the left hand side of the Ward–Takahashi identity arises from the fermion kinetic term in the Lagrangian (which is proportional to  $Z_\psi$ ).

The two point correlation functions on the right hand side give just the full fermion propagator, including all quantum corrections:

171

(Bold face is used here to emphasize that  $\mathbf{S}(x_1 - x_2)$  is the full propagator.) At leading order in renormalized perturbation theory, treating the Lagrangian terms  $\bar{\psi}_r \not{\partial} \psi_r - m_r \bar{\psi}_r \psi_r$  as the free Lagrangian and the counterterms  $(Z_\psi - 1) \bar{\psi}_r \not{\partial} \psi_r - (Z_m - 1) m_r \bar{\psi}_r \psi_r$  as a perturbation, we have  $\hat{\mathbf{S}}(q) = i/(\not{q} - m_r + i\epsilon)$ . This can be seen either by repeating the derivation of the fermion propagator from section 6.7 starting from the renormalized Lagrangian, or by considering that the free propagator should be a Green's function of the renormalized Dirac operator  $\bar{\psi}_r \not{\partial} \psi_r - m_r \bar{\psi}_r \psi_r$ .

We can also write the left hand side of eq. (8.128) in terms of Feynman diagrams. To do so, we recall from section 4.3.6 the momentum space Feynman rules for correlation functions. From the derivations there, we see, for the example of  $\phi^4$  theory, that the Feynman rules for correlation functions differ from the by now more familiar Feynman rules for scattering matrix elements in two ways: (1) also external lines are represented by propagators, and (2) external vertices come with a factor  $e^{ipx}$  for vertices attached to ingoing lines, and with a factor  $e^{-ipx}$  for vertices attached to outgoing lines. As usual, all momenta not determined by momentum conservation at the vertices are integrated over. We can decompose the left hand side of eq. (8.128) in the following way:

$$\langle 0 | T e_r \bar{\psi}_r(x) \gamma^\mu \psi_r(x) \psi_r(x_1) \bar{\psi}_r(x_2) | 0 \rangle = \frac{1}{Z_e} \cdot \begin{array}{c} x_1 \\ \nearrow \\ \text{blob} \\ \nwarrow \\ x_2 \end{array} \begin{array}{c} x, \mu \\ \nearrow \\ \text{blob} \\ \nwarrow \\ \vec{q} \end{array} \quad (8.130)$$

$$\frac{ie_r}{Z_e} \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} e^{-ik_1 x_1 + ik_2 x_2 + i(k_1 - k_2)x} \tilde{\mathbf{S}}(k_1) \tilde{\mathbf{\Gamma}}^\mu(k_1, k_2) \tilde{\mathbf{S}}(k_2).$$

Here, the blobs on the fermion lines stand for the full fermion propagators from eq. (8.129), and the blob in the center stands for all *1-particle irreducible* (1PI) vertex diagrams. A diagram is called 1-particle irreducible if it cannot be turned into a disconnected diagram by cutting a single propagator. For instance the diagram in fig. 8.1 (c) is 1PI, while the diagrams in fig. 8.1 (d), (e), (f) are not. Note that the 1PI vertex function  $\tilde{\mathbf{\Gamma}}^\mu(k_1, k_2)$  is just the vertex function discussed above in section 8.1, but now evaluated to all orders in perturbation theory, as indicated by the bold face symbol  $\tilde{\mathbf{\Gamma}}$ .

Since the Feynman rules in renormalized perturbation theory yield a factor  $Z_e e_r \gamma^\mu$  for the electron-photon vertex, but the left hand side of eq. (8.130) contains only a factor  $e_r \gamma^\mu$ , we need to explicitly divide the right hand side by  $Z_e$ .

We now plug eqs. (8.129) and (8.130) into eq. (8.128):

$$\begin{aligned} & -ie_r Z_\psi Z_e^{-1} q_\mu \delta^{(4)}(p - p' + q) \tilde{\mathbf{S}}(p') \tilde{\mathbf{\Gamma}}^\mu(p', p) \tilde{\mathbf{S}}(p) \\ & = -ie_r \delta^{(4)}(p - p' + q) [\tilde{\mathbf{S}}(p) - \tilde{\mathbf{S}}(p')]. \end{aligned} \quad (8.131)$$

Multiplying this expression from the left with  $\tilde{\mathbf{S}}^{-1}(p')$  and from the right with  $\tilde{\mathbf{S}}^{-1}(p)$  gives

$$Z_\psi Z_e^{-1}(p' - p)_\mu \tilde{\Gamma}^\mu(p', p) = \tilde{\mathbf{S}}^{-1}(p') - \tilde{\mathbf{S}}^{-1}(p). \quad (8.132)$$

We now take the fermion momenta close to the mass shell, i.e.  $p^2 \simeq m_r^2$  and  $p'^2 \simeq m_r^2$ , to obtain in on-shell renormalization

$$Z_\psi Z_e^{-1}(p' - p)_\mu \gamma^\mu = \not{p}' - \not{p}. \quad (8.133)$$

This proves that  $Z_\psi = Z_e$ .

### 8.2.7 Renormalization group evolution and the Callan–Symanzik equation

While the on-shell renormalization conditions eq. (8.106) are a very sensible choice, they are not unique. Consider for instance a particle production process at a high energy collider, e.g.  $e^+e^- \rightarrow \mu^+\mu^-$ , mediated by an off-shell photon. In this process, the momenta  $p'$  and  $p$  entering the QED vertices are very far from satisfying  $q = p' - p = 0$  or  $q^2 = 0$ . Therefore, the cancellation between loop diagrams and counterterms afforded by eq. (8.106) for low-energy processes will be grossly violated, and loop corrections to the amplitude will be sizeable. Therefore, for a process involving particles whose energies are of the order of some large scale  $M$ , it would make more sense to demand, for instance,

$$\Pi(q^2 = M^2) = 0, \quad \Gamma^\mu((p' - p)^2 = M^2) = \gamma^\mu. \quad (8.134)$$

Of course, the value of  $e_r$  obtained this way is different from the value obtained in on-shell renormalization.  $e_r$  thus becomes a function of  $M$ . This behavior is called *renormalization group evolution*.

To study it further, consider a correlation function

$$G^{n_\psi, n_{\bar{\psi}}, n_A} \equiv \langle 0 | T \psi_r(x_1) \cdots \psi_r(x_{n_\psi}) \bar{\psi}_r(y_1) \cdots \bar{\psi}_r(y_{n_{\bar{\psi}}}) A_r^\mu(z_1) \cdots A_r^\mu(z_{n_A}) | 0 \rangle. \quad (8.135)$$

Now, assume we shift the renormalization scale  $M$  by an infinitesimal amount:  $M \rightarrow M + \delta M$ . Then, also the renormalized quantities will change, and we write these changes as

$$\begin{aligned} \psi_r(x) &\rightarrow (1 + \delta x_\psi) \psi_r(x), \\ A_r^\mu(x) &\rightarrow (1 + \delta x_A) A_r^\mu(x), \\ m_r &\rightarrow m_r + \delta m_r, \\ e_r &\rightarrow e_r + \delta e_r. \end{aligned} \quad (8.136)$$

Consequently,  $G^{n_\psi, n_{\bar{\psi}}, n_A}$  transforms as

$$\delta G^{n_\psi, n_{\bar{\psi}}, n_A} = (n_\psi \delta x_\psi + n_{\bar{\psi}} \delta x_{\bar{\psi}} + n_A \delta x_A) G^{n_\psi, n_{\bar{\psi}}, n_A} \quad (8.137)$$

On the other hand, since the Feynman rules used to compute  $G^{n_\psi, n_{\bar{\psi}}, n_A}$  depend on the renormalization scheme only through the parameters  $M$ ,  $e_r$  and  $m_r$ , we can also write

$$\delta G^{n_\psi, n_{\bar{\psi}}, n_A} = \frac{\partial G^{n_\psi, n_{\bar{\psi}}, n_A}}{\partial M} \delta M + \frac{\partial G^{n_\psi, n_{\bar{\psi}}, n_A}}{\partial e_r} \delta e_r + \frac{\partial G^{n_\psi, n_{\bar{\psi}}, n_A}}{\partial m_r} \delta m_r. \quad (8.138)$$

Combining eqs. (8.137) and (8.138), and introducing the definitions

$$\beta \equiv \frac{M}{\delta M} \delta e_r \quad (8.139)$$

and

$$\begin{aligned} \gamma_\psi &\equiv -\frac{M}{\delta M} \delta x_\psi, \\ \gamma_A &\equiv -\frac{M}{\delta M} \delta x_A, \\ \gamma_m &\equiv \frac{M}{\delta M} \delta m_r, \end{aligned} \quad (8.140)$$

we obtain the *Callan–Symanzik equation*

$$\left[ M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial e_r} + \gamma_m \frac{\partial}{\partial m_r} + n_\psi \gamma_\psi + n_{\bar{\psi}} \gamma_\psi^* + n_A \gamma_A \right] G^{n_\psi, n_{\bar{\psi}}, n_A} = 0. \quad (8.141)$$

This equation specifies how the changes in the various renormalized quantities required by a change in the renormalization scale  $M$  are related to one another. Note that the coefficient functions  $\beta$ ,  $\gamma_m$ ,  $\gamma_\psi$ , and  $\gamma_A$  are universal, i.e. they do not depend on the correlation function  $G^{n_\psi, n_{\bar{\psi}}, n_A}$ .

### 8.2.8 The $\beta$ function

Of particular interest is the  $\beta$  function, which tells us how the coupling constant changes with  $M$ . To evaluate  $\beta$ , we should in principle repeat the renormalization procedure from section 8.2.4 with the modified renormalization conditions eq. (8.134). However, also these renormalization conditions are not the only possible way of defining the energy scale  $M$  at which we wish to renormalize. Other choices are possible, but of course the resulting  $\beta$  and  $\gamma$  functions may depend on this choice.

In practice, it is easiest to work with  $\overline{\text{MS}}$  renormalization, so this is what is usually done and what we will do now. Also the  $\overline{\text{MS}}$  prescription, which consists of choosing the counterterm such that the divergent term  $2/\epsilon - \gamma + \log 4\pi - \log \Delta$  in the loop amplitude gets replaced by  $-\log(\Delta/\mu^2)$ , introduces a renormalization scale: the unphysical parameter  $\mu$ . We have already emphasized in section 8.1.8 that the numerical value of  $e_r$  will depend on  $\mu$ . We can therefore identify  $\mu$  with the renormalization scale  $M$  in  $\overline{\text{MS}}$  renormalization:

$$M \equiv \mu. \quad (8.142)$$

In fact, just as eq. (8.134) implied that loop corrections vanish for a process in which  $q^2 = M^2$ , also  $\overline{\text{MS}}$  renormalization implies that loop corrections in such a process are small, if not exactly zero. To see this, consider that a typical logarithmically divergent loop amplitude, after  $\overline{\text{MS}}$  renormalization, has the structure

$$\int_0^1 dx dy \cdots \left( \log \frac{\Delta}{\mu^2} + \text{finite terms} \right). \quad (8.143)$$

Here, both  $\Delta$  and the finite terms are functions of the masses and momenta of the particles involved in the process. The finite terms are dimensionless and generically of order 1. This can be understood by considering these terms as rational functions of the masses and momenta, with the numerator and the denominator dominated by the largest energy scale in the problem, possibly multiplied by  $\mathcal{O}(1)$  Feynman parameters.  $\Delta$  has mass dimension 2 and is also typically dominated by the largest energy scale in the process. Therefore, if  $\mu$  is similar to that energy scale, the logarithm vanishes, while for very different scales, the logarithm can be very large and dominant. While these arguments may seem somewhat imprecise and handwaving, they work surprisingly well in practice in the sense that, for a process at an energy scale  $\mu$ , the amplitude can be calculated fairly accurately at tree level by choosing for the mass parameters and coupling constants the values obtained by renormalizing at the scale  $\mu$  using the  $\overline{\text{MS}}$  scheme.

Having established that  $\mu$  can be interpreted as the renormalization scale in  $\overline{\text{MS}}$  renormalization, we can finally calculate the  $\beta$  function of QED:

$$\beta = \mu \frac{\delta e_r}{\delta \mu} = \frac{\delta}{\delta \log \mu} \sqrt{Z_A} e_0 = \frac{e_0}{2\sqrt{Z_A}} \frac{\partial Z_A}{\partial \log \mu} \simeq e_r \frac{\partial Z_A}{\partial \log \mu^2}. \quad (8.144)$$

In the last step, we have used that, to lowest non-vanishing order in  $e_r$ ,  $Z_A$  can be replaced by 1 in the prefactor, and  $e_0$  by  $e_r$ . Equation (8.144) implies that we need to evaluate the  $\mu$ -dependence of the renormalization constant  $Z_A$ , or equivalently of the counterterm  $\delta_A = Z_A - 1$ . We can read off directly from the expression for the photon self energy  $\Pi_2(q^2)$ , eq. (8.103), that this counterterm is

$$\delta_A = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left( \frac{2}{\epsilon} - \gamma + \log 4\pi - \log \mu^2 \right). \quad (8.145)$$

Therefore,

$$\beta(e_r) = e_r \frac{2\alpha}{\pi} \int_0^1 dx x(1-x) = \frac{e_r^3}{12\pi^2}. \quad (8.146)$$

With the definition of  $\beta$  from eq. (8.139), we find now the  $\mu$  dependence of  $e_r$ . If  $\bar{e}_r$  is the value of  $e_r$  at renormalization scale  $\bar{\mu}$ , its value at any other scale is obtained from

$$\frac{[e_r(\mu)]^3}{12\pi^2} = \mu \frac{de_r(\mu)}{d\mu}$$

$$\begin{aligned}
 &\Leftrightarrow \quad \frac{d\mu}{\mu} = \frac{12\pi^2}{[e_r(\mu)]^3} de_r, \\
 &\Leftrightarrow \quad \log \frac{\mu}{\bar{\mu}} = -\frac{6\pi^2}{[e_r(\mu)]^2} + \frac{6\pi^2}{[\bar{e}_r]^2},
 \end{aligned} \tag{8.147}$$

which leads to the evolution equation

$$e_r^2(\mu) = \frac{\bar{e}_r^2}{1 - \frac{\bar{e}_r^2}{6\pi^2} \log \frac{\mu}{\bar{\mu}}}. \tag{8.148}$$

or, equivalently in terms of  $\alpha_r = e_r^2/(4\pi)$ ,

$$\alpha_r(\mu) = \frac{\bar{\alpha}_r}{1 - \frac{\bar{\alpha}_r^2}{3\pi} \log \frac{\mu}{\bar{\mu}}}. \tag{8.149}$$

This equation tells us that, for processes taking place at a very high energy,  $e_r$  is somewhat larger than for processes taking place at low energy. In fact, while at low energy, we have the familiar value  $\alpha_r(m_e) \simeq 1/137$ , the renormalized fine structure constant changes to  $\alpha_r(100 \text{ GeV}) \simeq 1/128$  at higher energy. The dependence is only logarithmic, and is moreover weakened by the smallness of  $\bar{\alpha}_r$  in the denominator, but it is still highly relevant in precision calculations.

## 8.3 Infrared Divergences

### 8.3.1 IR behavior of virtual (1-loop) corrections

After this extensive discussion of the problems associated with UV divergences in loop amplitudes, let us now turn to another class of problems that arise at the 1-loop level. Consider once again the 1-loop vertex correction after  $\overline{\text{MS}}$  renormalization:

$$\begin{array}{c} \diagup \\ | \\ \text{loop} \\ | \\ \diagdown \end{array} = \bar{u}(p') (-ie\Gamma^{(1)\mu}) u(p), \tag{8.150}$$

with

$$\Gamma^{(1),\mu} = \gamma^\mu F_1^{(1)}(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2^{(1)}(q^2) \tag{8.151}$$

and

$$\begin{aligned}
 F_1^{(1)}(q^2) = & \frac{\alpha}{2\pi} \int dx dy dz \delta(1-x-y-z) \\
 & \times \left[ -\log \frac{\Delta}{\mu^2} - 2 + \frac{q^2}{\Delta} (1-x)(1-y) + \frac{m^2}{\Delta} (1-4z+z^2) \right],
 \end{aligned} \tag{8.152}$$



$$F_2^{(1)}(q^2) = \frac{\alpha}{2\pi} \int dx dy dz \delta(1-x-y-z) \frac{2m^2}{\Delta} z(z-1). \quad (8.153)$$

The quantity  $\Delta$  appearing in the logarithm and in the denominators is given by

$$\Delta = m^2(1-z)^2 - xyq^2. \quad (8.154)$$

The superscript (1) indicates quantities evaluated at 1-loop order.

Note that at  $z \rightarrow 1$ ,  $x, y \rightarrow 0$ ,  $\Delta$  goes to zero, making the Feynman parameter integral divergent. We had elided this problem so far since we never had to actually carry out the Feynman parameter integral yet. Note that the problem affects only  $F_1(q^2)$ . In  $F_2(q^2)$ , the integrand of the Feynman parameter integral remains finite for all  $x, y, z$ . The physical reason for this new type of divergence is that, for loop momenta  $k \simeq p$  (see eq. (8.26)), the fermion propagators and the photon propagator in the loop are close to on-shell, so that the integrand goes to infinity very quickly in this limit. As the divergence arises in a region where the loop momentum is low, we refer to it as an *infrared (IR) divergence*, as opposed to the UV divergences we were dealing with previously.

To deal with the IR divergence, we consider for brevity only the limit  $q^2 \gg m^2$ . (See [1] for a more general discussion.) In this limit, we have

$$F_1^{(1)}(q^2) \simeq \frac{\alpha}{2\pi} \int_0^1 dx \int_0^{1-z} dy \frac{q^2(y+z)(1-y)}{m^2(1-z)^2 - q^2(1-y-z)y}. \quad (8.155)$$

Focusing only on the divergent region  $z \rightarrow 1$ ,  $y \rightarrow 0$ ,  $x = 1 - y - z \rightarrow 0$ , the term  $(y+z)(1-y)$  in the numerator is approximately 1. (We cannot make similar approximations in the denominator since this would alter the divergence structure.)

To make progress, we should regularize the IR divergence. We can do so by pretending that the photon has a small mass  $\rho$ . This leads to the shift

$$\Delta \rightarrow \Delta + z\rho^2 \simeq \Delta + \rho^2 \quad (8.156)$$

and thus to

$$F_1^{(1)}(q^2) \simeq \frac{\alpha}{2\pi} \int_0^1 dz \int_0^{1-z} dy \frac{q^2}{m^2(1-z)^2 - q^2(1-y-z)y + \rho^2}. \quad (8.157)$$

Ultimately, of course, we would like to send  $\rho \rightarrow 0$  again, in particular since introducing a mass for the photon violates gauge invariance. In the following, we will keep  $\rho$  only in those places where expressions would otherwise be infinite.

