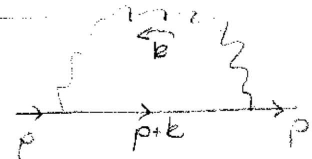
Blatt 7

① QCD Beta-Function

$Z_\psi \longrightarrow$ fermion self-energy 

$$= \int \frac{d^4k}{(2\pi)^4} (ig t^a \gamma^\mu) \frac{i(\not p + \not k + m)}{(p+k)^2 - m^2 + i\varepsilon} \frac{-i\delta^{ab} g_{\mu\nu}}{k^2 + i\varepsilon} (ig t^b \gamma^\nu)$$

$$= -g^2 \, t^a t^a \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not p + \not k + m)\gamma_\mu}{[(p+k)^2 - m^2 + i\varepsilon][k^2 + i\varepsilon]}$$

$$\begin{aligned} D &= [(p+k)^2 - m^2 + i\varepsilon]x + [k^2 + i\varepsilon]y \\ &= p^2x + 2pkx + k^2 - m^2x + i\varepsilon \\ &= (k+px)^2 + p^2x(1-x) - m^2x + i\varepsilon \\ &= \ell^2 - \Delta + i\varepsilon \end{aligned}$$

$$\ell = k + px \qquad \Delta = -p^2x(1-x) + m^2x$$

$$= -g^2 \, t^a t^a \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \, \frac{\gamma^\mu (\not p + \not \ell - \not p x - m)\gamma_\mu}{[\ell^2 - \Delta + i\varepsilon]^2}$$

drop odd ℓ
$m \approx 0$

$$\approx -ig^2 \, C(r) \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \frac{\gamma^\mu \not p (1-x)\gamma_\mu}{(\ell_E^2 + \Delta)^2}$$

$$= -ig^2 \, C(r) \int_0^1 dx \, (1-x) \underbrace{\int \frac{d^d\ell_E}{(2\pi)^d} (2-d) \frac{\not p}{(\ell_E^2 + \Delta)^2}}_{\not p \, (\varepsilon - 2) \cdot \frac{1}{(4\pi)^2}\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi)\right]}$$

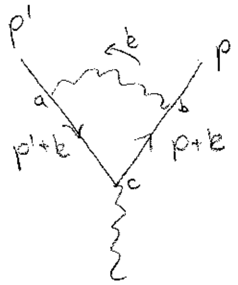
$$= -ig^2 \, C(r) \int dx \, (1-x) \not p \, \frac{1}{(4\pi)^2} \cdot (-2)\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) - 1\right]$$

$$= ig^2 \frac{2 \cdot C(r)}{(4\pi)^2} \int dx \, (1-x) \not p \, \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) - 1\right]$$

counterterm $i\not p \, \delta_\psi$

$$\Rightarrow \delta_\psi = g^2 \frac{C(r)}{8\pi^2} \int_0^1 dx \, (1-x) \left[\frac{2}{\varepsilon} - \log(\mu^2) + \ldots\right] = g^2 \frac{C_2(r)}{16\pi^2} \left[\frac{2}{\varepsilon} - \log(\mu^2) + \ldots\right]$$

$Z_{g\psi} \longrightarrow$  +  vertex correction



$$= \int \frac{d^4k}{(2\pi)^4} \left(igt^a\gamma^\mu\right) \frac{i(\not p' + \not k + m)}{(p'+k)^2 - m^2 + i\varepsilon} \left(igt^c\gamma^\nu\right) \frac{i(\not p + \not k + m)}{(p+k)^2 - m^2 + i\varepsilon} \left(igt^b\gamma^\rho\right) \cdot \frac{-i\delta^{ab} g_{\mu\rho}}{k^2 + i\varepsilon}$$

