problem sheet 6
to be handed by **Friday 1.7.2016 (12:00)** to the letterbox 37 (foyer of Staudingerweg 7)

1. **Kinetic Term of Non-Abelian Gauge Theories (40 P.)**

   In the lecture you deduced the kinetic term of non-abelian gauge theories

   $$\mathcal{L}_{gauge} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} \tag{1}$$

   calculating the commutator of covariant derivatives. Now derive the gauge kinetic term using the approach of a Wilson loop (compare chapter 9.1 of the lecture notes).

2. **Complex scalar field in a non-abelian gauge theory (20 P.)**

   Consider a complex scalar field ϕ in a non-abelian gauge theory. Write down the Lagrangian of the theory (you do not have to specify the self interaction of the scalar field). Then deduce the Feynman rules of this theory. (You can leave out the self interaction of the gauge bosons.)

3. **The Casimir Operator (40 P.)**

   In chapter 9.3.2 of the lecture notes you deduced the quantity $C_2(r)$. Since the proportionality factor $C(r)$ and the quadratic Casimir operator are related by

   $$d(G)C(r) = d(r)C_2(r) \tag{2}$$

   we can compute $C_2(r)$ by computing $C(r)$. Choose t^a and t^b to lie in a $SU(2)$ subgroup of the gauge group and use the identity

   $$tr[t^a_r t^b_r] = C(r)\delta^{ab} \,. \tag{3}$$

   Remember that a representation of $SU(2)$ can be described by a spin j with dimension $(2j+1)$.

   (a) (15 P.) First consider $SU(2)$ to be a subgroup of a general group G. An irreducible representation r of G will decompose into a sum of representations of $SU(2)$:

   $$r \to \sum j_i \,, \tag{4}$$

   where j_i are the spins of $SU(2)$ representations. Show that

   $$3C(r) = \sum_i j_i(j_i+1)(2j_i+1) \,. \tag{5}$$

   (b) (25 P.) Now consider $SU(2)$ to be a subgroup of $SU(N)$. Then the fundamental representation N of $SU(N)$ transforms as a two-component spinor ($j = 1/2$) and $N-2$ singlets. Use this relation to show $C(N) = \frac{1}{2}$.
   Then show that the adjoint representation of $SU(N)$ decomposes into one spin 1, $2(N-2)$ spin $\frac{1}{2}$'s, plus singlets, and use this decomposition to show $C(G) = N$.
   Hint: The adjoint representation of $SU(N)$ is the product $N \otimes \bar{N}$ of the fundamental and anti-fundamental representation. For $SU(2)$ the two represenations 2 and $\bar{2}$ are isomorphic.