problem sheet 4
to be handed by **Wednesday 7.6.2017 (12:00)** to the letterbox 37 (foyer of Staudingerweg 7)

Renormalisation of Yukawa theory

1. **Divergent Amplitudes of the Yukawa theory (30 P.)**

   The pseudoscalar Yukawa Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - m^2\phi^2 + \bar{\psi}(i\partial\!\!\!/ - M)\psi - ig\bar{\psi}\gamma^5\psi\phi - \frac{\lambda}{4!}\phi^4 \tag{1}$$

   where ϕ is a real scalar field and ψ a Dirac fermion.

   (a) (10 P.) Show that the Lagrangian is invariant under the parity transformation, where

$$\psi(t,\mathbf{x}) \quad \to \quad \gamma^0\psi(t,-\mathbf{x}) \tag{2}$$
$$\phi(t,\mathbf{x}) \quad \to \quad -\phi(t,-\mathbf{x})\,. \tag{3}$$

   (b) (20 P.) Deduce the equation for the superficial degree of divergence D in terms of the external legs of the scalars and the Dirac fermions.(Compare Peskin-Schroeder chapter 10.1 and lecture notes online.) From this equation deduce the four at one-loop level divergent diagrams in this theory (diagrams with $D \geq 0$). Note that the one-point function of the scalar would be divergent ($D = 3$), but it vanishes because of the parity invariance of $\mathcal{L}$. The same is true for the scalar three-point function with $D = 1$.

2. **Renormalisation of Yukawa theory (70 P.)**

   Now we want to renormalise the theory at one-loop level and therefore determine the counterterms for the above identified divergent diagrams. In order to keep the calculations simple we choose the following renormalisation conditions, called zero-momentum subtraction:

   Corrections to the two-point functions are supposed to vanish for both, the scalar and the fermion, and their derivatives vanish at zero momentum, too.

   For the amputated fermion-fermion-scalar three point function and the scalar four-point function we also demand the correction to vanish at zero external momenta. Note that when calculating the amplitudes for these processes, first set the external momenta to zero before solving the loop integral, this simplifies the calculation enormously.

   The calculation of the scalar and fermion two-point functions are similar to the calculations in the last two exercise sheets and we will therefore spare you the work and give you the final results:

   Scalar two-point function

   The scalar two-point function has two contributions, one from the fermion loop ($\Pi_\psi(p^2)$), and another from the scalar loop (Π_ϕ) that is independent from the external momentum p.

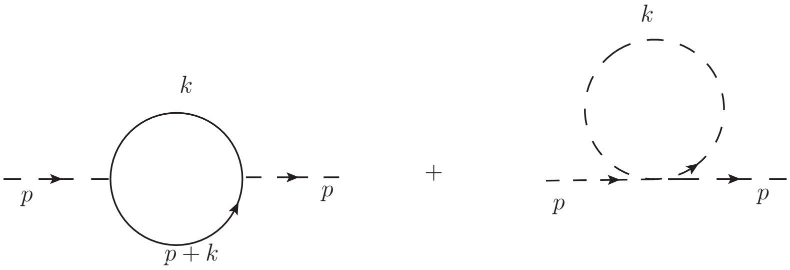


Figure 1: Scalar two-point function $\Pi_\psi(p^2)$ and Π_ϕ.

The contribution from the fermion loop is

$$-i\Pi_\psi(p^2) = g^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr[\gamma^5(\not{k} + M)\gamma^5(\not{p} + \not{k} + M)]}{(k^2 - M^2 + i\varepsilon)((k+p)^2 - M^2 + i\varepsilon)} \,. \tag{4}$$

When introducing Feynman parameters we get

$$\ell = k + xp \quad \text{and} \quad \Delta = -x(1-x)p^2 + M^2 \,. \tag{5}$$

This makes the trace in the numerator become (dropping odd powers of ℓ)

$$4(-\ell^2 + x(1-x)p^2 + M^2) \tag{6}$$

and we get

$$-i\Pi_\psi(p^2) = 4g^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{-\ell^2 + x(1-x)p^2 + M^2}{(\ell^2 - \Delta + i\varepsilon)^2} \,. \tag{7}$$

Wick rotation and dimensional regularisation give us

$$\begin{aligned} -i\Pi_\psi(p^2) &= 4ig^2 \int_0^1 dx \int \frac{d^d\ell_E}{(2\pi)^d} \left[\frac{\ell_E^2}{(\ell_E^2 + \Delta)^2} + \frac{x(1-x)p^2 + M^2}{(\ell_E^2 + \Delta)^2} \right] && (8) \\ &= \frac{ig^2}{4\pi^2} \int_0^1 dx \left\{ (-2\Delta) \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \frac{1}{2} + \mathcal{O}(\varepsilon) \right] \right. && (9) \\ &+ \left. (x(1-x)p^2 + M^2) \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon) \right] \right\} \end{aligned}$$

The contribution from the scalar loop is independent of the external momentum:

$$\begin{aligned} -i\Pi_\phi &= \frac{i\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} && (10) \\ &= \frac{i\lambda}{2} \int \frac{d_E^d k}{(2\pi)^d} \frac{1}{k_E^2 + m^2} && (11) \\ &= \frac{-i\lambda m^2}{32\pi^2} \left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log(m^2) + 1 + \mathcal{O}(\varepsilon) \right] && (12) \end{aligned}$$

Thus the scalar two-point function is given by the sum of the two contributions:

$$\begin{aligned}-i\Pi_\psi(p^2) - i\Pi_\phi &= \frac{ig^2}{4\pi^2}\int_0^1 dx\left\{(-2\Delta)\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \frac{1}{2} + \mathcal{O}(\varepsilon)\right]\right. \\ &+ \left.(x(1-x)p^2 + M^2)\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon)\right]\right\} \quad (13) \\ &- \frac{i\lambda m^2}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log(m^2) + 1 + \mathcal{O}(\varepsilon)\right],\end{aligned}$$

where $\Delta = -x(1-x)p^2 + M^2$.