$$= g^3\, t^a t^c t^a \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu (\not p' + \not k + m)\gamma^\nu (\not p + \not k + m)\gamma_\mu}{[(p'+k)^2 - m^2 + i\varepsilon][(p+k)^2 - m^2 + i\varepsilon][k^2 + i\varepsilon]}}$$

same integral as in QED

$\rightarrow$ use result from lecture

[alternative: set external mom $= 0$]

$$\delta_e = -\frac{\alpha}{2\pi}\int_0^1 dz\,(1-z)\left[\frac{2}{\varepsilon} + \text{finite}\right]$$

$$= -\frac{e^2}{16\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right]$$

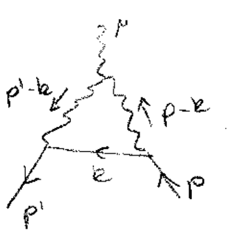
color factor:

$$t^a t^c t^a = t^a t^a t^c + t^a \underbrace{[t^c, t^a]}_{if^{cab}t^b}$$

$$= C(r)\,t^c + \underbrace{if^{cab}}_{\text{antisymm}}\, t^a t^b \quad \searrow = \tfrac{1}{2}[t^a, t^b] \qquad = C_2(r)\,t^c + \frac{i}{2} f^{cab} \underbrace{[t^a, t^b]}_{if^{abd}t^d}$$

$$= C(r)\,t^c - \frac{1}{2}\underbrace{f^{cab} f^{abd}}_{C_2(G)\,\delta^{cd}}\, t^d = \left[C_2(r) - \frac{1}{2}C_2(G)\right] t^c$$

$$\Rightarrow \quad \delta_{g\psi}^{(1)} = -\frac{g^2}{16\pi^2}\left[C_2(r) - \frac{1}{2}C_2(G)\right] \cdot \left[\frac{2}{\varepsilon} - \log(\mu^2) + \dots\right]$$



$$= \int \frac{d^4k}{(2\pi)^4} (ig\gamma_\nu t^b) \frac{i(\not k + m)}{k^2 - m^2 + i\varepsilon} (ig\gamma_\rho t^c) \frac{-i}{(p'-k)^2+i\varepsilon} \frac{-i}{(p-k)^2+i\varepsilon} \cdot$$

$$\cdot\, g f^{abc} \left[g^{\mu\nu}(2p'-p-k)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-p'-k)^\nu \right]$$

$$= ig^3 \underbrace{t^b t^c f^{abc}}_{\frac{1}{2} f^{abc} \cdot i f^{bcd} t^d = \frac{i}{2} C_2(G) t^a} \cdot \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu}{[(p'-k)^2+i\varepsilon][(p-k)^2+i\varepsilon][k^2-m^2+i\varepsilon]}$$

$$N^\mu = \gamma_\nu(\not k + m)\gamma_\rho \cdot \left[g^{\mu\nu}(2p'-p-k)^\rho + g^{\nu\rho}(-p'-p+2k)^\mu + g^{\rho\mu}(2p-p'-k)^\nu \right]$$

$$= \gamma^\mu(\not k + m)(2\not p' - \not p - \not k) + \gamma^\rho(\not k + m)\gamma_\rho(-p'-p+2k)^\mu + (2\not p - \not p' - \not k)(\not k + m)\gamma^\mu$$

To calculate the divergent part of this diagram we can again neglect masses & external momenta $\Rightarrow$

$$N^\mu = \gamma^\mu \not k(-\not k) + \gamma^\rho(\not k)\gamma_\rho(2k^\mu) + (-\not k)(\not k)\gamma^\mu$$