To evaluate the regularized Feynman parameter integral, and in particular to decouple the two integrals, we make the transformation

$$w^2 \equiv (1-z)^2 \quad \text{and} \quad \xi \equiv \frac{y}{1-z}, \quad (8.158)$$

with the Jacobian

$$J = \frac{d(w^2, \xi)}{d(y, z)} = \begin{pmatrix} 0 & -2(1-z) \\ \frac{1}{1-z} & \frac{y}{(1-z)^2} \end{pmatrix}. \quad (8.159)$$

It results in

$$\begin{aligned}
 F_1^{(1)}(q^2) &\simeq \frac{\alpha}{2\pi} \frac{1}{2} \int_0^1 d\xi \int_0^1 d(w^2) \frac{q^2}{w^2[m^2 - q^2\xi(1-\xi)] + \rho^2} \\
 &\simeq \frac{\alpha}{4\pi} \int_0^1 d\xi \frac{q^2}{m^2 - q^2\xi(1-\xi)} \log \left( \frac{m^2 - q^2\xi(1-\xi)}{\rho^2} \right). \tag{8.160}
 \end{aligned}$$

In the second step, we have set  $\rho^2 \rightarrow 0$  in the prefactor, where it is not needed to regularize a divergence. Next, we observe that, at  $q^2 \gg m^2$ , the largest contributions to the integral will come from the regions close to  $\xi = 0$  and  $\xi = 1$ . We can therefore restrict the integration domain to two small regions of size  $\epsilon$  around these points. Moreover, since the divergences at  $\xi = 0$  and  $\xi = 1$  are driven not by the logarithm, but by the prefactor, which goes to infinity much faster, we can, without making a large error, write

$$\begin{aligned}
 F_1^{(1)}(q^2) &\simeq \frac{\alpha}{4\pi} \log \left( -\frac{q^2}{\rho^2} \right) \left[ \int_0^\epsilon d\xi \frac{q^2}{m^2 - q^2\xi} + \int_{1-\epsilon}^1 d\xi \frac{q^2}{m^2 - q^2(1-\xi)} \right] \\
 &= \frac{\alpha}{4\pi} \log \left( -\frac{q^2}{\rho^2} \right) \left[ -\log \left( \frac{m^2 - q^2\xi}{m^2} \right) \Big|_0^\epsilon + \log \left( \frac{m^2 - q^2(1-\xi)}{m^2} \right) \Big|_{1-\epsilon}^1 \right] \\
 &\simeq -\frac{\alpha}{2\pi} \log \left( -\frac{q^2}{\rho^2} \right) \log \left( -\frac{q^2}{m^2} \right) \tag{8.161}
 \end{aligned}$$

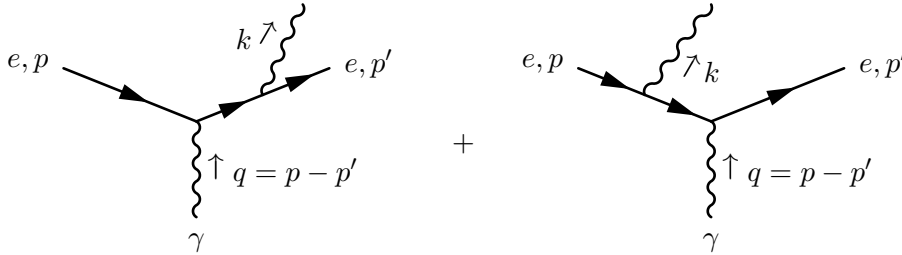
Overall, this correction implies that the 1-loop differential cross section  $d\sigma^{(1)}(p \rightarrow p')/d\Omega$  for the scattering of an electron on a photon field (for instance an external electrostatic potential) can be written as

$$\frac{d\sigma^{(1)}(p \rightarrow p')}{d\Omega} \simeq \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \left[ 1 - \frac{\alpha}{\pi} \log \left( -\frac{q^2}{\rho^2} \right) \log \left( -\frac{q^2}{m^2} \right) \right]. \tag{8.162}$$

Here, the subscripts (0) and (1) indicate tree level and (approximate) 1-loop results, respectively. We cannot work with this result any further for now, therefore, we will now study IR divergences in a different class of diagrams and show that in the end all IR divergences cancel each other.

### 8.3.2 IR behavior of real corrections

Consider higher-order (in  $\alpha$ ) processes in which a real photon is emitted:



$$\begin{aligned}
&= \bar{u}(p') \left[ (-ie\gamma^\nu) \frac{i(\not{p}' + \not{k} + m)}{(p' + k)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \right. \\
&\quad \left. + (-ie\gamma^\mu) \frac{i(\not{p} - \not{k} + m)}{(p - k)^2 - m^2 + i\epsilon} (-ie\gamma^\nu) \right] u(p) \epsilon_\nu^*(k) \tilde{A}_\mu(q) \quad (8.163)
\end{aligned}$$

Here, the hard photon is treated as a non-quantized external perturbation (such as a fixed electrostatic field), represented by the corresponding 4-potential in momentum space,  $\tilde{A}_\mu(q)$ . If this photon was described by a plane wave, we would have  $\tilde{A}_\mu(q) = \epsilon_\mu(q)$ , but for other external field configurations,  $\tilde{A}_\mu(q)$  has a more complicated form.

Let us focus in particular on the phase space region where  $k$  is very soft,  $\mathbf{k} \ll \mathbf{p}, \mathbf{p}', \mathbf{q}$ . In this region, the emission of an extra photon may easily go undetected. On the other hand, note that at  $k \rightarrow 0$ , the amplitude in eq. (8.163) becomes infinite as the intermediate fermion goes on-shell. To isolate this divergence, we simplify eq. (8.163), keeping  $k$  only in those places where it is needed as a regulator. We obtain

$$\begin{aligned}
\text{Diagram 1} + \text{Diagram 2} &= -ie^2 \bar{u}(p') \left[ \gamma^\nu \frac{\not{p}' + m}{2p' \cdot k + i\epsilon} \gamma^\mu + \gamma^\mu \frac{\not{p} + m}{-2p \cdot k + i\epsilon} \gamma^\nu \right] u(p) \epsilon_\nu^*(k) \tilde{A}_\mu(q) \\
&= -ie^2 \left[ \frac{p'^\nu}{p' \cdot k} - \frac{p^\nu}{p \cdot k} \right] \bar{u}(p') \gamma^\mu u(p) \epsilon_\nu^*(k) \tilde{A}_\mu(q) \\
&= i\mathcal{M}^{(0)} e \left[ \frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k} \right]. \quad (8.164)
\end{aligned}$$

In the last line,  $i\mathcal{M}^{(0)} = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(q)$  represents the matrix element for the leading order process without radiation of the extra soft photon. In computing the differential cross section for the radiative process, which we will denote by  $d\sigma(p \rightarrow p' + \gamma)/d\Omega$ , we need to integrate over the phase space of the extra photon:

$$\begin{aligned}
\frac{d\sigma(p \rightarrow p' + \gamma)}{d\Omega} &= \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \sum_{\text{polarizations}} \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|} e^2 \left| \frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k} \right|^2 \\
&= \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \int \frac{dk d\Omega_k}{(2\pi)^3 2k} e^2 k^2 \left[ \frac{2p \cdot p'}{(p \cdot k)(p' \cdot k)} - \frac{p'^2}{(p' \cdot k)^2} - \frac{p^2}{(p \cdot k)^2} \right]. \quad (8.165)
\end{aligned}$$

We focus again on the region where  $q^2 \gg m^2$ . There, then first term in square brackets in eq. (8.165) is much larger than the other two terms. To carry out the angular integral over  $d\Omega_k$ , we go to a frame in which  $E_{\mathbf{p}} = E_{\mathbf{p}'} \equiv E$ . In this frame, we can write

$$\begin{aligned}
p &= (E, \sqrt{E^2 - m^2} \hat{\mathbf{p}}) \simeq E(1, \mathbf{v}), \\
p' &= (E, \sqrt{E^2 - m^2} \hat{\mathbf{p}}') \simeq E(1, \mathbf{v}'), \\
k &= (|\mathbf{k}|, \mathbf{k}). \quad (8.166)
\end{aligned}$$

Here,  $\hat{\mathbf{p}}$  and  $\hat{\mathbf{p}}'$  denote unit vectors. Note that  $q^2 = -(E^2 - m^2)(\hat{\mathbf{p}} - \hat{\mathbf{p}}')^2 \gg m^2$  implies that  $|\mathbf{v}| = |\mathbf{v}'| \simeq 1$ . The angular integral is now

$$I_\Omega \equiv \int d\Omega_k \frac{2p \cdot p'}{(p \cdot k)(p' \cdot k)} = \int d\Omega_k \frac{2E^2(1 - \mathbf{v} \cdot \mathbf{v}')}{E^2 k^2 (1 - \hat{\mathbf{k}} \cdot \mathbf{v})(1 - \hat{\mathbf{k}} \cdot \mathbf{v}')} . \quad (8.167)$$

Again,  $\hat{\mathbf{k}}$  denotes a unit vector in the direction of  $\mathbf{k}$ . The integral is dominated by the phase space regions where either  $1 - \hat{\mathbf{k}} \cdot \mathbf{v} \simeq 0$  or  $1 - \hat{\mathbf{k}} \cdot \mathbf{v}' \simeq 0$ . Keeping only these contributions, and using  $|\mathbf{v}| = |\mathbf{v}'| \simeq 1$ , we can write

$$I_\Omega = \frac{4\pi}{k^2} \left[ \int_{1-\mathbf{v}' \cdot \mathbf{v}}^1 d\cos\theta \frac{1}{1 - |\mathbf{v}| \cos\theta} + \int_{1-\mathbf{v} \cdot \mathbf{v}'}^1 d\cos\theta \frac{1}{1 - |\mathbf{v}'| \cos\theta} \right] . \quad (8.168)$$

Here, we have chosen the spatial coordinate system in the first integral such that  $\cos\theta = 1$  corresponds to  $\hat{\mathbf{k}} \parallel \mathbf{v}$ , and in the second integral such that  $\cos\theta = 1$  corresponds to  $\hat{\mathbf{k}} \parallel \mathbf{v}'$ . The lower integration boundaries in eq. (8.168) are not really important. We will see in a moment that the value of the integral depends on them only logarithmically. The choice we are making here makes sense because the condition  $\cos\theta = 1 - \mathbf{v}' \cdot \mathbf{v}$  (or, equivalently,  $\hat{\mathbf{k}} \cdot \mathbf{v}^{(l)} = \mathbf{v}' \cdot \mathbf{v}$ ) denotes the boundary of the region where the radiated photon can be considered collinear with  $\mathbf{v}$  or  $\mathbf{v}'$ . Evaluating  $I_\Omega$ , we find

$$\begin{aligned} I_\Omega &= \frac{4\pi}{k^2} \left[ \log \frac{1 - \mathbf{v} \cdot \mathbf{v}'}{1 - |\mathbf{v}|} + \log \frac{1 - \mathbf{v} \cdot \mathbf{v}'}{1 - |\mathbf{v}'|} \right] \\ &= \frac{4\pi}{k^2} \log \frac{(p \cdot p')^2}{(1 - \sqrt{E^2 - m^2}/E)^2} \\ &\simeq \frac{8\pi}{k^2} \log \left( -\frac{q^2}{m^2} \right) . \end{aligned} \quad (8.169)$$

Putting this back into eq. (8.165) leads to

$$\frac{d\sigma(p \rightarrow p' + \gamma)}{d\Omega} \simeq \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \cdot \int_0^{|\mathbf{q}|} \frac{dk}{(2\pi)^3 2k} 8\pi e^2 \log \left( -\frac{q^2}{m^2} \right) . \quad (8.170)$$

This is another infinite integral! We can regularize it by pretending once again that the photon has a small mass  $\rho$ . Then,  $k$  in the denominator turns into  $\sqrt{k^2 + \rho^2}$  and we end up with

$$\frac{d\sigma(p \rightarrow p' + \gamma)}{d\Omega} \simeq \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \cdot \frac{\alpha}{\pi} \log \left( -\frac{q^2}{m^2} \right) \log \left( -\frac{q^2}{\rho^2} \right) . \quad (8.171)$$

Note that the choice of the upper integration boundary in eq. (8.170) is of the right order, but again somewhat arbitrary. In principle, we should do a full kinematic study to determine the appropriate boundary as a function of  $p$  and  $p'$ . However, our result depends on this boundary only logarithmically, and the divergence structure in the limit  $\rho \rightarrow 0$  that we are most interested in here is not affected at all.

Now compare eq. (8.171) to the IR divergence in the 1-loop amplitude, eq. (8.162), which we repeat here for convenience:

$$\frac{d\sigma^{(1)}(p \rightarrow p')}{d\Omega} \simeq \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \left[ 1 - \frac{\alpha}{\pi} \log \left( -\frac{q^2}{m^2} \right) \log \left( -\frac{q^2}{\rho^2} \right) \right]. \quad (8.172)$$

We see that  $\rho$  drops out of the cross section when both virtual (loop) corrections and real corrections are considered. In real life, this is precisely what one should do since the radiated photon can be arbitrarily soft, and even the best detector is not be able to detect arbitrarily soft photons. (The ultimate limitation here is given by the Heisenberg uncertainty.) If the detection threshold is  $E_{\text{thr}}$ , we can compute the contribution of soft photons that cannot be identified experimentally by choosing the upper integration boundary in eq. (8.170) at  $E_{\text{thr}}$  instead of  $|\mathbf{q}|$ . We then have

$$\frac{d\sigma^{(1)}(p \rightarrow p')}{d\Omega} \simeq \frac{d\sigma^{(0)}(p \rightarrow p')}{d\Omega} \left[ 1 - \frac{\alpha}{\pi} \log \left( -\frac{q^2}{m^2} \right) \log \left( -\frac{q^2}{E_{\text{thr}}^2} \right) \right]. \quad (8.173)$$



# 9

## Non-Abelian Gauge Theories

It is time that we move beyond quantum electrodynamics and extend our toolbox to include other quantum field theories, namely those based on non-Abelian gauge symmetries (as opposed to QED, which is based on the Abelian symmetry group  $U(1)$ ). Perhaps the most prominent example for a non-Abelian gauge symmetry is quantum chromodynamics (QCD), the theory of the strong interaction.

### 9.1 Gauge Interactions from Symmetry

To begin, let us develop a deeper understanding of the relation between the Lagrangian terms containing gauge fields and the underlying symmetry structure. Believing the title of this chapter, we will do this for an Abelian  $U(1)$  gauge symmetry first. Consider a Dirac field  $\psi(x)$  and impose a symmetry of the Lagrangian  $\mathcal{L}$  under transformations of the form

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \equiv U(x)\psi(x), \quad (9.1)$$

where  $U(x) \in U(1)$ . As we know, the mass term of the free Dirac theory is invariant under this transformation, but the kinetic term is not because, after applying eq. (9.1), the derivative acts on  $\alpha(x)$  as well as  $\psi(x)$ . The deeper reason the derivative is violating the symmetry is that it can be written as

$$n^\mu \partial_\mu \psi(x) = \lim_{\epsilon \rightarrow 0} \frac{\psi(x + \epsilon n) - \psi(x)}{\epsilon}, \quad (9.2)$$

where  $n$  is an arbitrary constant 4-vector. Since the transformation parameters  $\alpha(x + \epsilon n)$  and  $\alpha(x)$  are independent of one another, the two fields in the numerator transform differently in general.

To turn  $\partial_\mu \psi(x)$  into an object with well-defined gauge transformation properties, we introduce the *comparator*  $W(x, y)$ , a yet to be specified function with the transformation property

$$W(x, y) \rightarrow e^{i\alpha(x)} W(x, y) e^{-i\alpha(y)}. \quad (9.3)$$

It allows us to define the *covariant derivative*  $D_\mu$  via

$$n^\mu D_\mu \psi(x) \equiv \lim_{\epsilon \rightarrow 0} \frac{W(x, x + \epsilon n) \psi(x + \epsilon n) - \psi(x)}{\epsilon}. \quad (9.4)$$

By definition, it transforms as

$$D_\mu \psi(x) \rightarrow e^{i\alpha(x)} D_\mu \psi(x), \quad (9.5)$$

so that the modified fermion kinetic term  $\bar{\psi} i \not{D} \psi$  is gauge invariant. Of course, before this result can be of any use, we first have to determine  $W(x, y)$ . We demand that, in addition to eq. (9.3), the comparator should have the properties

$$\begin{aligned} W(y, y) &= 1 \\ |W(x, y)| &= 1. \end{aligned} \quad (9.6)$$

For infinitesimal displacements  $y^\mu - x^\mu = \epsilon n^\mu$ , it must then have the form

$$W(x, x + \epsilon n) = 1 + i e \epsilon n^\mu A_\mu(x), \quad (9.7)$$

where the *gauge connection* (or gauge field)  $A_\mu(x)$  is an arbitrary vector field, and we have arbitrarily pulled out a scalar constant  $e$ . At finite displacement, this integrates to the so-called *Wilson line*

$$W(x, y) = \exp \left[ i e \oint_y^x dz^\mu A_\mu(z) \right]. \quad (9.8)$$

This relation shows in particular that Wilson lines can be concatenated:

$$W(x, y) W(y, z) = W(x, z). \quad (9.9)$$

Note, however, that the Wilson line depends on the path that is chosen to go from  $y$  to  $x$ . With eq. (9.7), we can now write out the covariant derivative:

$$D_\mu \psi(x) = \partial_\mu \psi(x) + i e A_\mu(x) \psi(x). \quad (9.10)$$

Moreover, the transformation property eq. (9.3) that defines how  $D_\mu$  behaves under gauge transformations, allows us to deduce the transformation properties of  $A_\mu(x)$ :

$$\begin{aligned} W(x, x + \epsilon n) &\rightarrow e^{i\alpha(x)} W(x, x + \epsilon n) e^{-i\alpha(x + \epsilon n)} \\ &= e^{i\alpha(x)} [1 + i e \epsilon n^\mu A_\mu(x)] [1 - i \epsilon n^\mu \partial_\mu \alpha(x)] e^{-i\alpha(x)} \end{aligned}$$



$$= 1 + ie \epsilon n^\mu \left[ A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \right]. \quad (9.11)$$

Comparing to eq. (9.7), we conclude that  $A_\mu(x)$  must transform as

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x). \quad (9.12)$$

This should look vaguely familiar.

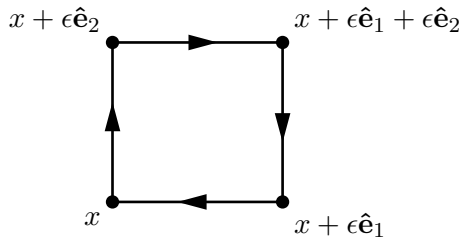
What have we achieved? Starting from the requirement that the quantum field theory of a fermion  $\psi$  should be invariant under  $U(1)$  gauge transformations, we have constructed a covariant derivative operator. Doing so required introducing new degrees of freedom—the vector field  $A^\mu(x)$ —and dictated its transformation properties. Note that the above derivation can be applied to any Lagrangian that is invariant under a global symmetry group  $G$  (with  $x$ -independent transformation operators), and that we want to render invariant under the corresponding local symmetry (where the transformation operators are  $x$ -dependent). The only problematic terms in this case are those involving derivatives, and they can be rendered gauge invariant by simply replacing each partial derivative by the covariant derivative.

To turn  $A^\mu(x)$  into a quantized field, we also need a kinetic term for it. In section 5.1, we had written down the kinetic term  $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  based on our knowledge of classical electrodynamics. However, we can also motivate it from symmetry arguments. This will be useful in particular in the context of more complicated gauge theories like QCD, where we have no classical theory to derive intuition from. We need to look for a gauge invariant term that depends on  $A_\mu(x)$  and its derivatives, but not on  $\psi(x)$ .

The comparator offers a way of constructing such a term: note that, for any set of spacetime points  $x_1, x_2, \dots, x_n$ , the quantity

$$W(x_1, x_2)W(x_2, x_3) \cdots W(x_{n-1}, x_n)W(x_n, x_1) \quad (9.13)$$

(called a *Wilson loop*) is gauge invariant. Since terms in the Lagrangian should be local (i.e. depend on only one spacetime coordinate  $x$ ) for our construction of propagators and vertices to work, consider the limit  $x_j \rightarrow x$  for all  $j = 1 \cdots n$ . More specifically, let us focus on the infinitesimal loop



The corresponding expression for this Wilson loop can be evaluated using eq. (9.8), expanded up to second order in  $\epsilon$ :

$$W(x, x + \epsilon \hat{e}_2) W(x + \epsilon \hat{e}_2, x + \epsilon \hat{e}_1 + \epsilon \hat{e}_2) W(x + \epsilon \hat{e}_1 + \epsilon \hat{e}_2, x + \epsilon \hat{e}_1) W(x + \epsilon \hat{e}_1, x)$$

$$\begin{aligned}
 &= \exp \left[ i e \epsilon \left( \hat{\mathbf{e}}_2^\mu A_\mu(x + \tfrac{\epsilon}{2} \hat{\mathbf{e}}_2) + \hat{\mathbf{e}}_1^\mu A_\mu(x + \tfrac{\epsilon}{2} \hat{\mathbf{e}}_1 + \epsilon \hat{\mathbf{e}}_2) \right. \right. \\
 &\quad \left. \left. - \hat{\mathbf{e}}_2^\mu A_\mu(x + \epsilon \hat{\mathbf{e}}_1 + \tfrac{\epsilon}{2} \hat{\mathbf{e}}_2) - \hat{\mathbf{e}}_1^\mu A_\mu(x + \tfrac{\epsilon}{2} \hat{\mathbf{e}}_1) \right) \right] \\
 &\simeq 1 + i e \epsilon \left( \hat{\mathbf{e}}_2^\mu \partial^\nu A_\mu(x) \tfrac{\epsilon}{2} \hat{\mathbf{e}}_{2\nu} + \hat{\mathbf{e}}_1^\mu \partial^\nu A_\mu(x) (\tfrac{\epsilon}{2} \hat{\mathbf{e}}_{1\nu} + \epsilon \hat{\mathbf{e}}_{2\nu}) \right. \\
 &\quad \left. - \hat{\mathbf{e}}_2^\mu \partial^\nu A_\mu(x) (\epsilon \hat{\mathbf{e}}_{1\nu} + \tfrac{\epsilon}{2} \hat{\mathbf{e}}_{2\nu}) - \hat{\mathbf{e}}_1^\mu \partial^\nu A_\mu(x) \tfrac{\epsilon}{2} \hat{\mathbf{e}}_{1\nu} \right) \\
 &= 1 + i e \epsilon^2 \hat{\mathbf{e}}_1^\mu \hat{\mathbf{e}}_2^\nu (-\partial_\mu A_\nu(x) + \partial_\nu A_\mu(x)).
 \end{aligned} \tag{9.14}$$

Instead of the Wilson loop in the  $\hat{\mathbf{e}}_1$ – $\hat{\mathbf{e}}_2$  plane, we could have also considered similar Wilson loops in any other plane. The result would differ from eq. (9.14) only in the unit vectors in the  $\epsilon^2$  term. Therefore, we conclude that the quantity

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{9.15}$$

is gauge invariant for any combination of  $\mu$  and  $\nu$ . This is once again a familiar results, now derived from geometrical arguments.