The counterterm is defined in Figure 3. The zero-momentum subtraction scheme gives us the following renormalisation conditions:

$$\left[\Pi_\psi(p^2) + \Pi_\phi + (Z_\phi - 1)p^2 + (Z_m - 1)m^2\right]_{p^2=0} = 0 \quad (14)$$

$$\frac{d}{dp^2}[\Pi_\psi(p^2) + \Pi_\phi + (Z_\phi - 1)p^2 + (Z_m - 1)m^2]\Big|_{p^2=0} = 0 \quad (15)$$

making

$$\begin{aligned}(Z_m - 1) &= \frac{-1}{m^2}(\Pi_\psi(p^2 = 0) + \Pi_\phi) \quad (16) \\ &= \frac{-g^2M^2}{4m^2\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log(M^2) + 1\right] \\ &\quad -\frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log(m^2) + 1\right] \quad (17) \\ &= \frac{-g^2M^2}{4m^2\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right] - \frac{\lambda}{32\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right] \quad (18) \\ (Z_\phi - 1) &= \frac{g^2}{8\pi^2}\left[\frac{2}{\varepsilon} - \log(M^2) - \gamma + \log(4\pi) + \frac{1}{3}\right] \quad (19) \\ &= \frac{g^2}{8\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right]. \quad (20)\end{aligned}$$

Fermion two-point function

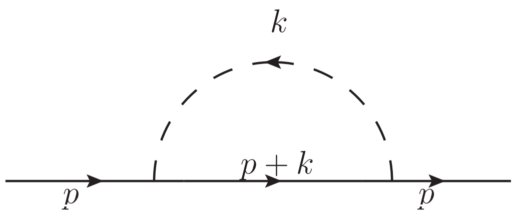
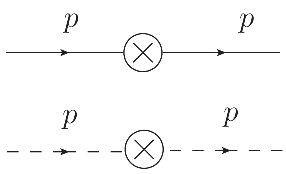


Figure 2: Fermion two-point function $-i\Sigma(\not p)$.

The fermion two-point function (see Figure 2) is given by:

$$-i\Sigma(\not p) = (ig)^2\int\frac{d^4k}{(2\pi)^4}\gamma^5\frac{i}{k^2 - m^2 + i\varepsilon}\gamma^5\frac{i(\not k + \not p + M)}{(k+p)^2 - M^2 + i\varepsilon}. \quad (21)$$



$$= i((Z_\psi - 1)\not{p} - (Z_M - 1)M)$$

$$= i[(Z_\phi - 1)p^2 + (Z_m - 1)m^2]$$

Figure 3: Counterterms for the fermion and scalar two-point functions.

After introducing Feynman parameters and shifting $k \to \ell$ we get

$$-i\Sigma(\not{p}) = g^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{(1-x)\not{p} + M}{(\ell^2 - \Delta + i\varepsilon)^2}\,, \tag{22}$$

with $\ell = k + xp$ and $\Delta = -x(1-x)p^2 + (1-x)m^2 + xM^2$. Wick rotation and dimensional regularisation give us

$$-i\Sigma(\not{p}) = \frac{-ig^2}{16\pi^2} \int_0^1 dx(\not{p}(1-x) + M)\left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\varepsilon)\right]. \tag{23}$$

In Figure 3 the counterterm for the fermion two-point function is defined. The renormalisation conditions we want to impose give us

$$\Sigma(0) + (Z_M - 1)M = 0 \tag{24}$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=0} - (Z_\psi - 1) = 0 \tag{25}$$

Therefore we get for the counterterms

$$(Z_M - 1) = \frac{-g^2}{16\pi^2} \int_0^1 dx \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi)\right] \tag{26}$$

$$= \frac{-g^2}{16\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right] \tag{27}$$

and

$$(Z_\psi - 1) = \frac{g^2}{16\pi^2} \int_0^1 dx(1-x) \left[\frac{2}{\varepsilon} - \log\Delta - \gamma + \log(4\pi)\right] \tag{28}$$

$$= \frac{g^2}{32\pi^2}\left[\frac{2}{\varepsilon} + \text{finite}\right] \tag{29}$$

with $\Delta = (1-x)m^2 + xM^2$.

(a) (35 P.) Write down the amplitude of the **fermion-fermion-scalar vertex** correction and calculate the counterterm in the zero-momentum subtraction scheme.

(b) (35 P.) Write down the **scalar four-point function** correction and also calculate the counterterm in the zero-momentum subtraction scheme.

(Remember you can set the external momenta to zero before solving the loop integral.)