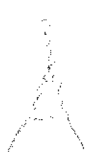
$$= -k^2\gamma^\mu + 2\gamma^\rho \not k \gamma_\rho k^\mu - k^2\gamma^\mu$$

$$= -2k^2\gamma^\mu + 2\underbrace{k^\sigma k^\mu}_{\frac{g^{\sigma\mu}k^2}{d}} \underbrace{\gamma^\rho\gamma_\sigma\gamma_\rho}_{(2-d)\gamma_\sigma}$$

$$= -2k^2\gamma^\mu + \frac{2}{d}k^2(2-d)\gamma^\mu$$

$$= 2k^2\left(-\gamma^\mu + \left(\frac{2}{d}-1\right)\gamma^\mu\right) = 2k^2\gamma^\mu\left(-1 + \left(\frac{2}{d}-1\right)\right) = 4k^2\gamma^\mu\left(\frac{1}{d}-1\right)$$

The denominator becomes $\sim \frac{1}{(k^2)^3}$

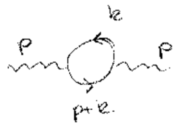
$\Rightarrow$  $= -\frac{1}{2} g^3 C_2(G) t^a \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2}}_{\frac{i}{(4\pi)^2}\left[\frac{2}{\varepsilon} + \text{finite}\right]} \gamma^\mu \underbrace{\left(\frac{4}{d}-4\right)}_{-\left(3+\frac{\varepsilon}{4}\right)}$

$$= \frac{3}{2}\frac{ig^3}{(4\pi)^2} C_2(G) t^a \left[\frac{2}{\varepsilon} + \text{finite} + \frac{1}{2}\right] \cdot \gamma^\mu$$

$$\Rightarrow \delta_{g\psi} = -\frac{g^2}{(4\pi)^2}\left[C(r) + C_2(G)\right] \cdot \left[\frac{2}{\varepsilon} - \log(\mu^2) + \ \right]$$

gauge boson self-energy

n^o fermions



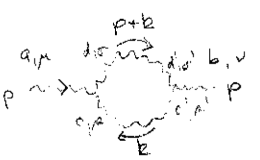
$$= (-1) \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[(ig t^a \gamma^\mu) \frac{i(\not k + m)}{(k^2 - m^2 + i\varepsilon)} (ig t^b \gamma^\nu) \frac{i(\not k + \not p + m)}{(k+p)^2 - m^2 + i\varepsilon} \right] \cdot n_f$$

$$= -g^2 \underbrace{tr[t^a t^b]}_{C(r)\delta^{ab}} \underbrace{\int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma^\mu (\not k + m) \gamma^\nu (\not k + \not p + m)]}{[k^2 - m^2 + i\varepsilon][(k+p)^2 - m^2 + i\varepsilon]}}_{\text{same integral as in QED (ex. sheet 3)}} \cdot n_f$$

$$\hookrightarrow \delta_A^{QED} = \frac{2\alpha}{\pi} \int dx \, x(1-x) \left[\frac{2}{\varepsilon} + \text{finite}\right]$$

$$= \frac{e^2}{12\pi^2} \left[\frac{2}{\varepsilon} + \text{finite}\right]$$

$$\Rightarrow \delta_A^{(1)} \overset{QCD}{=} -g^2 C(r) n_f \cdot \frac{1}{12\pi^2} \left[\frac{2}{\varepsilon} + \text{finite}\right]$$



$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\varepsilon} \frac{-i}{(k+p)^2 + i\varepsilon} g^2 f^{acd} f^{bc'd'} \delta^{cc'} \delta^{dd'} g_{\rho\rho'} g_{\sigma\sigma'} \cdot$$

$$\cdot \left[g^{\mu\rho} (p-k)^\sigma + g^{\rho\sigma} (2k+p)^\mu + g^{\sigma\mu} (-k-2p)^\rho \right] \times$$

$$\times \left[g^{\nu\rho'} (k-p)^{\sigma'} + g^{\rho'\sigma'} (-2k-p)^\nu + g^{\nu\sigma'} (k+2p)^{\rho'} \right]$$

$$= -\frac{1}{2} f^{acd} f^{bcd} g^2 \cdot \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \, \frac{N^{\mu\nu}}{D^2}$$