We can now construct the simplest possible gauge and Lorentz invariant Lagrangian involving only  $A^\mu$  and its derivatives:

$$\mathcal{L}_{\text{gauge kin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{9.16}$$

The prefactor  $-1/4$  follows from the requirement that the resulting gauge boson propagator should have the canonical normalization. (If  $A^\mu$  was not normalized in this way initially, it can always be rescaled appropriately, and the rescaling factor can be absorbed into the field strength renormalization  $Z_A$ .)

## 9.2 Non-Abelian Gauge Transformations

The gauge transformations  $e^{i\alpha(x)}$  from eq. (9.1) form the Abelian group  $U(1)$  (the unitary transformations in one dimension). We now generalize the arguments from the previous section to transformation operators forming non-Abelian groups. Consider a theory with  $N$  fermions, which we arrange into an  $N$ -dimensional vector:

$$\Psi \equiv (\psi_1, \dots, \psi_N). \tag{9.17}$$

We demand that the theory be invariant under transformations of the form

$$\Psi(x) \rightarrow U(x) \Psi(x), \tag{9.18}$$

where  $U(x)$  is an  $N \times N$  unitary matrix with  $|\det U| = 1$  (in order to preserve particle number). Since different  $U(x)$  in general do not commute, this is a *non-Abelian* symmetry transformation. If the set of all transformations  $U(x)$  we wish to allow forms a *Lie group*,

the transformation matrices in a neighborhood of the identity (the only case that will be relevant to us in this course) can be written as

$$U(x) = \exp[i\alpha^a(x)t^a]. \quad (9.19)$$

Here the  $t^a$  are a set of  $N \times N$  Hermitian matrices called *generators* that define the allowed transformations, and  $\alpha^a(x)$  are arbitrary smooth functions of  $x$ . The vector space spanned by the generators is called the *Lie algebra* corresponding to the symmetry group.

A few examples for non-Abelian gauge transformations are

- $U(x) \in SU(2)$ , the *special unitary group in two dimensions*, i.e. the group of unitary  $2 \times 2$  matrices with  $\det U = 1$ . Note that the last condition is stronger than the requirement  $|\det U| = 1$  mentioned below eq. (9.18).  $SU(2)$  has three generators, which are represented by the Pauli matrices divided by 2:

$$t^a = \frac{\sigma^a}{2}. \quad (9.20)$$

This symmetry group will be relevant in the unified theory of weak and electromagnetic interactions.

- $U(x) \in SU(3)$ , the *special unitary group in three dimensions*, i.e. the group of unitary  $3 \times 3$  matrices with  $\det U = 1$ .  $SU(3)$  has 8 generators, which can be written as

$$t^a = \frac{\lambda^a}{2}, \quad (9.21)$$

where  $\lambda^a$  are the *Gell-Mann matrices*:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad (9.22)$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad (9.23)$$

$$\lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \quad (9.24)$$

$SU(3)$  is the gauge group underlying the theory of strong interactions, QCD.

- $U(x) \in SU(N)$ , the *special unitary group in  $N$  dimensions*, i.e. the group of unitary  $N \times N$  matrices with  $\det U = 1$ . Note that the last property implies

$$1 = \det e^{i\alpha^a t^a} = e^{i\text{tr}(\alpha^a t^a)}, \quad (9.25)$$

which is equivalent to

$$\text{tr } t^a = 0. \quad (9.26)$$

The number of generators of  $SU(N)$  is

$$\underbrace{2N^2}_{\substack{N^2 \text{ real parts} \\ N^2 \text{ imaginary} \\ \text{parts}}} - \underbrace{2 \frac{N^2 - N}{2}}_{\substack{\text{constraints} \\ \text{on off-diagonal} \\ \text{elements}}} - \underbrace{N}_{\substack{\text{diagonal elements} \\ \text{real}}} - \underbrace{1}_{\text{tr } t^a = 0} = N^2 - 1 \quad (9.27)$$

To render the Lagrangian of the fermions  $\psi_1, \dots, \psi_n$  gauge invariant under a non-Abelian symmetry group, we can proceed as in the previous section. We first define the comparator  $W(x, y)$  by the requirement that it transforms as

$$W(x, y) \rightarrow e^{i\alpha^a(x)t^a} W(x, y) e^{-i\alpha^a(y)t^a} \quad (9.28)$$

and obeys the normalization requirement  $W(x, x) = 1$  and  $|W(x, y)| = 1$ . For infinitesimal displacements, the comparator can be written as

$$W(x, x + \epsilon n) = 1 - ig \epsilon t^a n^\mu A_\mu^a(x). \quad (9.29)$$

Once again,  $g$  is an arbitrary constant, which will define the strength of the gauge interactions, and  $A_\mu^a$  is a set of vector fields. If  $x$  and  $y$  are separated by a finite distance, the comparator becomes

$$W(x, y) = P \exp \left[ ig \oint_y^x dz^\mu t^a A_\mu(z) \right]. \quad (9.30)$$

The symbol  $P$  (for *path ordering*) implies that in each term of the Taylor series defining the exponential, the individual factors of  $t^a A_\mu^a(x)$  should be ordered along the integration contour. In other words, in a product of the form  $t^a A_\mu(x_m) \cdots t^b A_\mu(x_1)$ , the coordinate  $x_m$  that is closest to the end of the integration contour should appear on the left, and the coordinate  $x_1$  closest to the beginning of the contour should appear on the right. This is important because the matrices  $t^a$  do not commute.

We can now define the covariant derivative for a non-Abelian gauge theory:

$$\begin{aligned} n^\mu D_\mu \Psi(x) &\equiv \lim_{\epsilon \rightarrow 0} \frac{W(x, x + \epsilon n) \psi(x + \epsilon n) - \psi(x)}{\epsilon} \\ &= n^\mu (\partial_\mu - ig t^a A_\mu^a(x)) \Psi(x). \end{aligned} \quad (9.31)$$

With this definition, the fermionic part of the Lagrangian of a non-Abelian gauge symmetry has exactly the same structure as for an Abelian symmetry:

$$\mathcal{L} \supset \bar{\Psi}(i\not{D} - m)\Psi. \quad (9.32)$$

Things get more complicated, when it comes to the gauge transformation properties of the non-Abelian gauge fields  $A_\mu^a(x)$ . We begin again with the transformation of the comparator from eq. (9.28) in the case of infinitesimal displacement:

$$\begin{aligned} W(x, x + \epsilon n) &\rightarrow U(x) [1 - ig \epsilon t^a n^\mu A_\mu^a(x)] U^{-1}(x + \epsilon n) \\ &= 1 - ig \epsilon U(x) t^a U^{-1}(x) n^\mu A_\mu^a(x) + \epsilon n^\mu U(x) \partial_\mu U^{-1}(x) \\ &= 1 - ig \epsilon n^\mu [U(x) t^a A_\mu^a(x) U^{-1}(x) + \frac{i}{g} U(x) \partial_\mu U^{-1}(x)]. \end{aligned} \quad (9.33)$$

This leads us to conclude that  $A_\mu^a$  should transform according to

$$t^a A_\mu^a(x) \rightarrow U(x) [t^a A_\mu^a(x) + \frac{i}{g} \partial_\mu] U^{-1}(x). \quad (9.34)$$

To find the field strength tensor and the gauge kinetic term for a non-Abelian theory, we could follow the same procedure as in section 9.1: construct a closed Wilson loop of the form  $W(x_1, x_2)W(x_2, x_3) \cdots W(x_{n-1}, x_n)W(x_n, x_1)$  and study its transformation properties to find a gauge invariant tensor, the equivalent of  $F^{\mu\nu}$  in electromagnetism. This tensor could then be used to construct a gauge kinetic term. We leave this calculation as an exercise and instead present a different method to achieve the same goal. Note that the covariant derivative  $D_\mu$  by definition transforms as

$$D_\mu \rightarrow U(x) D_\mu U^{-1}(x). \quad (9.35)$$

The same transformation law holds also for products of covariant derivatives, and in particular for the commutator  $[D_\mu, D_\nu]$ . The reason to consider in particular this commutator is that it is actually not a derivative operator any more, but just a matrix-valued function:

$$\begin{aligned} [D_\mu, D_\nu] \Psi(x) &= (\partial_\mu - ig t^a A_\mu^a)(\partial_\nu - ig t^b A_\nu^b) \Psi(x) \\ &\quad - (\partial_\nu - ig t^b A_\nu^b)(\partial_\mu - ig t^a A_\mu^a) \Psi(x) \\ &= [-igt^b \partial_\mu A_\nu^b - g^2 t^a t^b A_\mu^a A_\nu^b + ig t^a \partial_\nu A_\mu^a + g^2 t^b t^a A_\nu^b A_\mu^a] \Psi(x) \\ &= [-ig(\partial_\mu A_\nu^a t^a - \partial_\nu A_\mu^a t^a - ig[t^a, t^b] A_\mu^a A_\nu^b)] \Psi(x). \end{aligned} \quad (9.36)$$

The last line can be simplified a little further using a bit of group theory. Namely, since the Lie algebra formed by the  $t^a$  is closed under the commutation operation, we can always write the commutator of two generators as a linear combination of generators:

$$[t^b, t^c] = it^a f^{abc}. \quad (9.37)$$

The  $f^{abc}$  are called *structure constants* and depend on the specific Lie algebra we are considering. (In fact, the structure constants are what determines the structure of the Lie algebra.) We thus have

$$[D_\mu, D_\nu] = -ig F_{\mu\nu}, \quad (9.38)$$

with

$$F_{\mu\nu} \equiv F_{\mu\nu}^a t^a \equiv [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c] t^a \quad (9.39)$$

the field strength tensor of the non-Abelian gauge theory. A gauge and Lorentz invariant Lagrangian term containing only the gauge fields and their derivatives, but no fermions, can now be constructed in the following way

$$\mathcal{L}_{\text{gauge}} \equiv \frac{c}{g^2} \text{tr}([D_\mu, D_\nu])^2 = -c \text{tr}(F_{\mu\nu} F^{\mu\nu}) = -c F_{\mu\nu}^a F^{\mu\nu,b} \text{tr}(t^a t^b). \quad (9.40)$$

A theory of gauge fields based on this Lagrangian is called *Yang–Mills theory*. The normalization constant  $c$  needs to be chosen once again such that correctly normalized gauge boson propagators are obtained. We will prove in section 9.3 below that one can always write  $\text{tr}(t^a t^b) = C \delta^{ab}$  with constant  $C$ . It turns out that the correctly normalized gauge kinetic term is

$$\mathcal{L}_{\text{gauge}} \equiv -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}. \quad (9.41)$$

We will discuss the quantization of non-Abelian gauge fields and the derivation of their propagators below in section 9.4.

### 9.3 Lie Algebras and Lie Groups

To work efficiently with non-Abelian gauge theories, we must learn some of the basic mathematical properties of the underlying symmetry groups, which here are always Lie groups. As mentioned above, an element  $U$  of a Lie group  $\mathfrak{G}$  can always be written as

$$U = \exp[i\alpha^a T^a]. \quad (9.42)$$

Here, the exponent is an element of the Lie algebra  $\mathfrak{g}$  corresponding to  $\mathfrak{G}$ . The generators  $T^a$  form a basis of  $\mathfrak{g}$ , and the  $\alpha^a$  are real parameters. For a gauge transformation,  $\alpha^a$  moreover depends on the spacetime coordinate  $x$ . The Lie algebra is characterized by the behavior of the generators under commutation:

$$[T^b, T^c] = iT^a f^{abc}. \quad (9.43)$$

#### Group Representations

For any given Lie algebra (characterized by the structure constants  $f^{abc}$ ), there are many sets  $(t_r^a)$  of matrices that satisfy eq. (9.43). The vector space spanned by each such set of matrices is called a *representations* of  $\mathfrak{g}$ . For example, the Lie algebra  $su(2)$  of the group  $SU(2)$  is represented in two dimensions by the Pauli matrices, divided by 2:  $t_2^a = \sigma^a/2$ . We know that the Pauli matrices satisfy

$$\left[ \frac{\sigma^a}{2}, \frac{\sigma^b}{2} \right] = i\epsilon^{abc} \frac{\sigma^c}{2}, \quad (9.44)$$

so the structure constants of  $su(2)$  are given by the totally antisymmetric tensor in two dimensions,  $\epsilon^{abc}$ . A different representation of  $su(2)$  is given for instance in three dimensions by the generators

$$t_3^1 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ & & 0 \end{pmatrix}, \quad t_3^2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t_3^3 = i \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}. \quad (9.45)$$

Note that in the previous section we had been fairly sloppy about the distinction between the elements of the Lie group and the Lie algebra and their representations, but had effectively always worked in a particular representation. To a mathematician, however, the group elements (denoted here with a capital  $T^a$ ) are abstract entities satisfying certain axioms, while the representation matrices in a  $d$ -dimensional representation (denoted here as  $t_r^a$ , where the subscript  $r$  identifies a particular representation) are concrete elements of  $\mathbb{C}^{d \times d}$ .

If, in a given representation  $r$ , all generators (basis vectors)  $t_r^a$  of the Lie algebra can be brought into the same block-diagonal shape simultaneously by a similarity transformation, the representation is called *reducible*. Correspondingly, a representation where this is not possible is called *irreducible*. For the following discussion, we focus on irreducible representations since they are the fundamental building blocks of representation theory.

As for any vector space, the choice of basis of a representation  $r$  is not unique. We can use this freedom to choose the  $t_r^a$  in such a way that they satisfy certain normalization conventions. In choosing these, we consider the matrix

$$D^{ab} \equiv \text{tr } t_r^a t_r^b. \quad (9.46)$$

$D^{ab}$  is real symmetric. In fact, using that the  $t_r^a$  are Hermitian and that the trace is cyclic, it is easy to see that

$$(D^{ab})^* = \text{tr } (t_r^a)^* (t_r^b)^* = \text{tr } (t_r^a)^T (t_r^b)^T = \text{tr } (t_r^b t_r^a)^T = \text{tr } t_r^b t_r^a = \text{tr } t_r^a t_r^b = D^{ab} \quad (9.47)$$

and

$$D^{ba} = \text{tr } t_r^b t_r^a = \text{tr } t_r^a t_r^b = D^{ab}. \quad (9.48)$$

Therefore,  $D^{ab}$  is diagonalizable, with all eigenvalues real. Diagonalization is achieved by an orthogonal transformation  $t_r^a \rightarrow R^{ab} t_r^b$ , which transforms  $D^{ab}$  according to

$$\begin{aligned} D^{ab} &\rightarrow \text{tr} (R^{ab} t_r^b R^{cd} t_r^d) = R^{ab} \text{tr} (t_r^b t_r^d) (R^T)^{dc} \\ &\equiv R^{ab} D_{\text{diag}}^{bd} (R^{-1})^{dc}, \end{aligned} \quad (9.49)$$

where  $D_{\text{diag}}^{bd}$  is diagonal. In the following, we assume that such a transformation has been applied, and we will drop the subscript “diag”. Next, we show that  $D$  is positive definite. In fact, if any of its diagonal entries was  $\leq 0$ , we would have  $\text{tr } t_r^a t_r^a \leq 0$  for at least one of the  $t^a$ . This is a contradiction since the eigenvalues of  $(t_r^a)^2$  must be  $\geq 0$ ,

and at least one of them has to be  $> 0$  (otherwise,  $t_r^a = 0$ ). By a suitable rescaling of the  $t_r^a$  we can now render  $D^{ab}$  proportional to the identity matrix:

$$D^{ab} = \text{tr } t_t^a t_r^b = C(r) \delta^{ab}. \quad (9.50)$$

The proportionality factor  $C(r)$  is a characteristic constant for each representation. Note that the transformations required to bring  $D^{ab}$  to this simple form, namely forming linear combinations of generators and rescaling them, change the structure constants in a well-defined way. In the following, we always assume that the basis of generators is chosen such that eq. (9.50) holds. One can show that, once this is done for one representation, it holds in all of them.

The normalization eq. (9.50) for the Lie algebra generators allows us to prove that the structure constants  $f^{abc}$  are totally antisymmetric. In fact, we can write the  $f^{abc}$  in terms of the generators as

$$\text{tr}([t_r^a, t_r^b] t_r^c) = i f^{abd} \text{tr } t_r^d t_r^c = i C(r) f^{abc}. \quad (9.51)$$

From this equation, one can show the antisymmetry of  $f^{abc}$  by using the antisymmetry of the commutator and the cyclic property of the trace.

### 9.3.1 Special Representations

For any given representation  $r$ , we can define the corresponding *conjugate representation*  $\bar{r}$  according to

$$t_{\bar{r}}^a \equiv -(t_r^a)^* = -(t_r^a)^T. \quad (9.52)$$

With this definition, the group elements generated by the two representations are the Hermitian conjugates of each other:

$$\exp[i\alpha^a t_r^a] \leftrightarrow \exp[i\alpha^a t_{\bar{r}}^a] = \exp[-i\alpha^a (t_r^a)^*]. \quad (9.53)$$

Note that  $r$  and  $\bar{r}$  may be equivalent if there is a unitary transformation  $V$  such that

$$t_{\bar{r}}^a = V t_r^a V^\dagger \quad \text{for all } a. \quad (9.54)$$

Such a representation is called *real*. For example, in  $SU(2)$ , the two-dimensional representation  $t_2^a = \sigma^a/2$  is real because

$$t_2^a = -\frac{\sigma^{a*}}{2} = (i\sigma^2) \frac{\sigma^a}{2} (i\sigma^2)^\dagger. \quad (9.55)$$

In  $SU(N)$ , the  $N$ -dimensional (lowest dimensional) representation is called *fundamental representation*.

Another special representation is the *adjoint representation*  $G$ , which is defined by the structure constants  $f^{abc}$  according to

$$(t_G^b)^{ac} \equiv i f^{abc}. \quad (9.56)$$



It is not immediately obvious that these matrices form a representation. To prove this, we start from the *Jacobi identity*

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0, \quad (9.57)$$

which follows from the definition of the commutator. (In maths text books, the Jacobi identity is one of the axioms that are part of the definition of a Lie algebra.) From the Jacobi identity and the definition of the structure constants,  $[T_r^a, T_r^b] = if^{abc}T_r^c$ , it follows that

$$\begin{aligned} [T^a, if^{bcd}T^d] + [T^b, if^{cad}T^d] + [T^c, if^{abd}T^d] &= 0 \\ \Leftrightarrow if^{bcd}if^{ade} + if^{bde}if^{cad} + if^{cde}if^{abd} &= 0 \\ \Leftrightarrow if^{bcd}if^{dea} - if^{bed}if^{dca} + if^{cde}if^{bda} &= 0 \\ \Leftrightarrow (t_G^c t_G^e)_{ba} - (t_G^e t_G^c)_{ba} &= -if^{cde}(t_G^d)_{ba}. \end{aligned} \quad (9.58)$$

For  $SU(N)$ , the dimensionality of the adjoint representation is  $N^2 - 1$ , equal to the number of generators.