$$N^{\mu\nu} = \left[g^{\mu\rho} (p-k)^\sigma + g^{\rho\sigma} (2k+p)^\mu - g^{\sigma\mu} (k+2p)^\rho \right] \cdot \left[\delta^\nu{}_\rho (k-p)_\sigma - g_{\rho\sigma} (2k+p)^\nu + \delta^\nu{}_\sigma (k+2p)_\rho \right]$$

$$D = x\left((k+p)^2 + i\varepsilon\right) + (1-x)(k^2 + i\varepsilon) = k^2 + xp^2 + 2kpx + i\varepsilon$$

$$= \underbrace{(k+xp)^2}_{\ell} - \underbrace{[-x(1-x)p^2]}_{\Delta} + i\varepsilon$$

$$N^{\mu\nu} = g^{\mu\nu} (p-k)^\sigma (k-p)_\sigma - (p-k)^\mu (2k+p)^\nu + (p-k)^\nu (k+2p)^\mu$$
$$+ (2k+p)^\mu (k-p)^\nu - \delta^\rho{}_\rho (2k+p)^\mu (2k+p)^\nu + (2k+p)^\mu (k+2p)^\nu$$
$$- (k+2p)^\nu (k-p)^\mu + (k+2p)^\mu (2k+p)^\nu - g^{\nu\mu} (k+2p)^\rho (k+2p)_\rho$$

$$f^{acd} f^{bcd} = C_2(G) \delta^{ab}$$

$$N^{\mu\nu} = g^{\mu\nu}\left(-p^2 - k^2 + 2p\cdot k - k^2 - 4p^2 - 4p\cdot k\right)$$
$$-2p^\mu k^\nu - p^\mu p^\nu + 2k^\mu k^\nu + k^\mu p^\nu + p^\nu k^\mu + 2p^\nu p^\mu - k^\nu k^\mu - 2k^\nu p^\mu$$
$$+ 2k^\mu k^\nu - 2k^\mu p^\nu + p^\mu k^\nu - p^\mu p^\nu - d\cdot\left[4k^\mu k^\nu + 2k^\mu p^\nu + 2p^\mu k^\nu + p^\mu p^\nu\right]$$
$$+ 2k^\mu k^\nu + 4k^\mu p^\nu + p^\mu k^\nu + 2p^\mu p^\nu - k^\nu k^\mu + k^\nu p^\mu - 2p^\nu k^\mu + 2p^\nu p^\mu$$
$$+ 2k^\mu k^\nu + k^\mu p^\nu + 4p^\mu k^\nu + 2p^\mu p^\nu$$

$$= g^{\mu\nu}\left(-5p^2 - 2k^2 - 2p\cdot k\right)$$
$$+ 3p^\mu k^\nu + 6p^\mu p^\nu + 6k^\mu k^\nu + 3k^\mu p^\nu$$
$$- d\left[4k^\mu k^\nu + p^\mu p^\nu + 2\left(k^\mu p^\nu + p^\mu k^\nu\right)\right]$$

$k = \ell - xp$

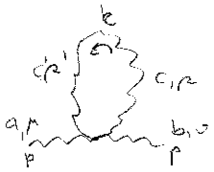
drop odd ℓ

$$= g^{\mu\nu}\left(-5p^2 - 2\ell^2 - 2x^2p^2 + 2p^2x\right)$$
$$- 3p^\mu p^\nu x + 6p^\mu p^\nu + 6\underbrace{\ell^\mu\ell^\nu}_{g^{\mu\nu}\frac{\ell^2}{d}} + 6p^\mu p^\nu x^2 - 3p^\mu p^\nu x$$
$$- d\left[4\underbrace{\ell^\mu\ell^\nu}_{g^{\mu\nu}\frac{\ell^2}{d}} + 4p^\mu p^\nu x^2 + p^\mu p^\nu + 2\left(-p^\mu p^\nu x - p^\mu p^\nu x\right)\right]$$
$$= g^{\mu\nu}\left(-2\ell^2 + \frac{6}{d}\ell^2 - 4\ell^2\right) - g^{\mu\nu}p^2\left(5 + 2x^2 - 2x\right)$$
$$- 6p^\mu p^\nu x + 6p^\mu p^\nu + 6p^\mu p^\nu x^2 - d\left[4p^\mu p^\nu x^2 + p^\mu p^\nu - 4p^\mu p^\nu x\right]$$

$$= -g^{\mu\nu}\ell^2\, 6\left(1 - \frac{1}{d}\right) - g^{\mu\nu}p^2\left[(2-x)^2 + (1+x)^2\right]$$
$$+ p^\mu p^\nu \underbrace{\left[6\left(-x + 1 + x^2\right) - d\left(4x^2 + 1 - 4x\right)\right]}$$
$$= (6-d) + x(1-x)\cdot\left[-6 + 4d\right]$$
$$= \left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right] \quad \text{(Notation Peskin)}$$

$$\Rightarrow \quad \text{(gluon loop)} = -\frac{1}{2}C_2(G)\delta^{ab}g^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{N^{\mu\nu}}{(\ell^2 - \Delta + i\epsilon)^2}$$
$$= -\frac{i}{2}C_2(G)\delta^{ab}g^2\int_0^1 dx\int\frac{d^4\ell_E}{(2\pi)^4}\frac{g^{\mu\nu}\ell_E^2(\ldots) - g^{\mu\nu}p^2(\ldots) + p^\mu p^\nu(\ )}{(\ell_E^2 + \Delta)^2}$$

$$= -\frac{ig^2\delta^{ab}}{2}C_2(G)\int_0^1 dx\left[g^{\mu\nu}6\left(1 - \frac{1}{d}\right)\int\frac{d^4\ell_E}{(2\pi)^4}\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} - \left(g^{\mu\nu}p^2(\ldots) - p^\mu p^\nu(\ )\right)\cdot\int\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{(\ell_E^2+\Delta)^2}\right]$$



$$= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{-i g_{\rho\rho'}\delta^{cc'}}{k^2+i\varepsilon}\,(-ig^2)\Big[f^{abe}f^{cc'e}\left(g^{\mu\rho}g^{\nu\rho'}-g^{\mu\rho'}g^{\nu\rho}\right)$$

$$+ f^{ace}f^{bc'e}\left(g^{\mu\nu}g^{\rho\rho'}-g^{\mu\rho'}g^{\nu\rho}\right) + f^{ac'e}f^{bce}\left(g^{\mu\nu}g^{\rho\rho'}-g^{\mu\rho}g^{\nu\rho'}\right)$$

$$= -\frac{1}{2}g^2\; 2\,\underbrace{f^{ace}f^{bce}}_{C_2(G)\delta^{ab}}\left(g^{\mu\nu}\underbrace{\delta^{\rho}_{\rho}}_{d}-g^{\mu\nu}\right)\cdot\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i\varepsilon}$$

$$= -g^2 C_2(G)\delta^{ab}g^{\mu\nu}(d-1)\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2+i\varepsilon}\cdot\frac{(k+p)^2}{(k+p)^2}$$

not divergent in $d=4$
but in $d=2$ → show
that div. cancels also in $d=2$
w/ other diagrams!