### 9.3.2 The Casimir Operator

Consider next the matrix

$$(t_r)^2 \equiv \sum_a t_r^a t_r^a. \quad (9.59)$$

It is familiar from the theory of spin in quantum mechanics, described by  $SU(2)$ . There,  $(t_r)^2$  is the square of the total spin,  $J^2$ . In general, note that  $(t_r)^2$  commutes with all generators of the algebra:

$$\begin{aligned} [t_r^b, t_r^a t_r^a] &= [t_r^b, t_r^a] t_r^a + t_r^a [t_r^b, t_r^a] \\ &= if^{bac} t_r^c t_r^a + if^{bac} t_r^a t_r^c \\ &= if^{bac} t_r^c t_r^a - if^{bac} t_r^c t_r^a \\ &= 0. \end{aligned} \quad (9.60)$$

This implies that  $(t_r)^2$  can be represented by

$$(t_r)^2 \equiv C_2(r) \mathbb{1}, \quad (9.61)$$

with a representation-dependent constant  $C_2(r)$  called the *quadratic Casimir operator*.

There is a relation between  $C_2(r)$  and the constant  $C(r)$  introduced in eq. (9.50). In particular, using  $\text{tr } t_r^a t_r^b = C(r) \delta^{ab}$ , we have

$$\sum_a \text{tr}(t_r^a t_r^a) = d(G) C(r), \quad (9.62)$$

where  $d(G)$  denotes the dimensionality of the adjoint representation  $G$ , i.e. the number of generators. On the other hand, we also have

$$\sum_a \text{tr}(t_r^a t_r^a) = C_2(r) \text{tr} \mathbb{1} = d(r) C_2(r), \quad (9.63)$$

with the dimensionality  $d(r)$  of representation  $r$ . Together, eqs. (9.62) and (9.63) imply

$$d(G) C(r) = d(r) C_2(r). \quad (9.64)$$

### 9.3.3 Product Representations

The *direct product* of two group representations  $r_1$  and  $r_2$ , written as  $r_1 \times r_2$  is a representation of dimension  $d(r_1) \cdot d(r_2)$ . A vector transforming in the product representation can be written as a rank 2 tensor  $\Xi_{pq}$ , i.e. a tensor with two indices, the first of which transforms according to  $r_1$ , the second according to  $r_2$ . The generators of  $r_1 \times r_2$  are rank-4 tensors of the form

$$t_{r_1 \times r_2}^a = t_{r_1}^a \otimes \mathbb{1} + \mathbb{1} \otimes t_{r_2}^a, \quad (9.65)$$

where the first matrix in the outer product  $\cdot \otimes \cdot$  acts on the index  $p$  of  $\Xi_{pq}$ , and the second acts on the index  $q$ . Note that  $\mathbb{1}$  in the first product is meant to be the identity matrix in  $d(r_2)$  dimensions,  $\mathbb{1}$  in the second product the identity matrix in  $d(r_1)$  dimensions. If we want to avoid higher-rank tensors and write  $\Xi$  as an ordinary vector, we can choose a set of basis tensors  $\{\hat{e}_{pq}\}$  and decompose  $\Xi$  according to  $\Xi_{pq} = \sum_j c_j \hat{e}_{pq}^j$ . The  $c_j$  then form a  $d(r_1) \cdot d(r_2)$ -dimensional vector. The  $t_{r_1 \times r_2}^a$  can be turned into matrices of dimension  $d(r_1)d(r_2) \times d(r_1) \cdot d(r_2)$  in the same way.

Like every representation of a Lie group, also  $r_1 \times r_2$  can be decomposed into irreducible representations. We write formally

$$r_1 \times r_2 = \sum_i r_i. \quad (9.66)$$

This means that, by a suitable similarity transformation, all generators of  $r_1 \times r_2$  can be brought to the same block-diagonal shape, with each block corresponding to the generators of one of the  $r_i$ .

The quadratic Casimir operator in the product representation is

$$\sum_a t_{r_1 \times r_2}^a t_{r_1 \times r_2}^a = \sum_a \left[ (t_{r_1}^a)^2 \otimes \mathbb{1} + 2 t_{r_1}^a \otimes t_{r_2}^a + \mathbb{1} \otimes (t_{r_2}^a)^2 \right]. \quad (9.67)$$

Once again, the dimensions of the unit matrices are implicit. Taking the trace on both sides, and using that  $t_{r_1}^a$  and  $t_{r_2}^a$  are traceless, leads to

$$\sum_a \text{tr}(t_{r_1 \times r_2}^a t_{r_1 \times r_2}^a) = (C_2(r_1) + C_2(r_2)) d(r_1) d(r_2). \quad (9.68)$$

Using eq. (9.66), we can also write

$$\sum_a \text{tr}(t_{r_1 \times r_2}^a t_{r_1 \times r_2}^a) = \sum_i C_2(r_i) d(r_i). \quad (9.69)$$

Therefore,

$$(C_2(r_1) + C_2(r_2)) d(r_1) d(r_2) = \sum_i C_2(r_i) d(r_i). \quad (9.70)$$

### 9.3.4 The $SU(N)$ Groups

For the most important gauge symmetry groups in particle physics—the  $SU(N)$  groups—we will now explicitly give the characteristic constants  $C(r)$  and  $C_2(r)$  for the fundamental representation  $N$  and the adjoint representation  $G$ .

First, note that eq. (9.50) leaves us the freedom to choose  $C(r)$  for one particular representation by rescaling the generators. This comes at the expense of changing the structure constants, and since the structure constants must be the same in each representation, the  $C(r)$  of all other representations are then fixed. Let us choose in particular for the fundamental representation

$$C(N) = \frac{1}{2}. \quad (9.71)$$

This is motivated by the fundamental representation of  $SU(2)$ , which is given by the Pauli matrices, for which  $\text{tr } t_2^a t_2^b = \frac{1}{4} \text{tr } \sigma^a \sigma^b = \frac{1}{8} \text{tr} \{ \sigma^a, \sigma^b \} = \frac{1}{2} \delta^{ab}$ . From eqs. (9.27) and (9.64), we then obtain also  $C_2(N)$ :

$$C(N) = \frac{1}{2}, \quad C_2(N) = \frac{N^2 - 1}{2N}. \quad (9.72)$$

To compute  $C(G)$  and  $C_2(G)$  for the adjoint representation of  $SU(N)$ , we write that representation as the *direct product* of the fundamental representation and its conjugate. Consider specifically the product representation  $N \times \bar{N}$ . We now show that  $N \times \bar{N} = \mathbf{1} + G$ , where  $\mathbf{1}$  denotes the trivial (singlet) representation of  $SU(N)$  (all generators zero). First, we show that  $N \times \bar{N}$  must contain the singlet representation. Indeed, consider the  $N \times N$  unit tensor  $\delta_{pq}$ . The action of an infinitesimal  $SU(N)$  transformation, written in the  $N \times \bar{N}$  representation, on  $\delta_{pq}$  is

$$(1 + i\alpha^a t_N^a)_{pr} (1 + i\alpha^a t_{\bar{N}}^a)_{qs} \delta_{rs} \simeq \delta_{pq} + i\alpha^a [(t_N^a)_{pr} \delta_{qs} - (t_{\bar{N}}^a)_{sq} \delta_{pr}] \delta_{rs} = \delta_{pq}. \quad (9.73)$$

In other words, the space of  $N \times N$  tensors  $\Xi_{pq}$  has a subspace, spanned by  $\delta_{pq}$ , that is closed under the action of the  $N \times \bar{N}$  representation of  $SU(N)$ . This subspace corresponds to the singlet representation. Next consider the transformation properties of  $t_{N,pq}^b$ , where  $t_N^b$  are the generators of  $SU(N)$  in the fundamental representation  $N$ :

$$(1 + i\alpha^a t_N^a)_{pr} (1 + i\alpha^a t_{\bar{N}}^a)_{qs} t_{N,rs}^b \simeq t_{N,pq}^b + i\alpha^a [t_{N,pr}^a t_{N,rq}^b - t_{N,sq}^a t_{N,ps}^b]$$

$$= t_{N,pq}^b + i\alpha^a i f^{abc} t_{N,pq}^c. \quad (9.74)$$

This shows that also the subspace spanned by the  $t_N^b$  is closed under  $SU(N)$  transformations, and that transformations in this subspace are effected by the adjoint representation of  $SU(N)$ . Therefore, we can write

$$N \times \bar{N} = \mathbf{1} + G. \quad (9.75)$$

Since the dimensionality of the product representation on the left hand side is  $N^2$ , which equals  $d(1) + d(G)$ , there is no room for additional terms on the right hand side. From eq. (9.70), we then find

$$\begin{aligned} [C_2(N) + C_2(\bar{N})][d(N)]^2 &= 0 + C_2(G)d(G), \\ \Leftrightarrow 2 \frac{(N^2 - 1)}{2N} N^2 &= C_2(G) \cdot (N^2 - 1). \end{aligned} \quad (9.76)$$

Together with eq. (9.64), this leads to

$$C(G) = N, \quad C_2(G) = N. \quad (9.77)$$

## 9.4 Quantization of Non-Abelian Gauge Fields

We have already seen in our derivation of the photon propagator in section 6.6 that the quantization of gauge is non-trivial. In particular, field configurations that differ only by a gauge transformation should be counted only once in the path integral instead of being integrated over. In QED, this was achieved by removing longitudinal photon modes from the path integrals. In non-Abelian theories, a more complicated approach—the *Faddeev-Popov procedure*—is necessary because the manifold of gauge-equivalent field configurations is more complicated.

Let us consider pure Yang–Mills theory with the Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}, \quad (9.78)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \quad (9.79)$$

(see eqs. (9.39) and (9.41)). The partition function for this theory in the presence of an external source  $J_\mu^a$  is

$$Z_0[J] = \int \mathcal{D}A \exp \left[ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + J^{\mu,a} A_\mu^a \right) \right]. \quad (9.80)$$

For an Abelian gauge field, removing the gauge degrees of freedom was achieved by simply restricting the path integral to the components of  $\tilde{A}(p)$  (the Fourier transformation of

$A(x)$ ) orthogonal to  $k^\mu$ . This could be seen by noting that Abelian gauge transformations act on the gauge field as  $\tilde{A}_\mu(k) \rightarrow \tilde{A}_\mu(k) + \frac{i}{e} k_\mu \tilde{\alpha}(k)$ . Non-Abelian gauge fields, however, transform in a non-linear way, see eq. (9.34). To remove the gauge degrees of freedom, Fadeev and Popov introduced a gauge-fixing  $\delta$  function in the path integral.

To understand how this works, let us first consider as a toy example an ordinary integral of the form

$$Z = \int dz dy e^{iS(z)}, \quad (9.81)$$

with some function  $S(z)$  that does not depend on  $y$ .  $S(z)$  is the analog to the action in eq. (9.80),  $z$  is a physical degree of freedom, and  $y$  is an unphysical (gauge) degree of freedom. To remove the gauge degree of freedom  $y$ , we could of course simply drop the  $y$  integral, but we could equivalently introduce a  $\delta$  function:

$$Z = \int dz dy \delta(y - f(z)) e^{iS(z)}. \quad (9.82)$$

Here,  $f(z)$  can be an arbitrary function of  $z$ , and the condition  $y = f(z)$  is a gauge fixing condition. This condition could also be given in implicit form  $G(z, y) = 0$ , where in this particular case  $G(z, y) \equiv y - f(z)$ . While this seems trivial in the simple example given here, some gauge fixing conditions are so complicated that they can only be given in implicit form. Written in terms of  $G(z, y)$ , the  $\delta$  function becomes

$$\delta(y - f(z)) = \delta(G(z, y)) \left| \frac{\partial G}{\partial y} \right|, \quad (9.83)$$

and the integral eq. (9.82) turns into

$$Z = \int dz dy \delta(G(z, y)) \left| \frac{\partial G}{\partial y} \right| e^{iS(z)}. \quad (9.84)$$

Promoting  $y$  and  $z$  to vectors of length  $n$ ,  $Z$  generalizes to

$$Z = \int dz^n dy^n \left[ \prod_i \delta(G_i(z, y)) \right] \left| \det \left( \frac{\partial G_i}{\partial y_j} \right) \right| e^{iS(z)}. \quad (9.85)$$

Note that now,  $n$  gauge fixing conditions of the form  $G_i(z, y) = 0$  are required to fix all  $n$  components of  $y$ . We are now ready to implement gauge fixing in the path integral eq. (9.80). The components of the vector  $z$  are now the physical components of  $A_\mu^a(x)$  at each spacetime point, and the components of  $y$  are the unphysical components of  $A_\mu^a(x)$  that can be removed by gauge transformations. We will call these components  $\theta^a(x)$  in the following. (We will not concern ourselves with the fact that the lengths of  $y$  and  $z$  then become uncountably infinite.) We take the gauge fixing function to have the form

$$G^a(x) \equiv \partial^\mu A_\mu^a(x) - \omega^a(x), \quad (9.86)$$

with an arbitrary smooth function  $\omega^a(x)$ . This is a generalization of the Lorenz gauge condition  $\partial^\mu A_\mu(x) = 0$ .

To evaluate the gauge fixed path integral

$$Z_0[J] = \int \mathcal{D}A \left[ \prod_{x,a} \delta(G^a(x)) \right] \left| \det \left( \frac{\delta G^a(x)}{\delta \theta^b(x')} \right) \right| \times \exp \left[ i \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} + J^{\mu,a} A_\mu^a \right) \right], \quad (9.87)$$

we need to evaluate the functional derivative  $\det(\delta G/\delta \theta)$ . To do so, we need to investigate how  $G^a(x)$  behaves under infinitesimal variations of  $\theta^a(x')$ , or, in other words, how  $G^a(x)$  behaves under an infinitesimal gauge transformation

$$U(x) = 1 + ig\theta^a(x)t^a. \quad (9.88)$$

We find

$$\begin{aligned} G^a t^a &\rightarrow G^a t^a + \partial^\mu [U A_\mu^a t^a U^\dagger + \frac{i}{g} U (\partial_\mu U^\dagger) - A_\mu^a t^a] \\ &= G^a t^a + \partial^\mu [ig\theta^a t^a A_\mu^b t^b - ig A_\mu^b t^b \theta^a t^a + \frac{i}{g} (-ig) \partial_\mu \theta^a t^a] \\ &= G^a t^a + \partial^\mu [ig\theta^a A_\mu^b i f^{abc} t^c + \partial_\mu \theta^a t^a] \\ &= G^a t^a + \partial^\mu [-ig\theta^a A_\mu^b (t_g^b)^{ca} t^c + \partial_\mu \theta^a t^a] \\ &= G^a t^a + \partial^\mu D_{\mu,G}^{ac} \theta^c t^a. \end{aligned} \quad (9.89)$$

In the last line, we have used the covariant derivative in the adjoint representation,  $D_{\mu,G} = \partial_\mu - ig t_G^b A_\mu^b = \partial_\mu \delta^{ac} + g f^{abc} A_\mu^b$ . It follows that

$$\frac{\delta G^a(x)}{\delta \theta^c(x')} = \partial^\mu D_{\mu,G}^{ac} \delta^{(4)}(x - x'). \quad (9.90)$$

The determinant of this infinite-dimensional matrix can be computed by using a result from section 6.7. There, we had shown in eq. (6.78) that the determinant of a complex  $n \times n$  matrix  $B$  can be written in terms of an integral over Grassmann variables  $\theta_j, \theta_j^*$  according to

$$\det B = \int d\theta_1^* d\theta_1 \cdots d\theta_n^* d\theta_n e^{-\theta_i^* B_{ij} \theta_j}. \quad (9.91)$$

Sending  $n$  to infinity, and identifying the components of the Grassmann vectors  $\theta, \theta^*$  with the values of a complex scalar fields  $c$  at different spacetime point, this integral turns into a path integral. With this in mind, we can now write

$$\det \left( \frac{\delta G^a(x)}{\delta \theta^c(x')} \right) = \int \mathcal{D}c \mathcal{D}\bar{c} \exp \left[ i \int d^4x \mathcal{L}_{\text{gh}}(c, \bar{c}) \right], \quad (9.92)$$

with

$$\mathcal{L}_{\text{gh}} \equiv - \int d^4x' \bar{c}^a(x) \partial^\mu D_{\mu,G}^{ab}(x) \delta^{(4)}(x-x') c^b(x'). \quad (9.93)$$

The new auxiliary field  $c^a$  transforming in the adjoint representation of the gauge group is called *ghost field*. Its Lagrangian can still be simplified a little further by using the delta function to remove the integral over  $x'$ :

$$\begin{aligned} \mathcal{L}_{\text{ghost}} &= -\bar{c}^a(x) \partial^\mu D_{\mu,G}^{ab}(x) c^b(x) \\ &= (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b. \end{aligned} \quad (9.94)$$

We see that the ghost field indeed behaves almost like a regular quantum field, with a Lagrangian consisting of a kinetic term and a gauge interaction vertex. We said *almost* because the kinetic term has a negative sign. This signals the fact that Feynman diagrams involving  $c$  cancel contributions from other diagrams. In particular, they cancel precisely the contributions of the unphysical degrees of freedom of internal gauge bosons.

We are not quite done yet with rewriting the gauge fixed partition function eq. (9.87)—we still need to deal with the  $\delta$  functions  $\delta(G^a(x))$ . To rewrite it, we use that the gauge fixing function  $\omega^a(x)$  appearing in  $G^a(x)$  is completely arbitrary. Without loss of generality, we can therefore carry out a functional integral (path integral) over all possible functions, with arbitrary weighting factors. We choose in particular Gaussian weighting factors. We thus replace, in eq. (9.87),

$$\begin{aligned} \left[ \prod_{x,a} \delta(G^a(x)) \right] &\rightarrow \int \mathcal{D}\omega \exp \left[ -\frac{i}{2\xi} \int d^4x (\omega^a)^2 \right] \left[ \prod_{x,a} \delta(G^a(x)) \right] \\ &= \exp \left[ -\frac{i}{2\xi} \int d^4x (\partial^\mu A_\mu^a)^2 \right]. \end{aligned} \quad (9.95)$$

Here,  $\xi$  is an arbitrary constant, and necessary overall normalization factors are not explicitly written out. (We assume that, later, they are absorbed into the path integral measures  $\mathcal{D}A$ ,  $\mathcal{D}c$  and  $\mathcal{D}\bar{c}$ .)

Putting all the pieces together, the partition function of Yang–Mills theory is now

$$Z_0[J] = \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \exp \left[ i \int d^4x \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{ghost}} + \mathcal{L}_{\text{gauge-fix}} + J^{\mu,a} A_\mu^a \right], \quad (9.96)$$

with

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a}, \quad (9.97)$$

$$\mathcal{L}_{\text{ghost}} = (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b, \quad (9.98)$$

$$\mathcal{L}_{\text{gauge-fix}} = -\frac{1}{2\xi} (\partial^\mu A_\mu^a)(\partial^\nu A_\nu^a). \quad (9.99)$$

The last term implements the gauge fixing conditions.

## 9.5 The Feynman Rules for Non-Abelian Gauge Theories

To determine the Feynman rules for Yang–Mills theory, we expand the Lagrangian, starting with the gauge kinetic term from eq. (9.97):

$$\begin{aligned}
 \mathcal{L}_{\text{YM}} &= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,a} \\
 &= -\frac{1}{4} \left[ \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c \right] \left[ \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} + g f^{ade} A^{\mu,d} A^{\nu,e} \right] \\
 &= -\frac{1}{2} (\partial_\mu A_\nu^a) (\partial^\mu A^{\nu,a}) + \frac{1}{2} (\partial_\mu A_\nu^a) (\partial^\nu A^{\mu,a}) \\
 &\quad - g f^{abc} A_\mu^b A_\nu^c \partial^\mu A^{\nu,a} - \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu,d} A^{\nu,e}. \quad (9.100)
 \end{aligned}$$

### 9.5.1 The Gauge Boson Propagator

Together with  $\mathcal{L}_{\text{gauge-fix}} = -\frac{i}{2\xi} (\partial^\mu A_\mu^a) (\partial^\nu A_\nu^a)$ , the terms quadratic in  $A_\mu^a$  become (after integration by parts)

$$\mathcal{L} \supset \frac{1}{2} A_\mu^a \left[ g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \left( 1 - \frac{1}{\xi} \right) \right] A_\nu^a. \quad (9.101)$$

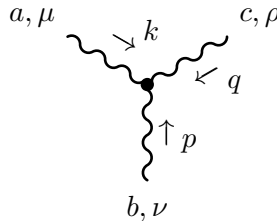
The gauge boson propagator is as usual the inverse of the operator in square brackets (see e.g. sections 6.3, 6.6 and 6.7). Explicitly, the propagator is given by<sup>1</sup>

$$D_{\mu\nu}^{ab}(x-y) = \int \frac{d^4 k}{(2\pi)^4} \frac{-i\delta^{ab}}{k^2 + i\epsilon} \left[ g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right] e^{-ik(x-y)}. \quad (9.103)$$

Since  $\xi$  is an unphysical gauge fixing parameter, observables cannot depend on its value. A particularly convenient choice is obviously  $\xi = 1$ , the *Feynman-'t Hooft gauge*.

### 9.5.2 Gauge Boson Self-Interactions

The terms in the second and third line of eq. (9.100) lead to couplings of three and four gauge bosons—a feature that was not present in Abelian gauge theories. The corresponding Feynman rules are

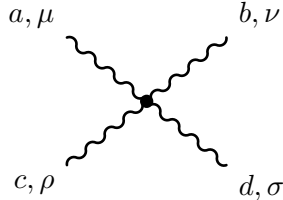


$$= -ig f^{abc} \left[ -ik^\nu g^{\mu\rho} + \begin{array}{c} \text{5 permutations} \\ \text{of } (a, \mu, k), (b, \nu, p), (c, \rho, q) \end{array} \right] \quad (9.104)$$

<sup>1</sup>It is easy to check explicitly that

$$\frac{1}{k^2} \left[ g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right] \cdot k^2 \left[ g_{\nu\rho} - \frac{k_\nu k_\rho}{k^2} \left( 1 - \frac{1}{\xi} \right) \right] = \delta^\mu{}_\rho \quad (9.102)$$





$$\begin{aligned}
 &= -ig^2 \left[ f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \right. \\
 &\quad + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad \left. + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right] \quad (9.105)
 \end{aligned}$$

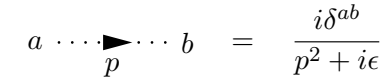
In reading these rules off the Lagrangian, we need to be careful to account for all the different ways of permuting the photon fields. Note that all momenta in eq. (9.104) are defined as ingoing. If a momentum is outgoing, its sign needs to be flipped.

### 9.5.3 Ghost Interactions

From the ghost Lagrangian

$$\mathcal{L}_{\text{ghost}} = (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - gf^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b, \quad (9.106)$$

(see eq. (9.98)) we can directly read off the following Feynman rules for ghost fields:



$$a \cdots \overset{p}{\longrightarrow} \cdots b = \frac{i\delta^{ab}}{p^2 + i\epsilon} \quad (9.107)$$



$$= gf^{abc} p^\mu \quad (9.108)$$

The arrows on the ghost lines are used to keep track of momentum flow, which is relevant for the sign of  $p^\mu$  at the vertex. Note that ghost cannot appear as external lines in Feynman diagrams because they are not physical fields, but rather a mathematical tool to cancel unwanted degrees of freedom of the internal gauge bosons. For external gauge bosons, appropriately chosen polarization vectors ensure that only physical degrees of freedom appear. If we did allow for external ghosts and unphysical gauge boson degrees of freedom to appear in Feynman amplitudes, we would find that ghosts have a negative normalization and their contribution to the cross section cancels exactly the contribution of unphysical gauge boson degrees of freedom.

### 9.5.4 Coupling to Fermions

While pure Yang–Mills theory, defined by eq. (9.97) is of great theoretical interest, a viable description of nature requires the inclusion of fermions. A fermion  $n$ -plet  $\psi$ , transforming in some  $n$ -dimensional representation  $r$  of the gauge group, is described by the Lagrangian

$$\mathcal{L} \supset \bar{\psi}(i\not{D} - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi + g\bar{\psi}\gamma^\mu t_r^a \psi A_\mu^a. \quad (9.109)$$

This leads to the gauge interaction vertex

$$a, \mu \text{ --- wavy line --- } \bullet \begin{array}{l} \nearrow \text{arrow} \\ \searrow \text{arrow} \end{array} = ig\gamma^\mu t_r^a. \quad (9.110)$$

In the specific case that the gauge group is  $SU(3)$  and  $r$  is the fundamental (triplet) representation, this vertex describes the QCD interaction of quarks. The three components of  $\psi$  then correspond to red, green and blue quarks. The gauge fields of QCD are called *gluons*. We will discuss QCD below in detail in section 9.7.

## 9.6 The Beta Function for Non-Abelian Gauge Theories

Just like quantum electrodynamics, also non-Abelian gauge theories need to be renormalized to deal with UV divergences of loop diagrams. Just as for QED, the renormalization group evolution of the coupling constant (i.e. the  $\beta$  function) is obtained from the multiplicative renormalization constants, evaluated in the  $\overline{\text{MS}}$  scheme. The renormalized Lagrangian for a non-Abelian gauge theory reads

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}Z_A(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}) \\ & - gZ_{3A}f^{abc}A_\mu^b A_\nu^c \partial^\mu A^{\nu,a} \\ & - \frac{1}{4}Z_{4A}g^2 f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu,d} A^{\nu,e} \\ & + Z_c(\partial^\mu \bar{c}^a)(\partial_\mu c^a) - Z_{gc}gf^{abc}A_\mu^c(\partial^\mu \bar{c}^a)c^b \\ & + Z_\psi \bar{\psi} i \not{D} \psi - Z_m m \bar{\psi} \psi + Z_{g\psi} g \bar{\psi} \gamma^\mu t_r^a \psi A_\mu^a. \end{aligned} \quad (9.111)$$

Here, all fields and coupling constants are interpreted as renormalized quantities, but the subscript  $r$  we used to denote this in chapter 8 is omitted now for simplicity, and to avoid confusion with the index  $r$  we used to denote a group representation.

Note that there are eight counterterms  $Z_j - 1$  in eq. (9.111), but only 5 free parameters: the normalizations of  $\psi$ ,  $A_\mu^a$ , and  $c$ , the coupling constant  $g$ , and the mass  $m$ . Therefore, not all counterterms are independent. In fact, one can show that  $g$  gets renormalized in the same way in all the vertices where it appears (see eq. 73.2 and sec. 74 in [2]). This implies

$$\begin{aligned} Z_{3A} &= Z_A Z_{g\psi} Z_\psi^{-1}, \\ Z_{4A} &= Z_A Z_{g\psi}^2 Z_\psi^{-2}, \\ Z_{gc} &= Z_{g\psi} Z_\psi^{-1} Z_c. \end{aligned} \quad (9.112)$$

Since

$$g = \frac{Z_\psi Z_A^{1/2}}{Z_{g\psi}} g_0, \quad (9.113)$$

(with  $g_0$  the bare gauge coupling), the  $\beta$  function is given by

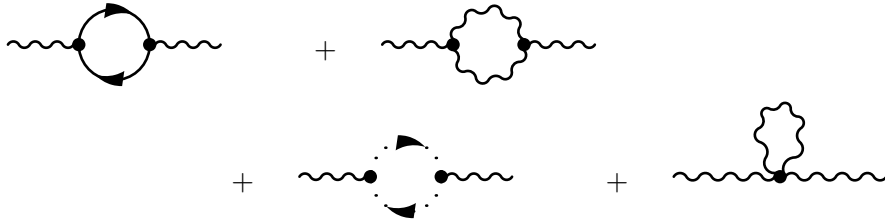
$$\begin{aligned}\beta &= \mu \frac{\partial g}{\partial \mu} \\ &= g_0 \mu \frac{\partial}{\partial \mu} \frac{Z_\psi Z_A^{1/2}}{Z_{g\psi}} \\ &\stackrel{1\text{-loop}}{\simeq} g_0 \mu \frac{\partial}{\partial \mu} \left[ Z_\psi - Z_{g\psi} + \frac{1}{2} Z_A \right].\end{aligned}\tag{9.114}$$

The approximation in the last line holds at the 1-loop level only.


We now need to evaluate the divergent pieces of the 1-loop contributions to the fermion self-energy,


(9.115)

to get  $Z_\psi$ ; the 1-loop contributions to the gauge boson self-energy,


(9.116)

to get  $Z_A$ ; and to the vertex correction


(9.117)

to get  $Z_{g\psi}$ . The result for the  $\beta$  function is

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} \sum_f C(r_f) \right].\tag{9.118}$$

Here, the sum in the second term runs over all fermion species in the theory, and  $r_f$  denotes the representation under which species  $f$  transforms. Equation (9.118) implies the following scale dependence for the coupling constant:

$$g^2(\mu) = \frac{\bar{g}^2}{1 + \frac{2\bar{g}^2}{(4\pi)^2} \left[ \frac{11}{3} C_2(G) - \frac{4}{3} \sum_f C(r_f) \right] \log \frac{\mu}{\bar{\mu}}}.\tag{9.119}$$

The crucial observation here is that  $\beta(g)$  can be *negative* if the number of fermion species and the corresponding  $C(r_f)$  terms are not too large. In this case, the renormalized coupling constant *decreases* with increasing energy. Since this means that, at  $\mu \rightarrow \infty$ , the gauge interaction becomes infinitely weak, this phenomenon is called *asymptotic freedom*.

For example, in QCD at intermediate energies of order 100 GeV, we have five species of quarks, each transforming the fundamental representation of  $SU(3)$ . (At  $i\mathcal{O}(100 \text{ GeV})$ , the up, down, strange, charm and bottom quarks can be treated as massless, and the top quark can be neglected since loops involving the top are suppressed by  $m_t^{-1}$ .) With  $C(3) = 1/2$  and  $C_2(G) = 3$  for  $SU(3)$  (see eqs. (9.72) and (9.77)), we find

$$\beta(g_s) = -\frac{g_s^3}{(4\pi)^2} \left[ 11 - \frac{10}{3} \right]. \quad (9.120)$$

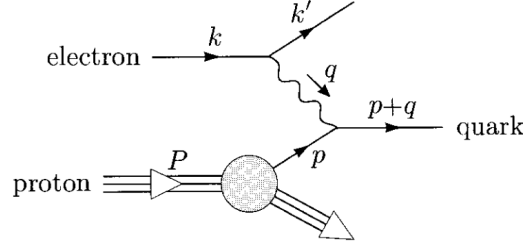
Thus, QCD is asymptotically free: at low energy, quarks interact very strongly, and this is the reason why the forces holding nucleons together are so strong. At high energies, for example at the LHC, the QCD coupling becomes perturbative ( $\alpha_s = g_s^2/(4\pi) \sim 0.1$  at 100 GeV). Expressed in coordinate space rather than momentum space, asymptotic freedom means that the potential experienced by quarks resembles a bag: at short distances, quarks move freely with respect to each other, but separating them by more than  $\sim 1 \text{ fm}$  requires huge amounts of energy. In fact, when one attempts to separate a quark and an antiquark by that much, the potential energy they carry becomes so strong that quark–antiquark pairs can be produced from the vacuum. These combine with the original quark and antiquark into two mesons (quark–antiquark bound states), which have no net color charge and therefore do not feel the strong QCD potential. This is the origin of *confinement*, the observation that strongly interacting particles exist only in color-neutral bound states.

## 9.7 Phenomenology of Quantum Chromodynamics (QCD)

We now discuss some elementary phenomenological properties of quantum chromodynamics. In order to do so, we recapitulate the Lagrangian, including for completeness also electromagnetic interactions:

$$\begin{aligned} \mathcal{L}_{\text{QCD}} = & -\frac{1}{4}G_{\mu\nu}^a G^{\mu\nu,a} - \frac{1}{2\xi}(\partial^\mu A_\mu^a)(\partial^\nu A_\nu^a) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ & + (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b \\ & + \sum_q [\bar{\psi}_q (i\not{\partial} - m_q) \psi_q + g \bar{\psi}_q \gamma^\mu t^a \psi_q A_\mu^a + e Q_q \bar{\psi}_q \gamma^\mu \psi_q A_\mu]. \end{aligned} \quad (9.121)$$

Here, we distinguish the QED and QCD field strength tensors by calling the former  $F_{\mu\nu}$  and the latter  $G_{\mu\nu}$ . Moreover, it is understood that gauge fields  $A_\mu^a$  with an  $SU(3)$  index  $a$  are gluon fields (the force carriers of QCD), while those without a group index mean



**Figure 9.1:** Deep-inelastic scattering: a lepton scatters off a prton contained in a nucleon. Figure taken from [1].

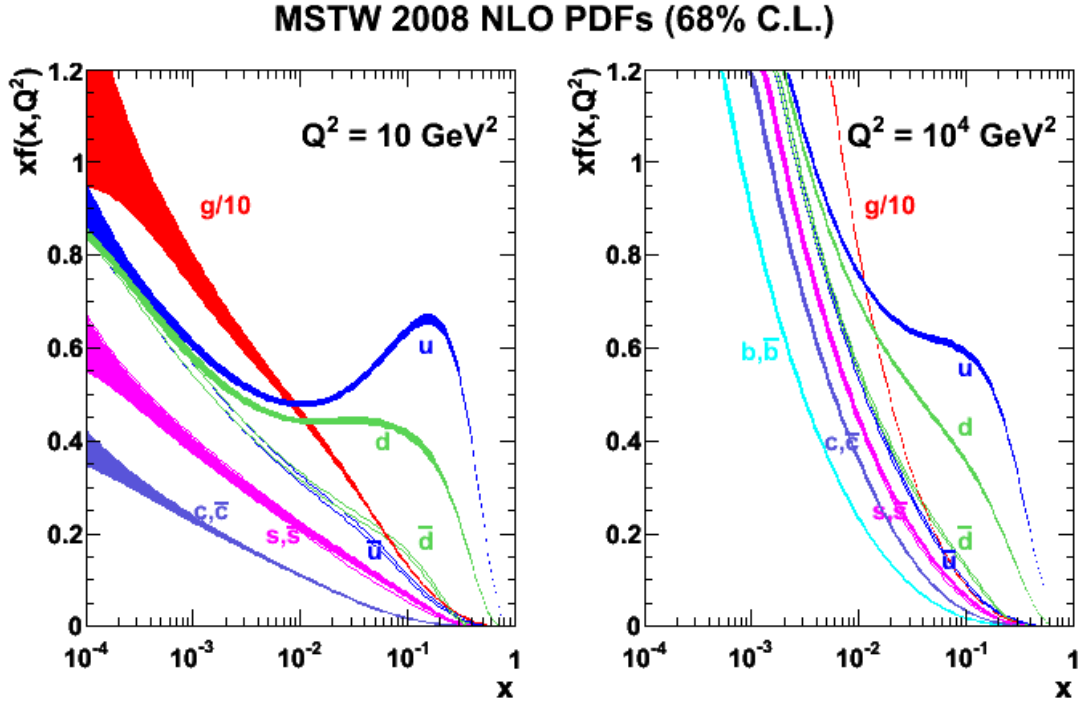
the photon field. The sum in the third line runs over all quark species (or “flavors”), and each field  $\psi_q$  is understood to be a triplet of  $SU(3)$ , with electric charge  $Q_q$ . To distinguish gluons from photons in Feynman diagrams, we use curly lines for gluons and wiggly lines for photons:

$$\begin{aligned} \text{~~~~~} &= \text{gluon,} \\ \text{~~~~~} &= \text{photon.} \end{aligned}$$

### 9.7.1 Deep-Inelastic Scattering

As a first QCD process, let us discuss *deep-inelastic scattering (DIS)*—the scattering of a lepton (electron, muon, or neutrino) off a nucleon (proton or neutron). This process is shown schematically in fig. 9.1. At the fundamental level, of course, the lepton scatters off one of the quarks inside the proton. At sufficiently high center of mass energy ( $E_{\text{cm}} \gtrsim \text{GeV}$ ), the quark is kicked out of the proton at high speed. As explained in the previous section, subsequent soft processes create  $q\bar{q}$  pairs from the vacuum, and color neutral hadrons form. These can be either  $q_r q'_g q''_b$  bound states (baryons) or  $q_c \bar{q}_{\bar{c}}$  bound states (mesons). Here, the subscripts indicate color charge (red, green, or blue for quarks, which transform in the 3 representation of  $SU(3)$ , and anti-red, anti-blue, anti-green for antiquark, which transform in the  $\bar{3}$  representation). We will in the following always assume the high energy limit ( $E_{\text{cm}} \gg \text{GeV}$ ), so that both the proton mass and the lepton mass are negligible.

Since it is impossible to describe the internal structure of the proton analytically, we need to parameterize it phenomenologically. We need to know in particular the momentum of the quark that directly interacts with the lepton. We introduce the *Bjorken scale variable*  $x$ , which denotes the ratio of the quark’s momentum to the proton’s momentum, assuming the masses of both are negligible. The distribution of  $x$  is given by the *parton*



**Figure 9.2:** Parton distribution functions at two different values of  $Q^2$  (the negative of the momentum transfer in the interaction) [4].

*distribution function (PDF):*<sup>2</sup>

$$f_q(x) dx = \begin{array}{l} \text{probability of finding a constituent of type } q \\ \text{with momentum fraction in } [x, x + dx]. \end{array} \quad (9.122)$$

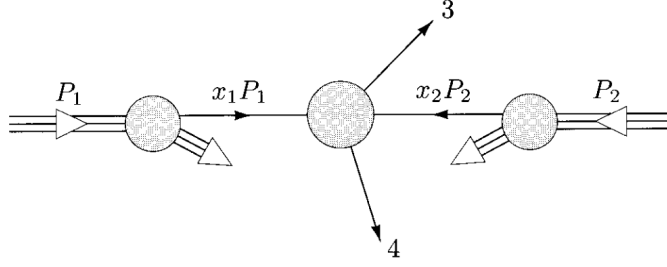
The subscript  $q$  here can be one of  $u, \bar{u}, d, \bar{d}, s, \bar{s}, c, \bar{c}, b, \bar{b}$ , or  $g$  (for the gluon). A few example for PDFs are plotted in fig. 9.2. Note that the valence quarks  $u$  and  $d$  tend to carry a larger fraction of the proton momentum than the sea quarks and the gluons. Moreover, note that the PDFs depend on the momentum transfer  $Q^2 \equiv -q^2$  in the scattering process.

The cross section of deep inelastic scattering can now be written as

$$\begin{aligned} \sigma(e^-(k) + p^+(P) \rightarrow e^-(k') + X) \\ = \int_0^1 dx \sum_q f_q(x) \sigma(e^-(k) + q(xP) \rightarrow e^-(k') + q'(p')) \end{aligned} \quad (9.123)$$

Here, the symbol  $X$  in the final state denotes any kind of hadronic debris and the sum runs over all quark flavors. (The gluons inside the nucleon do not contribute because they

<sup>2</sup>Quarks and gluon are collectively referred to as *partons*.



**Figure 9.3:** Proton–proton scattering at high energy effectively corresponds to the scattering of two partons (quarks or gluons). Figure taken from [1].

are electrically neutral.) The momentum  $p'$  of the outgoing quark is given by  $p' = xP + q$  with  $q = k' - k$ . The PDFs on the right hand side of eq. (9.123) are taken from fits to data. (In fact, DIS is used to fit the PDFs in the first place.) The parton level cross section  $\sigma(e^-(k) + q(xP) \rightarrow e^-(k') + q'(p'))$  can be computed in perturbation theory, provided the energy of the scattering is high enough for QCD to be perturbative. The result is

$$\frac{d}{d \cos \theta_{\text{cms}}} \sigma(e^- q \rightarrow e^- q') = \frac{\pi \alpha^2 Q_q^2}{\hat{t}^2} \left( \frac{\hat{s}^2 + \hat{u}^2}{4} \right), \quad (9.124)$$

where  $\theta_{\text{cms}}$  is the scattering angle in the center of mass frame, and  $\hat{s}$ ,  $\hat{t}$ ,  $\hat{u}$  are the Mandelstam variables corresponding to the partonic momenta (as indicated by the hat):

$$\begin{aligned} \hat{s} &= (p + k)^2 = 2pk = 2xPk = xs, \\ \hat{t} &= q^2 = -Q^2, \\ \hat{u} &= (k' - p)^2. \end{aligned} \quad (9.125)$$

### 9.7.2 Proton–Proton Interactions at the LHC

Processes at a hadron–hadron collider involve two partons in the initial state, as illustrated in fig. 9.3. Correspondingly, the relation between the hadron level cross section and the parton level cross section is

$$\sigma(pp \rightarrow X) = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{q_1, q_2} f_{q_1}(x_1) f_{q_2}(x_2) \sigma(q_1 + q_2 \rightarrow \hat{X}). \quad (9.126)$$

Again,  $X$  denotes one or many arbitrary hadronic final state, and  $\hat{X}$  denotes the corresponding parton level processes. At high energy, there are typically *many* partonic processes contributing on the right hand side. This is first because, to a collider detector, all partons look alike (with the exception of the top quark and to some extent the bottom quark): all of them hadronize by surrounding themselves with more partons

created from the vacuum in order to form color-neutral hadronic final states. At high energy, these hadronic final states contain a lot of mesons and baryons, all emitted into about the same direction. Such a hadron shower is called a *jet*. Consider for instance one of the simplest final states at the LHC: dijet production  $pp \rightarrow jj$ . The process can be initiated by two quarks/antiquarks, a quark/antiquark and a gluon, or by two gluons. The partonic final state can also consist of two quarks/antiquarks, a quark/antiquarks and a gluon, or two gluons.