$$= \int\frac{d^4k}{(2\pi)^4}\,\frac{(k+p)^2}{(k^2+i\varepsilon)((k+p)^2+i\varepsilon)}$$

$$D = (k+p)^2x + k^2(1-x) + i\varepsilon = k^2 + xp^2 + i\varepsilon = (k+xp)^2 - x^2p^2 + xp + i\varepsilon$$

$$= \ell^2 - \Delta + i\varepsilon$$

$\ell = k + xp$
$\Delta = -x(1-x)p^2$ ← same Δ as in 

$$= -g^2 C_2(G)\delta^{ab}g^{\mu\nu}(d-1)\int\frac{d^4\ell}{(2\pi)^4}\int dx\,\frac{(\ell+p(1-x))^2}{(\ell^2-\Delta+i\varepsilon)^2}$$

$$= \text{---}\,''\,\text{---}\int dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\ell^2+p^2(1-x)^2}{(\ell-\Delta+i\varepsilon)^2}$$

$$= \text{---}\,''\,\text{---}\,(-i)\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{\ell_E^2-p^2(1-x)^2}{(\ell_E^2+\Delta)^2}$$

$$= ig^2C_2(G)\delta^{ab}g^{\mu\nu}(d-1)\int_0^1 dx\left[\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} - p^2(1-x)^2\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta)^2}\right.$$

$$= \frac{ig^2C_2(G)\delta^{ab}}{(4\pi)^{d/2}}\,g^{\mu\nu}\int_0^1 dx\,(d-1)\left(\frac{1}{\Delta}\right)^{2-d/2}\Big[-\Gamma(1-d/2)\cdot x(1-x)p^2\cdot\frac{d}{2} - p^2(1-x)^2\,\Gamma(2-$$

$$= \frac{ig^2C_2(G)\delta^{ab}}{(4\pi)^{d/2}}\,g^{\mu\nu}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-d/2}p^2\Big[-\Gamma(1-d/2)\,d(d-1)\frac{x}{2}(1-x) - \Gamma(2-d/2)(d-1)(1-x)$$

$$= (-1)\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2+i\varepsilon}\,\frac{i}{(k+p)^2+i\varepsilon}\,(-g)^2 f^{dac}(k+p)^\mu f^{cbd}k^\nu$$

$$= g^2\underbrace{f^{dac}f^{cbd}}_{-C_2(G)\delta^{ab}}\int\frac{d^4k}{(2\pi)^4}\,\frac{k^\mu k^\nu + p^\mu k^\nu}{(k^2+i\varepsilon)((k+p)^2+i\varepsilon)}$$

$$D = (k+p)^2x + k^2(1-x) + i\varepsilon = (k+px)^2 - p^2x^2 + p^2x + i\varepsilon$$

$$= \ell^2 - \Delta + i\varepsilon \qquad \ell = k+px$$

$$\Delta = -x(1-x)p^2 \quad \leftarrow \text{as before!}$$

$$= -g^2 C_2(G)\delta^{ab}\int\frac{d^4\ell}{(2\pi)^4}\int dx\,\frac{\ell^\mu\ell^\nu - p^\mu p^\nu x(1-x)}{(\ell^2-\Delta+i\varepsilon)^2}$$

$$= ig^2 C_2(G)\delta^{ab}\int_0^1 dx\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{\frac{1}{d}\ell_E^2 g^{\mu\nu} + x(1-x)p^\mu p^\nu}{(\ell_E^2+\Delta)^2}$$

$$= ig^2 C_2(G)\delta^{ab}\int_0^1 dx\left[\frac{g^{\mu\nu}}{d}\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{\ell_E^2}{(\ell_E^2+\Delta)^2} + x(1-x)p^\mu p^\nu\int\frac{d^4\ell_E}{(2\pi)^4}\,\frac{1}{(\ell_E^2+\Delta}\right.$$

$$= \frac{ig^2 C_2(G)\delta^{ab}}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-d/2}\left[-\Gamma(1-d/2)\frac{g^{\mu\nu}}{d}x(1-x)p^2\frac{d}{2} + x(1-x)p^\mu p^\nu\cdot\Gamma(2-d/2)\right.$$

$$= \frac{ig^2 C_2(G)\delta^{ab}}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-d/2}\left[-\Gamma(1-d/2)g^{\mu\nu}p^2\frac{1}{2}x(1-x) + p^\mu p^\nu\Gamma(2-d/2)x(1\right.$$

$$= -\frac{ig^2\delta^{ab}C_2(G)}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-d/2}\Big[-\frac{1}{2}g^{\mu\nu}\,6\left(1-\frac{1}{d}\right)\Gamma(1-d/2)\,x(1-x)p^2\frac{d}{2}$$