Moreover, initial final state partons have a high probability of radiating additional partons. A quark/antiquark can radiate a gluon, a gluon can split into a quark–antiquark pair. Such extra radiation is particularly strong when the radiated parton is very soft or its direction of flight is collinear with that of the original parton. In both cases, the intermediate parton propagator is almost on-shell. The former type of enhancement (*soft divergence*) is equivalent to the infrared divergence discussed in section 8.3, which is regularized once loop diagrams are included, but still lead to a potentially large logarithmic enhancement. The second type of enhancement (*collinear divergence*) is regularized by the parton masses and will be studied further in sections 9.7.3 and 9.7.5. In practice, it is in most processes only possible to fully compute the amplitude for the emission of a single extra parton (*next-to-leading order*, *NLO*) or two extra partons (*next-to-next-to-leading order*, *NNLO*). The limitation is the high-dimensional final state phase space, which needs to be integrated over and the calculation of a large number of loop diagrams, which must be included to keep the order of the perturbative expansion consistent and to cancel soft divergences. Of course, all of these calculations are not done by hand, but in a fully automated or partly automated way using numerical methods and/or computer algebra.

### 9.7.3 Infrared divergences in QCD

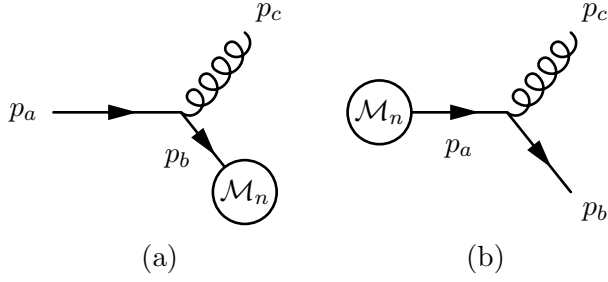
As emphasized above, the amplitude for a process in which an initial or final state parton radiates an extra, collinear, parton, is strongly enhanced. The reason is that the intermediate parton propagator in such a process is nearly on-shell.

Our goal in the following is to demonstrate that, in the case of collinear radiation, the cross section for a scattering process factorizes into a term describing the “hard process” without radiation, and a term describing the radiation. To do so, we need to consider all possible splitting processes:  $q \rightarrow qg$ ,  $g \rightarrow q\bar{q}$ ,  $g \rightarrow gg$ . We begin with the first one, which is shown in fig. 9.4. Denoting the 4-momentum of the incoming quark as  $p_a$ , and the momenta of the outgoing quark and gluon as  $p_b$ ,  $p_c$ , we define the energy fraction

$$z = \frac{E_c}{E_a} = 1 - \frac{E_b}{E_a}. \quad (9.127)$$

If the incoming gluon is in the initial state of the hard process (fig. 9.4 (a)), we can assume that  $p_b^2 \gg p_a^2, p_c^2$ . If the gluon is in the final state (fig. 9.4 (b)) of the hard process, we have  $p_a^2 \gg p_b^2, p_c^2$ .





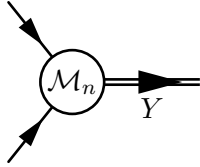
**Figure 9.4:** Splitting of (a) an initial state gluon and (b) a final state gluon.

We will focus in the following on splittings of initial state partons, and write

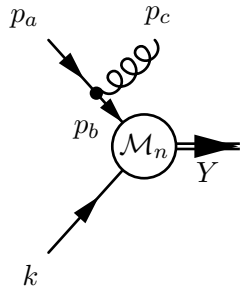
$$p_a = \begin{pmatrix} p \\ 0 \\ 0 \\ -p \end{pmatrix}, \quad p_b = \begin{pmatrix} (1-z)p \\ -p_\perp \\ 0 \\ (1-z)p + \frac{p_\perp^2}{2zp} \end{pmatrix}, \quad p_c = \begin{pmatrix} zp \\ p_\perp \\ 0 \\ zp - \frac{p_\perp^2}{2zp} \end{pmatrix}, \quad (9.128)$$

It is easy to see that, with these definitions,  $p_a^2 = 0$  and  $p_c^2 = 0$  (up to a term of order  $p_\perp^4$ , which is negligible in the collinear limit), while  $p_b^2 = p_\perp^2/z$ .

We now consider specifically a quark–quark scattering process. The leading order diagram is



while at next-to-leading order, we encounter for instance the diagram



Here,  $Y$  denotes the  $n$ -particle final state, and  $\mathcal{M}_n$  is again the corresponding matrix element. The cross section  $\sigma_{n+1}$  for the process with  $n+1$  particles in the final state can be written in terms of the matrix element  $\mathcal{M}_n$  and cross section  $\sigma_n$  for the  $n$ -particle

process:

$$\begin{aligned}
 \sigma_{n+1} &= \frac{1}{8pk^0} \int \frac{d^3p_c}{(2\pi)^3 2zp} \int d\Phi_n |\mathcal{M}_n|^2 \left( \frac{1}{p_b^2} \right)^2 |\mathcal{M}_{g \leftarrow q}|^2 \\
 &= \int \frac{pdz p_\perp dp_\perp d\phi}{(2\pi)^3 2zp} \left( \frac{1}{p_b^2} \right)^2 |\mathcal{M}_{g \leftarrow q}|^2 \frac{1-z}{8(1-z)pk^0} \int d\Phi_n |\mathcal{M}_n|^2 \\
 &= \int \frac{pdz d(p_\perp^2) d\phi}{(2\pi)^3 4p} \frac{z(1-z)}{p_\perp^4} |\mathcal{M}_{g \leftarrow q}|^2 \sigma_n.
 \end{aligned} \tag{9.129}$$

In these expressions  $\mathcal{M}_{g \leftarrow q}$  denotes the matrix element for the splitting process  $q \rightarrow qg$ . The ordering of the indices comes from the idea that  $\mathcal{M}_{i \leftarrow j}$  is a  $z$ -dependent matrix that multiplies from the left a vector containing the amplitudes of initial state partons and yields the amplitudes of the outgoing partons with momentum fraction  $z$ . Following this logic, we denote for instance the matrix element for  $q \rightarrow gq$  (which is obtained from  $\mathcal{M}_{g \leftarrow q}$  by replacing  $z \leftrightarrow 1-z$ ) as  $\mathcal{M}_{q \leftarrow g}$ . We consider first the splitting  $q \rightarrow qg$  with left-handed quarks in the initial and final states:

$$i\mathcal{M}_{g \leftarrow q} = \bar{u}_L(p_b) [ig\gamma_\mu (t^a)^{c_1 c_2}] u_L(p_a) \epsilon^{*\mu}(p_c). \tag{9.130}$$

Here,  $a$  is an index of the adjoint representation of  $SU(3)$ , and  $c_1, c_2 = 1 \dots 3$  are the color indices of the incoming and outgoing quarks. We write as usual for massless particles

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad \text{and} \quad u_L(p) = \sqrt{2p^0} (\xi(p)) \tag{9.131}$$

and obtain

$$i\mathcal{M}_{g \leftarrow q} = -\sqrt{2(1-z)p} \sqrt{2p} ig \xi^\dagger(p_b) \sigma^j \xi(p_a) \epsilon^{*j}(p_c) (t^a)^{c_1 c_2}. \tag{9.132}$$

Here, we have already used the fact that real gluons must be transversely polarized, so that only the spatial components  $\epsilon^j(p_c)$  of the polarization vector are nonzero. In fact, up to order  $p_\perp$ , the two possible gluon polarizations are

$$\epsilon_L^{*j}(p_c) \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ -\frac{p_\perp}{zp} \end{pmatrix} \quad \text{and} \quad \epsilon_R^{*j}(p_c) \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ -\frac{p_\perp}{zp} \end{pmatrix}. \tag{9.133}$$

Similarly, the fermion spinors are

$$\xi(p_a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \xi(p_b) = \begin{pmatrix} \frac{p_\perp}{2(1-z)p} \\ 1 \end{pmatrix}. \tag{9.134}$$

To derive these expressions formally, start with the polarization vector or spinor in a frame in which the parton momentum is aligned with the  $z$  axis and then Lorentz

transform to the frame in which  $p_a, p_b, p_c$  are defined in eq. (9.128). When plugging eqs. (9.133) and (9.134) into eq. (9.132), we note that we obtain the combination  $\sigma^j \epsilon^{*j}(p_c) = \sigma^1 \pm i\sigma^2 - \frac{p_\perp}{z p} \sigma^3$ . This leads to

$$i\mathcal{M}_{g \leftarrow q}(q_L \rightarrow q_L + g_R) = -ig \frac{\sqrt{2(1-z)}}{z} p_\perp (t^a)^{c_1 c_2}, \quad (9.135)$$

$$\begin{aligned} i\mathcal{M}_{g \leftarrow q}(q_L \rightarrow q_L + g_L) &= -2pig \sqrt{2(1-z)} \frac{p_\perp}{2(1-z)p} - ig \frac{\sqrt{2(1-z)}}{z} p_\perp (t^a)^{c_1 c_2} \\ &= -ig \frac{\sqrt{2(1-z)}}{z(1-z)} p_\perp (t^a)^{c_1 c_2}. \end{aligned} \quad (9.136)$$

Since QCD conserves parity, the matrix elements involving right-handed quarks are immediately obtained by flipping all the helicities in eqs. (9.135) and (9.136). Processes in which the quark flips its helicity have a zero amplitude. For the spin- and color-averaged matrix element appearing in eq. (9.129), we therefore have

$$|\overline{\mathcal{M}_{g \leftarrow q}}|^2 = \frac{4}{3} \frac{2g^2 p_\perp^2}{z^2(1-z)} [(1-z)^2 + 1] \quad (9.137)$$

Note that the factor  $1/2$  from the averaging over initial state spins cancels the factor of two from adding the contributions from left-handed and right-handed quarks. Note also the color factor  $\frac{1}{3} \sum_{c_1 c_2} \sum_a (t^a)^{c_1 c_2} (t^{a*})^{c_1 c_2} = \frac{1}{3} \sum_a \text{tr}(t^a t^a) = \frac{1}{3} C_2(r) \text{tr} \mathbb{1}_{3 \times 3} = \frac{4}{3}$ . This finally gives for the cross section

$$\sigma_{n+1} = \int_0^1 dz \int \frac{d(p_\perp^2)}{p_\perp^2} \frac{4}{3} \frac{\alpha_s}{2\pi} \left[ \frac{(1-z)^2 + 1}{z} \right] \sigma_n. \quad (9.138)$$

We note a few salient features of this expression. The proportionality to the strong fine structure constant  $\alpha_s = g/(4\pi)$  just encodes the strength of the interaction: at large  $\alpha_s$ , splittings are more likely. The integrand diverges in the collinear limit  $p_\perp \rightarrow 0$ , and also in the soft limit  $z \rightarrow 0$ , where the extra gluon carries very little energy.

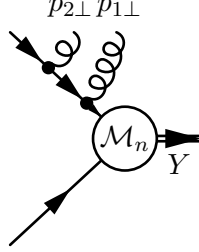
#### 9.7.4 Multiple Splittings

Equation (9.138) looks quite similar to the cross section for deep-inelastic scattering, eq. (9.123). In particular, it suggests that

$$\int \frac{d(p_\perp^2)}{p_\perp^2} \frac{4}{3} \frac{\alpha_s}{2\pi} \left[ \frac{(1-z)^2 + 1}{z} \right] \quad (9.139)$$

can be interpreted as a parton distribution function: it is folded with the cross section to account for the momentum distribution of an initial state quark. However, this interpretation is not complete yet. It does not account for the possibility that there is no splitting, or that there is more than one. It also does not account for the possibility that gluons themselves can also split into  $q\bar{q}$  pairs. Finally, it still has unregularized divergences.

To regularize the divergence in  $p_\perp^2$ , we can simply reintroduce  $m^2$  as a lower cutoff in the integral over  $p_\perp^2$ . To deal with multiple gluon emission, consider the next-to-next-to-leading order process in which two gluons are emitted from the same incoming quark line:



If the outer emission process is more collinear than the inner one,  $p_{2\perp}^2 \ll p_{1\perp}^2$ , the cross section receives a contribution

$$\begin{aligned} \sigma_{n+2} &= \int_0^1 dz_1 \int_0^{z_1} dz_2 \int_{m^2}^s \frac{d(p_{1\perp}^2)}{p_{1\perp}^2} \int_{m^2}^{p_{1\perp}^2} \frac{d(p_{2\perp}^2)}{p_{2\perp}^2} \left( \frac{4}{3} \frac{\alpha_s}{2\pi} \right)^2 \left[ \frac{(1-z_1)^2 + 1}{z_1} \right] \left[ \frac{(1-z_2)^2 + 1}{z_2} \right] \sigma_n \\ &= \int_0^1 dz_1 \int_0^{z_1} dz_2 \left( \frac{4}{3} \frac{\alpha_s}{2\pi} \right)^2 \left[ \frac{(1-z_1)^2 + 1}{z_1} \right] \left[ \frac{(1-z_2)^2 + 1}{z_2} \right] \frac{1}{2} \log^2 \frac{s}{m^2} \sigma_n. \end{aligned} \quad (9.140)$$

Note that we have introduced a lower cutoff  $m^2$  in the transverse momentum integrals to regularize the collinear singularity. If instead  $p_{2\perp}^2 \gg p_{1\perp}^2$ , the transverse momentum integrals take the form

$$\int_{m^2}^s \frac{d(p_{2\perp}^2)}{p_{2\perp}^2} \int_{m^2}^{p_{2\perp}^2} \frac{d(p_{1\perp}^2)}{p_{1\perp}^2} \simeq \log \frac{s}{m^2}. \quad (9.141)$$

This expression is of lower order in the large logarithm  $\log(s/m^2)$  than eq. (9.140) and therefore negligible. In other words, the most relevant radiation processes are those where the emissions are transverse momentum ordered in such a way that the softest emission happens first and the hardest emission last. We say that the quark momenta are *strongly ordered*.

The above procedure can be continued to arbitrary numbers of emitted gluons. For  $n$  gluon emissions, strongly ordered in  $p_\perp$ , the transverse momentum integrals yield a factor

$$\frac{1}{n!} \left( \frac{4}{3} \frac{\alpha_s}{2\pi} \right)^n \log^n \frac{s}{m^2}. \quad (9.142)$$

### 9.7.5 The DGLAP Equations

The physical interpretation of strong ordering is that, as the quark is probed at smaller and smaller distance scales (corresponding to larger and larger energy scales  $s$ ), the quark

resolves into more and more partons. These partons have a virtuality (or off-shellness) of up to  $p_\perp^2 \sim s$ . It is useful to explicitly write out this energy scale dependence in the parton distribution functions. The quark PDF  $f_q(x, Q)$  then gives the probability of finding inside a hadron a quark with longitudinal momentum fraction  $x$  and virtuality  $p_\perp < Q$ . The antiquark PDF  $f_{\bar{q}}(x, Q)$  and the gluon PDF  $f_g(x, Q)$  have analogous interpretations.

If  $Q$  is slightly increased to  $Q + dQ$ , additional splitting processes need to be taken into account. For instance, we have seen above in eq. (9.138) that the differential probability that a quark radiates a gluon with longitudinal momentum fraction  $z$  is

$$\frac{4}{3} \frac{\alpha_s}{2\pi} \frac{(1-z)^2 + 1}{z} \frac{dp_\perp^2}{p_\perp^2} dz. \quad (9.143)$$

This modifies the PDF for the gluon in the following way:

$$\begin{aligned} f_g(x, Q + dQ) &= f_g(x, Q) + \int_0^1 dx' \int_0^1 dz \left[ \frac{4}{3} \frac{\alpha_s}{2\pi} \frac{(1-z)^2 + 1}{z} \right] \frac{d(Q^2)}{Q^2} f_q(x', Q) \delta(x - zx') \\ &= f_g(x, Q) + \int_x^1 \frac{dz}{z} \left[ \frac{4}{3} \frac{\alpha_s}{\pi} \frac{(1-z)^2 + 1}{z} \right] \frac{dQ}{Q} f_q(x/z, Q). \end{aligned} \quad (9.144)$$

Written as a differential equation, this becomes

$$\frac{d}{d \log Q} f_g(x, Q) = \frac{\alpha_s}{\pi} \int_x^1 \frac{dz}{z} P_{g \leftarrow q}(z) f_q(x/z, Q), \quad (9.145)$$

with the *splitting kernel*

$$P_{g \leftarrow q}(z) \equiv \frac{4}{3} \left[ \frac{(1-z)^2 + 1}{z} \right]. \quad (9.146)$$

Of course, eq. (9.145) is not complete yet. Namely,  $f_g(x, Q)$  should receive contributions also from the splitting of antiquarks,  $\bar{q} \rightarrow \bar{q}g$ , and from  $g \rightarrow gg$ . Moreover, we also need evolution equations for  $f_q(x, Q)$  and  $f_{\bar{q}}(x, Q)$  to obtain a complete, solvable, system. Finally, we should take into account that there is more than one quark flavor  $f$ , and each of them has a separate PDF  $f_f(x, Q)$  and a corresponding antiquark PDF,  $f_{\bar{f}}(x, Q)$ . It is clear what structure the full set of equations must have:

$$\begin{aligned} \frac{d}{d \log Q} f_g(x, Q) &= \frac{\alpha_s}{\pi} \int_x^1 \frac{dz}{z} \left[ P_{g \leftarrow q}(z) \sum_f (f_f(x/z, Q) + f_{\bar{f}}(x/z, Q)) \right. \\ &\quad \left. + P_{g \leftarrow g}(z) f_g(x/z, Q) \right], \\ \frac{d}{d \log Q} f_f(x, Q) &= \frac{\alpha_s}{\pi} \int_x^1 \frac{dz}{z} \left[ P_{q \leftarrow q}(z) f_f(x/z, Q) + P_{q \leftarrow g}(z) f_g(x/z, Q) \right], \\ \frac{d}{d \log Q} f_{\bar{f}}(x, Q) &= \frac{\alpha_s}{\pi} \int_x^1 \frac{dz}{z} \left[ P_{\bar{q} \leftarrow q}(z) f_{\bar{f}}(x/z, Q) + P_{\bar{q} \leftarrow g}(z) f_g(x/z, Q) \right]. \end{aligned} \quad (9.147)$$

These equations are called the *Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP)* equations, or, for short, the *Altarelli-Parisi* equations. (The short form was used historically because the works by Dokshitzer and by Gribov and Lipatov was not known west of the iron curtain for a long time.)