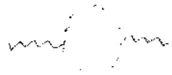
$$-\frac{1}{2}g^{\mu\nu}p^2\left[(2-x)^2+(1+x)^2\right]\Gamma\left(2-\frac{d}{2}\right)$$

$$+\frac{1}{2}p^\mu p^\nu\left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]\Gamma(2-d/2)\Big]$$

$$= \frac{ig^2\delta^{ab}C_2(G)}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{1}{\Delta}\right)^{2-d/2}\Big[\Gamma(1-d/2)g^{\mu\nu}p^2\overbrace{\frac{6}{4}d\left(1-\frac{1}{d}\right)}^{\frac{3}{2}(d-1)}x(1-x)$$

$$+ g^{\mu\nu}p^2\Gamma(2-d/2)\left[\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2\right]$$

$$- p^\mu p^\nu\Gamma(2-d/2)\left[(1-d/2)(1-2x)^2 + (1+x)(2-x)\right]\Big]$$

combine  +  + 

$$= \frac{ig^2\delta^{ab}C_2(G)}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-d/2} \cdot \Big\{ \Gamma(1-d/2)\cdot g^{\mu\nu}p^2 \overset{(1)}{\left[\frac{3}{2}(d-1)x(1-x) - d(d-1)\frac{x}{2}(1-x) - \frac{1}{2}x(1-x)\right]}$$

$$+ \Gamma(2-d/2)\, g^{\mu\nu}p^2 \overset{(2)}{\left[\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 - (d-1)(1-x)^2\right]}$$

$$+ \Gamma(2-d/2)\, p^\mu p^\nu \overset{(3)}{\left[-(1-d/2)(1-2x)^2 \underbrace{-(1+x)(2-x) + x(1-x)}_{\substack{-2-2x+x+x^2+x-x^2 \\ = -2}}\right]} \Big\}$$

$$[(1)] = x(1-x)\left[\frac{3}{2}(d-1) - \frac{d}{2}(d-1) - \frac{1}{2}\right]$$

$$= x(1-x)\left[\frac{3}{2}d - \frac{3}{2} - \frac{1}{2}d^2 + \frac{d}{2} - \frac{1}{2}\right]$$

$$= x(1-x)\left[2d - 2 - \frac{1}{2}d^2\right] = x(1-x)(d-2)\left(1-\frac{d}{2}\right)$$

$$\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \Gamma(2-d/2) \Rightarrow$$

$$(1) \rightarrow \Gamma(2-d/2)\, g^{\mu\nu}p^2\, x(1-x)(d-2)$$

$$\Rightarrow (2)+(1) = \Gamma(2-d/2)\, g^{\mu\nu}p^2 \Big[\underbrace{(d-2)x(1-x) + \frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 - (d-1)(1-x)^2}_{(*)}\Big]$$

$$(*) = (d-2)(x-x^2) + \frac{1}{2}(4-4x+x^2) + \frac{1}{2}(1+2x+x^2) - (d-1)(1-2x+x^2)$$

$$= dx - dx^2 - 2x + 2x^2 + 2 - 2x + \frac{x^2}{2} + \frac{1}{2} + x + \frac{x^2}{2} - d + 2dx - dx^2 + 1 - 2x + x^2$$

$$= -2dx^2 + 3dx - d + 4x^2 - 5x + \frac{7}{2}$$

since Δ is symmetric in $x \leftrightarrow (1-x)$ we can substitute for terms linear in x the variable $\frac{1}{2}x + \frac{1}{2}(1-x)$ in both expressions, $(*)$ and (3). Then we can see that both are (opposite) equal under the integration $\int_0^1 dx$.

Thus we get

 +  + 

$$= \frac{ig^2 \delta^{ab} C_2(G)}{(4\pi)^{d/2}} \int_0^1 dx \left(\frac{1}{\Delta}\right)^{2-d/2} \cdot \Gamma(2-\tfrac{d}{2}) \cdot (g^{\mu\nu}p^2 - p^\mu p^\nu)$$

$$\cdot \left[(1-\tfrac{d}{2})(1-2x)^2 + 2\right]$$

$$\overset{d=4-\varepsilon}{=} \underbrace{\frac{ig^2 \delta^{ab} C_2(G)}{(4\pi)^2} \int_0^1 dx\, (g^{\mu\nu}p^2 - p^\mu p^\nu)}_{(\#)} \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon)\right]$$

$$\cdot \left[\left(\frac{\varepsilon}{2}-1\right)(1-2x)^2 + 2\right]$$

$$= (\#) \cdot \Big\{ \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) - 1\right](1-2x)^2(-1)$$

$$+ 2\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon)\right]\Big\}$$

$$= (\#) \cdot \left\{ \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi)\right] \cdot \left[-(1-2x)^2 + 2\right] + \overset{1-4x+4x^2}{\overset{\shortparallel}{(1-2x)^2}} \right\}$$

$$= \frac{ig^2 \delta^{ab} C_2(G)}{(4\pi)^2} (g^{\mu\nu}p^2 - p^\mu p^\nu) \left\{ \left[\frac{2}{\varepsilon} + \text{finite}\right] \underbrace{\left(-\overbrace{\left(1 - 4\tfrac{1}{2} + 4\cdot\tfrac{1}{3}\right)}^{\frac{1}{3}} + 2\right)}_{5/3} + \frac{1}{3} \right\}$$

$$= \frac{ig^2 \delta^{ab} C_2(G)}{(4\pi)^2} (g^{\mu\nu}p^2 - p^\mu p^\nu) \left\{ \left[\frac{2}{\varepsilon} + \text{finite}\right] \cdot \frac{5}{3} + \frac{1}{3} \right\}$$

$\Rightarrow$ counterterm $\quad i(g^{\mu\nu}p^2 - p^\mu p^\nu)\delta^{ab}\,\delta_A^{(2)}$

$$\delta_A^{(2)} = \frac{g^2}{16\pi^2} C_2(G) \frac{5}{3} \left[\frac{2}{\varepsilon} - \log(\mu^2) + \dots\right]$$

$$\Rightarrow \delta_A = \frac{g^2}{16\pi^2} \left[\frac{5}{3} C_2(G) - \frac{4}{3} C(r) \cdot n_f\right] \cdot [\,\dots\,]$$

QCD Beta - fct.

$$\beta \overset{\text{1-loop}}{\approx} g_0 \, \mu \frac{\partial}{\partial \mu} \left[Z_{\psi} - Z_{g\psi} + \frac{1}{2} Z_A \right]$$

$$\approx 2g \frac{\partial}{\partial \log(\mu^2)} \left[\delta_{\psi} - \delta_{g\psi} + \frac{1}{2} Z_A \right]$$

$$\approx 2g \Big[-g^2 \frac{C_2(r)}{16\pi^2} + \frac{g^2}{16\pi^2} \left[C_2(r) + C_2(G) \right] + \frac{g^2}{2 \cdot 16\pi^2} \Big[\frac{5}{3} C_2(G) - \frac{4}{3} C(r)$$

$$\approx - \frac{2g^3}{16\pi^2} \Big[- C_2(r) + \left(C_2(r) + C_2(G) \right) + \frac{1}{2} \Big[\frac{5}{3} C_2(G) + \frac{4}{3} C(r) \cdot n_f \Big] \Big]$$

$$\approx - \frac{g^3}{16\pi^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f \, C(r) \right]$$