The only task that remains is to compute the remaining splitting kernels  $P_{q \leftarrow q}(z)$ ,  $P_{q \leftarrow g}(z)$  and  $P_{g \leftarrow g}(z)$ . The first of these, which describes the splitting of a quark into a quark with momentum fraction  $z$  and a gluon with momentum fraction  $1 - z$  is obtained directly from  $P_{g \leftarrow q}(z)$  by replacing  $z \rightarrow 1 - z$ . However, there is a problem: the resulting expression

$$\frac{4}{3} \left[ \frac{z^2 + 1}{1 - z} \right] \quad (9.148)$$

diverges at the upper boundary of the  $z$  integral in eq. (9.147). This phase space region corresponds to the emission of very soft gluons. But as we have discussed in section 8.3, such soft divergences are cancelled exactly by similarly divergent terms in loop diagrams. We do not include loop diagrams explicitly here, but we simply remove the divergence, knowing that it is unphysical. The recipe used to achieve this is called the *plus prescription*. We define  $1/(1 - z)_+$  by the requirement that, for an arbitrary function  $f(z)$ ,

$$\int_0^1 dz \frac{f(z)}{(1 - z)_+} \equiv \int_0^1 dz \frac{f(z) - f(1)}{1 - z}. \quad (9.149)$$

By replacing  $1/(1 - z)$  by  $1/(1 - z)_+$  in  $P_{q \leftarrow q}$ , we can remove the associated soft divergence from the DGLAP equations. In practice, one usually includes also the numerator  $1 + z^2$  in the plus prescription and defines

$$P_{q \leftarrow q} \equiv \frac{4}{3} \left[ \frac{1 + z^2}{1 - z} \right]_+ = \frac{4}{3} \left[ \frac{1 + z^2}{1 - z} - \delta(1 - z) \int_0^1 dy \frac{1 + y^2}{1 - y} \right]. \quad (9.150)$$

It is easy to check that the difference between including and not including the numerator in the plus prescription is

$$\begin{aligned} & \left( \frac{1 + z^2}{1 - z} \right)_+ - (1 + z^2) \left( \frac{1}{1 - z} \right)_+ \\ &= \frac{1 + z^2}{1 - z} - \delta(1 - z) \int_0^1 dy \frac{1 + y^2}{1 - y} - \frac{1 + z^2}{1 - z} + \delta(1 - z)(1 + z^2) \int_0^1 dy \frac{1}{1 - y} \\ &= \delta(1 - z) \int_0^1 dy \frac{1 - y^2}{1 - y} \\ &= \frac{3}{2} \delta(1 - z). \end{aligned} \quad (9.151)$$

This illustrates that the regularization procedure for the soft divergences is not unique. This ambiguity could only be removed if there was a way of including virtual corrections (loop diagrams) in the DGLAP formalism.

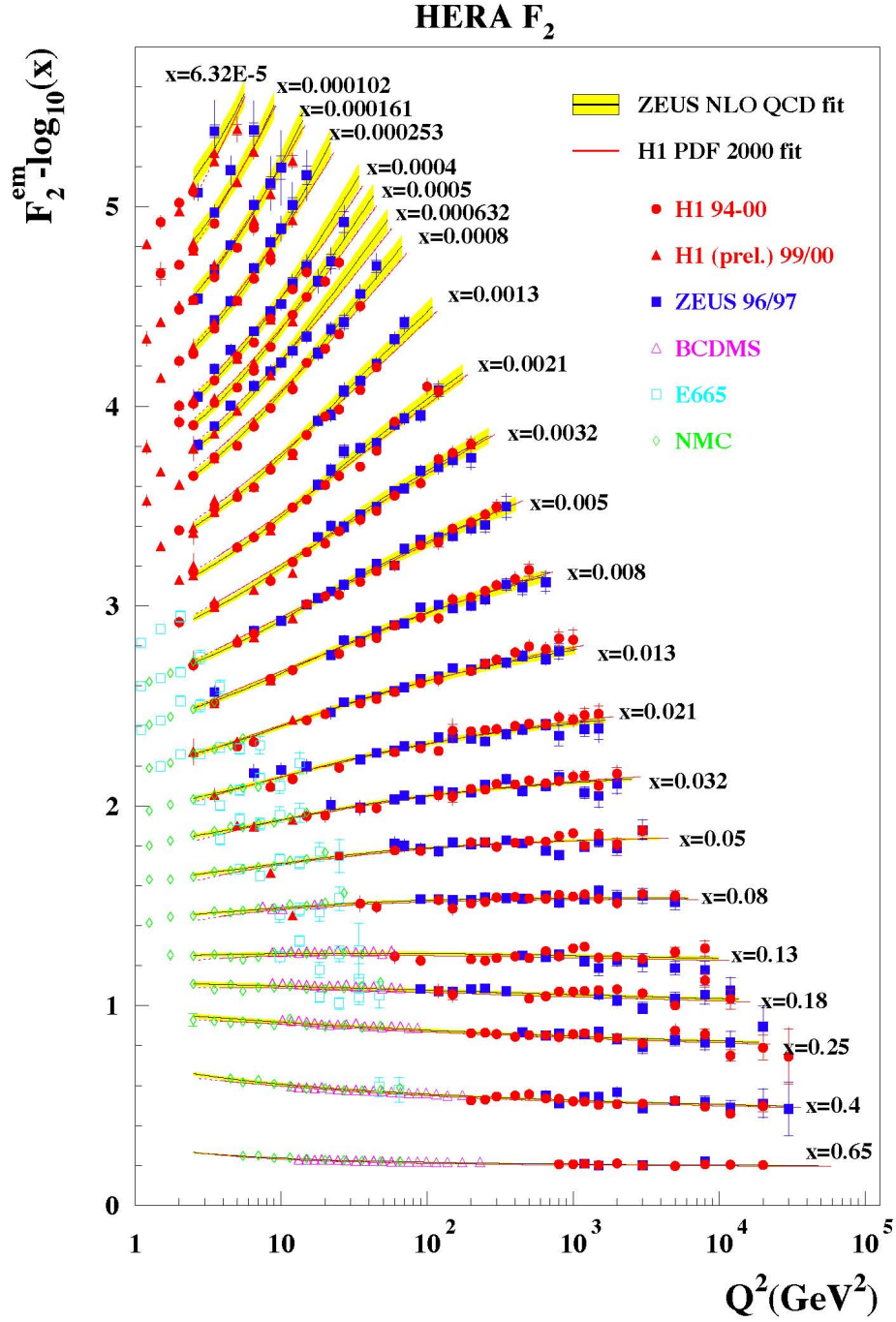
The expressions for the remaining splitting kernels, which we are not going to compute explicitly here, are

$$P_{q \leftarrow g} \equiv \frac{1}{2} [z^2 + (1-z)^2] \quad (9.152)$$

$$P_{g \leftarrow g} \equiv 6 \left[ \frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left( \frac{11}{12} - \frac{n_f}{18} \right) \delta(1-z) \right]. \quad (9.153)$$

Note that the plus prescription is used again in  $P_{g \leftarrow g}$ .

This completes our derivation of the DGLAP equations. If, at a given value of  $Q$ , the PDFs are measured, the DGLAP equations allow us to determine parton distributions at any other scale. For instance, PDFs can be measured in deep inelastic scattering, where  $Q$  (the maximum off-shellness of a parton) corresponds to the momentum transfer from the lepton to the hadronic system). Given measurements at a particular value of  $Q$ , the DGLAP equations allow us to extrapolate to other values. Figure 9.5 shows just how spectacularly well this works.



**Figure 9.5:** A fit of the parton distribution functions for quarks, expressed here in terms of  $F_2(x, Q^2) \equiv \sum_f x q_f^2 f_f(x, Q^2)$ , to various experimental data sets. The functional form of the PDFs is determined by the DGLAP equations.



# 10

## Spontaneous Symmetry Breaking

With quantum electrodynamics and quantum chromodynamics, we already have two of the main ingredients of modern elementary particle physics. What is missing is the theory of weak interactions, and its unification with QED. One salient feature of the resulting *electroweak* theory is that the  $W$  and  $Z$  gauge bosons are massive—a feature that is forbidden in a gauge invariant theory since a gauge boson mass term like  $m_A^2 A^\mu A_\mu$  violates the gauge symmetry. The solution to this dilemma is the concept of *spontaneous symmetry breaking*, which we will now introduce, considering first a toy model with an Abelian gauge symmetry.

### 10.1 The Abelian Higgs Mechanism

Let us study a complex scalar field  $\phi$  coupled to a  $U(1)$  gauge symmetry. The Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - V(\phi). \quad (10.1)$$

We choose the scalar potential is

$$V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{2}(\phi^\dagger\phi)^2. \quad (10.2)$$

This is the most general renormalizable potential: a trilinear term could be absorbed into a shift of the scalar field,  $\phi \rightarrow \phi + a$ , a constant term has no physical consequences since only energy differences are measurable (except in cosmology). Terms with more than four factors of  $\phi$  or  $\phi^\dagger$  are not renormalizable.

The new feature here compared to the scalar potentials we considered in previous chapters is the minus sign in front of the  $\mu^2$  term. It implies that the potential has its

minimum not at  $\phi = 0$ , but at a nonzero value

$$\langle \phi \rangle \equiv v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (10.3)$$

This value is called the *vacuum expectation value* (*vev*) of  $\phi$ . Let us therefore write

$$\phi(x) \equiv v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)). \quad (10.4)$$

Here,  $\phi_1(x)$  and  $\phi_2(x)$  are real scalar fields, and the factor  $1/\sqrt{2}$  is just a convention. In terms of  $\phi_1(x)$  and  $\phi_2(x)$ , the potential becomes

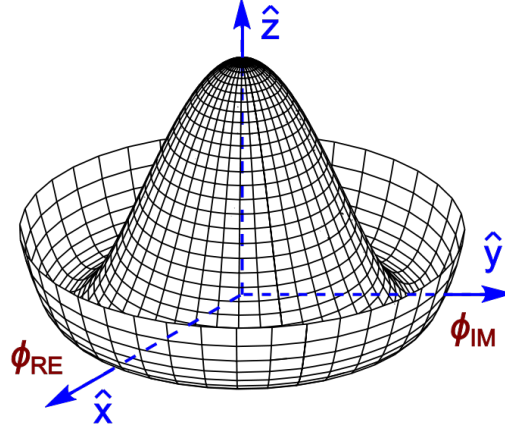
$$\begin{aligned} V(\phi) &= -\mu^2 \left[ v + \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x)) \right] \left[ v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \right] \\ &\quad + \frac{\lambda}{2} \left( \left[ v + \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x)) \right] \left[ v + \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \right] \right)^2 \\ &= -\frac{\mu^4}{\lambda} - 2\mu^2 \sqrt{\frac{\mu^2}{\lambda}} \frac{1}{\sqrt{2}} \phi_1 - \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) \\ &\quad + \frac{\mu^4}{2\lambda} + 4\frac{\lambda}{2\sqrt{2}} \left( \frac{\mu^2}{\lambda} \right)^{3/2} \phi_1 + \frac{\lambda}{4} \frac{\mu^2}{\lambda} (6\phi_1^2 + 2\phi_2^2) + \text{cubic and quartic terms} \\ &= -\frac{\mu^4}{2\lambda} + \mu^2 \phi_1^2 + \text{cubic and quartic terms}. \end{aligned} \quad (10.5)$$

This implies that  $\phi_1$  has mass  $\sqrt{2}\mu$ , while  $\phi_2$  is *massless*. To understand this, it is helpful to plot  $V$ , see fig. 10.1. The plot also explains why such a potential is called *Mexican hat potential*. If the field is expanded around the minimum at  $\text{Re}\phi = v$ ,  $\text{Im}\phi = 0$ , the radial excitation corresponds to  $\phi_1$ . The second derivative of the potential in the radial direction is nonzero, therefore  $\phi_1$  is massive. Orthogonal excitations along the azimuthal direction correspond to  $\phi_2$ . Since the potential is flat in this direction,  $\phi_2$  is massless.

Note that the expansion in eq. (10.4) is not unique. We could equivalently have expanded around any other point  $\phi = e^{i\alpha}v$ . At all these points,  $V(\phi)$  has the same value, they all physically equivalent vacuum states. We thus have a theory that respects a  $U(1)$  gauge symmetry, but has a nontrivial vacuum state, for which the symmetry is no longer manifest and which is not invariant under symmetry transformations. We say that  $\phi$  *spontaneously breaks* the gauge symmetry. The massless boson  $\phi_2$  is called a *Nambu–Goldstone boson* or just *Goldstone boson*. The appearance of massless bosons in connection with spontaneously broken symmetries is generic—the is the so-called Goldstone theorem, which we will discuss in section 10.2.

Consider now the kinetic term of  $\phi$ :

$$\begin{aligned} |D_\mu \phi|^2 &= \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \sqrt{2}ev A_\mu \partial^\mu \phi_2 + e^2 v^2 A_\mu A^\mu \\ &\quad + \text{cubic and quartic terms}. \end{aligned} \quad (10.6)$$



**Figure 10.1:** The “Mexican hat” potential, eq. (10.2). Excitations along the radial direction correspond to the massive real scalar field  $\phi_1$ , excitations along the azimuthal direction correspond to the Goldstone boson  $\phi_2$ .

We have thus generated a gauge boson mass term in a gauge-invariant theory. While the mass term itself is not gauge invariant, its combination with the other new terms is.

Note that one can always make a gauge transformation  $\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)$  such that afterwards  $\phi_2(x) = 0$  for all  $x$ . For this particular gauge choice,  $\phi_2$  disappears from the theory, but a new degree of freedom has appeared: the longitudinal polarization of the gauge boson. We say that  $\phi_2$  gets *eaten* by  $A^\mu$ . The interchangeability of  $\phi_2$  and one degree of freedom of  $A^\mu$  can be understood also from the fact that  $k^\mu\phi_2$  and  $A^\mu$  have the same quantum numbers and are related by gauge transformations.

## 10.2 Goldstone's Theorem

Above, we have already alluded to Goldstone's theorem which states that any spontaneously broken continuous symmetry in a QFT leads to the appearance of a massless boson in the theory. Intuitively, this results can be understood in the following way: in a spontaneously broken theory, there must be infinitely many physically equivalent vacuum states, related to each other by gauge transformations. Field excitations along this flat direction in the potential correspond to a massless boson. We will now prove this statement more rigorously.

Consider a theory with  $n$  scalar fields  $\Phi \equiv (\phi^1, \dots, \phi^n)$ . The scalar potential  $V(\Phi)$  has its global minimum at a point  $\Phi_0 \equiv (\phi_0^1, \dots, \phi_0^n)$ , which must satisfy

$$\left. \frac{\partial}{\partial \phi^a} V(\Phi) \right|_{\Phi(x)=\Phi_0} = 0 \quad (10.7)$$

for all  $a = 1 \cdots n$ . We now expand  $V(\Phi)$  around this minimum:

$$V(\Phi) = V(\Phi_0) + \frac{1}{2}(\phi^a - \phi_0^a)(\phi^b - \phi_0^b) \left[ \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right]_{\Phi_0} + \cdots \quad (10.8)$$

The matrix

$$(m^{ab}) = \left( \frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right) \Big|_{\Phi_0} \quad (10.9)$$

is just the mass matrix of the physical scalar states.

Now assume that the theory is invariant under continuous symmetry transformations, the infinitesimal version of which is

$$\phi^a(x) \rightarrow \phi^a(x) + \alpha(x) \Delta^a(\Phi). \quad (10.10)$$

Here,  $\alpha(x)$  is the infinitesimal transformation parameter, and  $\Delta^a(\Phi)$  is the variation of the scalar fields under symmetry transformations. Invariance under eq. (10.10) implies in particular that  $V(\Phi)$  is invariant:

$$V(\Phi) = V(\Phi + \alpha \Delta(\Phi)), \quad (10.11)$$

(All other terms containing  $\Phi$  will contain either derivatives or other fields and therefore cannot compensate for a possible non-invariance of  $V(\Phi)$ .) In infinitesimal form, eq. (10.11) implies

$$V(\Phi) = V(\Phi) + \alpha \Delta^a(\Phi) \frac{\partial V(\Phi)}{\partial \phi^a}, \quad (10.12)$$

and thus

$$\Delta^a(\Phi) \frac{\partial V(\Phi)}{\partial \phi^a} = 0. \quad (10.13)$$

Differentiating this relation once more leads to

$$\frac{\partial \Delta^a(\Phi)}{\partial \phi^b} \frac{\partial V(\Phi)}{\partial \phi^a} + \Delta^a(\Phi) \frac{\partial^2 V(\Phi)}{\partial \phi^a \partial \phi^b} = 0. \quad (10.14)$$

Now set  $\Phi = \Phi_0$ . Then, the first term vanishes since  $\Phi_0$  is a minimum. The second one thus has to vanish on its own, which implies that either  $\Delta^a(\Phi_0) = 0$ , as for a vacuum state that is invariant under the symmetry, or that  $\Delta^a(\Phi_0)$  is an eigenvector of the matrix  $(m^{ab})$  from eq. (10.9) with eigenvalue 0. The latter case is realized in a theory with spontaneous symmetry breaking, where the vacuum state is not invariant under symmetry transformations. The state  $\Delta^a(\Phi_0)\phi^a$  is thus a scalar field with mass zero—the Nambu–Goldstone boson.

## 10.3 The Glashow–Salam–Weinberg Theory of Electroweak Interactions

### 10.3.1 Higgs Mechanism

We are now ready to introduce the state-of-the-art theory of electroweak interactions: the Glashow–Salam–Weinberg (GSW) model, which is part of the Standard Model (SM) of particle physics. The GSW model is based on the gauge group  $SU(2) \times U(1)$ , i.e. particles may transform non-trivially under both  $SU(2)$  transformations and  $U(1)$  transformations. The charge associated with the  $U(1)$  gauge group is called *hypercharge* and is conventionally denoted by the letter  $Y$ .

The theory also contains a new scalar field  $\phi$ —the Higgs fields—which transforms as a doublet of  $SU(2)$  and carries a hypercharge of  $1/2$ . Its gauge transformation properties are thus

$$\phi(x) \rightarrow e^{i\alpha^a(x)\sigma^a/2} e^{i\beta(x)/2} \phi(x), \quad (10.15)$$

where as usual  $\sigma^a(x)/2$  are the generators of  $SU(2)$  in the fundamental representation. The Lagrangian of the Higgs field is

$$\mathcal{L}_{\text{Higgs}} \equiv (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2. \quad (10.16)$$

Note that the sign of the  $\mu^2$  term is chosen such that the theory will feature spontaneous symmetry breaking. The Higgs potential has its minimum at

$$v^2 \equiv 2 \langle \phi^\dagger \phi \rangle = \frac{\mu^2}{\lambda}. \quad (10.17)$$

(The factor 2 in the definition of  $v$  is a convention.) We write

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \quad (10.18)$$

but of course any  $SU(2) \times U(1)$  transformation of this vector would offer an equivalently valid definition of  $v$ .

Note that the nonzero vacuum expectation value  $\langle \phi \rangle$  breaks  $SU(2) \times U(1)$  except for transformations with  $\alpha^1 = \alpha^2 = 0$ ,  $\alpha^3 = \beta$ . As can be seen from eq. (10.15), such transformations leave  $\langle \phi \rangle$  invariant. There is thus a residual  $U(1)$  symmetry, which we will soon identify with electromagnetism. In addition, there should be three Goldstone bosons, since  $SU(2) \times U(1)$  has four generators, and the residual unbroken  $U(1)$  corresponds to one generator.

### 10.3.2 Gauge Boson Masses

To make further progress and to determine the gauge boson masses in analogy with the Abelian Higgs mechanism, we plug the vev from eq. (10.18) into the Lagrangian

eq. (10.16). We use

$$D_\mu \phi = \partial_\mu \phi - igW_\mu^a \frac{\sigma^a}{2} \phi - i\frac{1}{2}g'B_\mu \phi, \quad (10.19)$$

where  $W_\mu$  and  $B_\mu$  are the  $SU(2)$  and  $U(1)$  gauge boson fields, respectively, and  $g, g'$  are the corresponding gauge coupling constants. This leads to the gauge boson mass terms

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &\supset \frac{1}{2}(0, v) \left( gW_\mu^a \frac{\sigma^a}{2} + \frac{1}{2}g'B_\mu \right) \left( gW_\mu^b \frac{\sigma^b}{2} + \frac{1}{2}g'B_\mu \right) \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2} \frac{v^2}{4} \left( g^2(W_\mu^1)^2 + g^2(W_\mu^2)^2 + g^2(W_\mu^3)^2 - 2gg'W_\mu^3 B_\mu + g'^2(B_\mu)^2 \right) \\ &= \frac{1}{2} \frac{g^2 v^2}{4} (W_\mu^1)^2 + \frac{1}{2} \frac{g^2 v^2}{4} (W_\mu^2)^2 + \frac{1}{2} \frac{v^2}{4} (W_\mu^3, B_\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}. \end{aligned} \quad (10.20)$$

We see that the two gauge boson  $W^1$  and  $W^2$  have mass

$$m_W = \frac{gv}{2}, \quad (10.21)$$

while  $W^3$  and  $B$  mix with each other. The states of definite mass are

$$Z_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (gW_\mu^3 - g'B_\mu) \quad \text{with mass } m_Z = \sqrt{g^2 + g'^2} \frac{v}{2} \quad (10.22)$$

and

$$A_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (g'W_\mu^3 + gB_\mu) \quad \text{with mass } m_A = 0. \quad (10.23)$$

It makes sense that there is a massless gauge boson: as we argued above, a  $U(1)$  subgroup of  $SU(2) \times U(1)$  remains unbroken, and an unbroken gauge symmetry needs a massless gauge boson.

It is useful to also define linear combinations of  $W^1$  and  $W^2$ :

$$W_\mu^\pm \equiv \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2). \quad (10.24)$$

We will see shortly that the superscript  $+$  or  $-$  on these fields specifies indeed their electric charge. This is also the reason for defining these linear combinations: we want states of definite electric charge.

We now rewrite the covariant derivative in terms of the physical (mass and charge

eigenstate) fields  $W^\pm$ ,  $Z$  and  $A$ . Using

$$\begin{aligned} W_\mu^1 &= \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-), \\ W_\mu^2 &= \frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-), \\ W_\mu^3 &= \frac{1}{\sqrt{g^2 + g'^2}}(gZ_\mu + g'A_\mu), \\ B_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}(-g'Z_\mu + gA_\mu), \end{aligned} \tag{10.25}$$

we find

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{i}{\sqrt{2}}gt^1(W_\mu^+ + W_\mu^-) + \frac{1}{\sqrt{2}}gt^2(W_\mu^+ - W_\mu^-) \\ &\quad - ig t^3 \frac{1}{\sqrt{g^2 + g'^2}}(gZ_\mu + g'A_\mu) - ig'Y \frac{1}{\sqrt{g^2 + g'^2}}(-g'Z_\mu + gA_\mu) \\ &= \partial_\mu - \frac{ig}{\sqrt{2}}(t^+W_\mu^+ + t^-W_\mu^-) - \frac{i}{\sqrt{g^2 + g'^2}} \left[ Z_\mu(g^2t^3 - g'^2Y) + A_\mu gg'(t^3 + Y) \right]. \end{aligned} \tag{10.26}$$

In the last step, we have defined

$$t^\pm \equiv t^1 \pm it^2. \tag{10.27}$$

The coupling to the massless gauge boson—the photon—is

$$e \cdot Q \equiv \frac{gg'}{\sqrt{g^2 + g'^2}} \cdot (t^3 + Y). \tag{10.28}$$

The first factor on the right hand side is independent of the  $SU(2) \times U(1)$  representation of which  $D_\mu$  is acting and is therefore identified with the electromagnetic gauge coupling  $e$ . The second factor is representation dependent and gives the electric charge  $Q$  in terms of the hypercharge  $Y$  and the eigenvalue of  $t^3$ , which we call the *third component of the weak isospin*.<sup>1</sup> Remember that in the fundamental representation  $t^3$  is diagonal, with eigenvalues  $+1/2$  and  $-1/2$ , so that the components of an  $SU(2)$  doublet have well-defined electric charges, which differ by one unit.

Here are a few examples: in the Standard Model of particle physics, the left-chiral up and down quarks form an  $SU(2)$  doublets of Weyl fermions with hypercharge  $Y = 1/6$ :

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \tag{10.29}$$

<sup>1</sup>The weak isospin 3-vector is given by the operator  $(t^1, t^2, t^3)$ .

The LH up quark has weak isospin  $+1/2$ , the LH down quark has weak isospin  $-1/2$ . Their electric charges are therefore

$$\begin{aligned} Q_{u_L} &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} && \text{for the LH up quark} \\ Q_{d_L} &= -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3} && \text{for the LH down quark.} \end{aligned} \quad (10.30)$$

The right-handed quarks form  $SU(2)$  singlets, i.e. they carry no weak isospin, and their electric charge is equal to their hypercharge,  $Q = Y$ . We choose  $Y = 2/3$  for the RH up quark and  $Y = -1/3$  for the RH down quark, so that LH and RH quarks of the same flavor have identical electric charge.

As a final remark, we introduce the *Weinberg angle*  $\theta_w$ , the mixing angle of  $W_\mu^3$  and  $B_\mu$ . We write

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos \theta_w & -\sin \theta_w \\ \sin \theta_w & \cos \theta_w \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}. \quad (10.31)$$

From the definitions of  $Z_\mu$  and  $A_\mu$  in eqs. (10.22) and (10.23), we read off

$$\sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad \cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}}. \quad (10.32)$$

With the Weinberg angle and the electric charge from eq. (10.28), the covariant derivative can be rewritten as

$$D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}}(t^+ W_\mu^+ + t^- W_\mu^-) - \frac{ig}{\cos \theta_w} Z_\mu(t^3 - Q \sin^2 \theta_w) - ie A_\mu Q. \quad (10.33)$$

### 10.3.3 Gauge Boson Self-Couplings

We have not demonstrated yet that the  $W^\pm$  bosons indeed have electric charge  $\pm 1$ . To tie up this loose end, we consider the gauge boson self-couplings to determine how  $W^\pm$  couple to the photon. The triple gauge boson couplings from eq. (9.100) are

$$\begin{aligned} \mathcal{L}_{\text{YM}} &\supset -g W_\mu^2 W_\nu^3 \partial^\mu W^{\nu,1} + g W_\mu^3 W_\nu^2 \partial^\mu W^{\nu,1} - g W_\mu^3 W_\nu^1 \partial^\mu W^{\nu,2} + g W_\mu^1 W_\nu^3 \partial^\mu W^{\nu,2} \\ &\quad - g(W_\mu^1 W_\nu^2 - W_\mu^2 W_\nu^1) \partial^\mu W^{\nu,3} \\ &= g W_\nu^3 (W_\mu^1 \partial^\mu W^{\nu,2} - W_\mu^2 \partial^\mu W^{\nu,1}) - g W_\mu^3 (W_\nu^1 \partial^\mu W^{\nu,2} - W_\nu^2 \partial^\mu W^{\nu,1}) \\ &\quad - g(\partial^\mu W^{\nu,3})(W_\mu^1 W_\nu^2 - W_\mu^2 W_\nu^1) \end{aligned} \quad (10.34)$$

We can use the relation  $W_\mu^1 W_\nu^2 = \frac{i}{2}(W_\mu^+ W_\nu^+ + W_\mu^- W_\nu^+ - W_\mu^+ W_\nu^- - W_\mu^- W_\nu^-)$ , which follows from eq. (10.25). We will in the following keep only terms containing  $W_\mu^\pm$  and  $A_\mu$ , and drop terms involving the  $Z$  boson. We obtain

$$\mathcal{L}_{\text{YM}} \supset ie A_\nu (W_\mu^- \partial^\mu W^{+\nu} - W_\mu^+ \partial^\mu W^{-\nu}) - ie A_\mu (W_\nu^- \partial^\mu W^{+\nu} - W_\nu^+ \partial^\mu W^{-\nu})$$



$$-ie(\partial^\mu A^\nu)(W_\mu^- W^{+\nu} - W_\mu^+ W^{-\nu}). \quad (10.35)$$

This shows that the  $W^\pm$  bosons couple to the photon with a coupling of  $\pm e$  and are therefore charge  $\pm 1$  particles. The coupling structure is somewhat more complicated than for couplings of scalars or fermions to the photon to maintain gauge invariance.

The other triple and quartic gauge boson vertices can be derived in a similar way. The calculation is a bit tedious and tends to turn into a battle with indices. The Feynman rules for the Standard Model are summarized in the appendices of many textbooks (sometimes with mistakes), see for instance [5] (where some of the couplings to fermions are incorrect, though).

### 10.3.4 Gauge Boson Coupling to Fermions

One of the salient features of the Standard Model is that it is a *chiral gauge theory*, i.e. left-handed and right-handed fermions are in different representations of the gauge group. (Theories in which LH and RH fermions are in the same representation are called *vector-like theories*.)

The following table summarizes the gauge quantum number assignment for the SM fermions:

	$SU(3)$	$SU(2)$	$U(1)_Y$
LH leptons $L_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	1	2	$-\frac{1}{2}$
RH charged leptons $e_R$	1	1	$-1$
LH quarks $Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	$\frac{1}{6}$
RH up-type quarks $u_R$	3	1	$\frac{2}{3}$
RH down-type quarks $d_R$	3	1	$-\frac{1}{3}$

All of these fermion representations come in three copies, or *families*. For instance, the charged leptons are the  $e$ ,  $\mu$ , and  $\tau$ , the corresponding neutrinos are the  $\nu_e$ ,  $\nu_\mu$ , and  $\nu_\tau$ . The up-type quarks are the up, charm, and top, while the down type quarks are the down, strange, and bottom quarks. Note that we have omitted right-handed neutrinos from the above table. Even though we know that neutrinos have mass, we do not know if their mass term is Dirac or Majorana, therefore we do not know whether right-handed neutrinos exist. If they do, they would not be charged under any of the SM gauge groups.

To write out the structure of the fermion kinetic terms, which contain in particular the fermion couplings to gauge bosons, we use the covariant derivative in the form given by eq. (10.33). Moreover, we use that, in the fundamental representation of  $SU(2)$ , the matrices  $t^\pm$  and  $t^3$  are simply

$$t^+ = \frac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (10.36)$$

$$t^- = \frac{1}{2}(\sigma^1 + i\sigma^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (10.37)$$

and

$$t^3 = \frac{1}{2}\sigma^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \quad (10.38)$$

We obtain

$$\begin{aligned} \mathcal{L}_{\text{fermion,kin}} &= \bar{L}_L i \not{D} L_L + \bar{e}_R i \not{D} e_R + \bar{Q}_L i \not{D} Q_L + \bar{u}_R i \not{D} u_R + \bar{d}_R i \not{D} d_R \\ &= \bar{L}_L i \not{\partial} L_L + \bar{e}_R i \not{\partial} e_R + \bar{Q}_L i \not{\partial} Q_L + \bar{u}_R i \not{\partial} u_R + \bar{d}_R i \not{\partial} d_R \\ &\quad + g(W_\mu^+ J_W^{\mu+} + W_\mu^- J_W^{\mu-} + Z_\mu J_Z^\mu) + e A_\mu J_{\text{em}}^\mu, \end{aligned} \quad (10.39)$$

with the currents

$$J_W^{\mu+} \equiv \frac{1}{\sqrt{2}}(\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L) \quad (10.40)$$

$$J_W^{\mu-} \equiv \frac{1}{\sqrt{2}}(\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L) \quad (10.41)$$

$$\begin{aligned} J_Z^\mu &\equiv \frac{1}{\cos \theta_w} \left[ \frac{1}{2} \bar{\nu}_L \gamma^\mu \nu_L + \left( -\frac{1}{2} + \sin^2 \theta_w \right) \bar{e}_L \gamma^\mu e_L + \sin^2 \theta_w \bar{e}_R \gamma^\mu e_R \right. \\ &\quad + \left( \frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right) \bar{u}_L \gamma^\mu u_L + \left( -\frac{2}{3} \sin^2 \theta_w \right) \bar{u}_R \gamma^\mu u_R \\ &\quad \left. + \left( -\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right) \bar{d}_L \gamma^\mu d_L + \left( \frac{1}{3} \sin^2 \theta_w \right) \bar{d}_R \gamma^\mu d_R \right] \end{aligned} \quad (10.42)$$

$$J_{\text{em}}^\mu \equiv -\bar{e} \gamma^\mu e + \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d. \quad (10.43)$$

When working in 4-component notation, it is sometimes useful to rewrite  $J_Z^\mu$  in the form

$$\begin{aligned} J_Z^\mu &\equiv \frac{1}{4 \cos \theta_w} \left[ \bar{\nu} \gamma^\mu (1 - \gamma^5) \nu + \bar{e} \gamma^\mu (-1 + 4 \sin^2 \theta_w + \gamma^5) e \right. \\ &\quad \left. + \bar{u} \gamma^\mu \left( 1 - \frac{8}{3} \sin^2 \theta_w - \gamma^5 \right) u + \bar{d} \gamma^\mu \left( -1 + \frac{4}{3} \sin^2 \theta_w + \gamma^5 \right) d \right]. \end{aligned} \quad (10.44)$$

### 10.3.5 Fermion Masses

Since the left-handed and right-handed fermions in the SM carry different gauge charges, conventional mass terms of the form  $m \bar{f} f = m \bar{f}_L f_R + h.c.$  are forbidden. But of course, we know that quarks and leptons have masses. Fortunately, the Higgs field is there to save the day. Consider first the charged leptons, which couple to the Higgs field via the Yukawa term

$$\mathcal{L}_Y \supset -y_e \bar{L}_L \phi e_R + h.c.. \quad (10.45)$$

It is easy to check that this coupling is allowed by the gauge quantum numbers: all involved fields are color singlets, and under  $SU(2) \times U(1)$ ,  $L_L$  transforms as  $(2, -\frac{1}{2})$ ,  $\phi$  transforms as  $(2, \frac{1}{2})$ , and  $e_R$  transforms as  $(1, -1)$ . We write the Higgs field as

$$\phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}, \quad (10.46)$$

where  $v$  is the vev, and  $G^\pm$ ,  $G^0$  and  $h$  are the field excitations around this vev. (We will check below that  $G^\pm$ ,  $G^0$  correspond to the massless Goldstone modes that are eaten by the  $W$  and  $Z$  bosons, while  $h$  is a physical real scalar, the Higgs boson.) Keeping only the vev term, eq. (10.45) becomes

$$\mathcal{L}_Y \supset -m_e \bar{e}_L e_R + h.c., \quad (10.47)$$

with the lepton mass

$$m_e \equiv \frac{y_e v}{\sqrt{2}}. \quad (10.48)$$

Note that we have again suppressed family (or flavor) indices here. Reintroducing them,  $y_e$  and  $m_e$  turn into  $3 \times 3$  matrices.

The mass terms for the down quarks are generated in exactly the same way. For up quarks, a Yukawa coupling of the form of eq. (10.45) is not allowed:  $Q_L$  transforms as  $(2, \frac{1}{6})$ ,  $\phi$  as  $(2, -\frac{1}{2})$ , and  $u_R$  as  $(1, \frac{2}{3})$ . Moreover, even if a Yukawa coupling like eq. (10.45) was allowed,  $\langle \phi \rangle$  would project out the lower components of  $Q_L$  and thus not give mass to up quarks. The solution is to use instead a Yukawa coupling of the form

$$\mathcal{L}_Y \supset -y_u \bar{Q}_L \tilde{\phi} u_R + h.c., \quad (10.49)$$

with

$$\tilde{\phi} \equiv i\sigma^2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^*. \quad (10.50)$$

Such a coupling leads to a mass term  $-m_u \bar{u}_L u_R$  with  $m_u \equiv (y_u v)/2$ . To check that eq. (10.49) is gauge invariant, we note that  $\tilde{\phi}$  has hypercharge  $+\frac{1}{2}$  (the opposite of  $\phi$ ) because of the complex conjugation involved. We then only have to demonstrate that  $\tilde{\phi}$  transforms as a 2 of  $SU(2)$ , i.e.

$$\tilde{\phi} \rightarrow e^{i\alpha^a \sigma^a / 2} \tilde{\phi}. \quad (10.51)$$

It is easiest to show this by considering infinitesimal transformations:

$$\begin{aligned} \tilde{\phi} &\rightarrow i\sigma^2 \left[ \left( 1 + i\alpha^a \frac{\sigma^a}{2} \right) \phi \right]^* \\ &= i\sigma^2 \left( 1 - i\alpha^a \frac{\sigma^{a*}}{2} \right) (-i\sigma^2) \tilde{\phi} \end{aligned}$$

$$\begin{aligned}
 &= \left(1 - i\alpha^a \sigma^2 \frac{\sigma^{a*}}{2} \sigma^2\right) \tilde{\phi} \\
 &= \left(1 + i\alpha^a \frac{\sigma^a}{2}\right) \tilde{\phi}.
 \end{aligned} \tag{10.52}$$

In the last line, we have used that  $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$  and that  $\sigma^1 = \sigma^{1*}$ ,  $\sigma^2 = -\sigma^{2*}$ ,  $\sigma^3 = \sigma^{3*}$ .

### 10.3.6 The Higgs Boson

We have already argued above that the Higgs field can be rewritten as

$$\phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}. \tag{10.53}$$

Let us now show that, indeed,  $G^\pm$  and  $G^0$  correspond to the Goldstone modes. To do so, we expand the Higgs potential in components, keeping only terms with up to two fields:

$$\begin{aligned}
 V(\phi) &= -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \\
 &= -\mu^2 \left( G^+ G^- + \frac{1}{2}(v^2 + 2vh + h^2) + \frac{1}{2}(G^0)^2 \right) \\
 &\quad + \frac{\mu^2}{v^2} \left( G^+ G^- + \frac{1}{2}(v^2 + 2vh + h^2) + \frac{1}{2}(G^0)^2 \right)^2 \\
 &= -\frac{1}{2}\mu^2 v^2 - \mu^2 vh - \frac{1}{2}\mu^2 h^2 + \frac{1}{4}\frac{\mu^2}{v^2}(v^4 + 4v^3h + 2v^2h^2 + 4v^2h^2) \\
 &\quad + \text{cubic and quartic terms} \\
 &= -\frac{1}{4}\mu^2 v^2 + \frac{1}{2}(2\mu^2)h^2 + \text{cubic and quartic terms}.
 \end{aligned} \tag{10.54}$$

We see that  $G^\pm$  and  $G^0$  have no mass term, i.e. their masses are zero. The constant term can be ignored since it has no physical consequences (except in cosmology, where it contributes to the cosmological constant), and the mass of the physical Higgs boson  $h$  is  $\sqrt{2}\mu = \sqrt{2\lambda}v$ .

The fact that  $G^\pm$  and  $G^0$  are unphysical can also be seen from the fact that they can always be rotated away by a suitable  $SU(2) \times U(1)$  transformation. (Use a  $U(1)$  rotation combined with an  $SU(2)$  transformation generated by  $t^3$  to make both components of  $\phi$  real; then use a transformation generated by  $t^1$  to set the upper one to zero.) This gauge is called *unitary gauge*. In unitary gauge, the Higgs Lagrangian is

$$\begin{aligned}
 \mathcal{L}_{\text{Higgs}} &\equiv (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \\
 &= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{4} W_\mu^+ W^{-\mu} (v + h)^2 + \frac{g^2}{8 \cos \theta_w} Z_\mu Z^\mu (v + h)^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\mu^2}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4 \\
 & = \frac{1}{2}(\partial_\mu h)^2 + m_W^2 W_\mu^+ W^{-\mu} \left(1 + \frac{2h}{v} + \frac{h^2}{v^2}\right) + \frac{1}{2}m_Z^2 Z_\mu Z^\mu \left(1 + \frac{2h}{v} + \frac{h^2}{v^2}\right) \\
 & \quad - \frac{1}{2}(2\mu^2)h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4.
 \end{aligned} \tag{10.55}$$

This Lagrangian leads in particular to the Feynman rules

$$\begin{array}{c}
 W^-, \nu \\
 \text{---} h \text{---} \bullet \text{---} \text{wavy line} \\
 W^+, \mu
 \end{array} = \frac{2im_W^2}{v} g^{\mu\nu} \tag{10.56}$$

$$\begin{array}{c}
 Z, \nu \\
 \text{---} h \text{---} \bullet \text{---} \text{wavy line} \\
 Z, \mu
 \end{array} = \frac{i2m_Z^2}{v} g^{\mu\nu} \tag{10.57}$$

$$\begin{array}{c}
 h \\
 \diagdown \text{---} \bullet \text{---} \text{wavy line} \\
 h \quad W^-, \nu \\
 \diagup \text{---} \bullet \text{---} \text{wavy line} \\
 h \quad W^+, \mu
 \end{array} = \frac{i2m_W^2}{v^2} g^{\mu\nu} \tag{10.58}$$

$$\begin{array}{c}
 h \\
 \diagdown \text{---} \bullet \text{---} \text{wavy line} \\
 h \quad Z, \nu \\
 \diagup \text{---} \bullet \text{---} \text{wavy line} \\
 h \quad Z, \mu
 \end{array} = \frac{i2m_Z^2}{v^2} g^{\mu\nu} \tag{10.59}$$

$$\begin{array}{c}
 h \\
 \diagdown \text{---} \bullet \text{---} \text{dashed line} \\
 h \text{---} \text{---} \bullet \text{---} \text{dashed line} \\
 h \quad h
 \end{array} = -6i\lambda v \tag{10.60}$$

$$\begin{array}{c}
 h \quad h \\
 \diagdown \text{---} \bullet \text{---} \text{dashed line} \\
 h \quad h
 \end{array} = -6i\lambda \tag{10.61}$$

In these diagrams, we have taken all particles to be incoming. Note that extra numerical factors appear for those vertices containing identical particles on some of their legs.

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