problem sheet 5
to be handed by **Friday 21.6.2017 (12:00)** to the letterbox 37 (foyer of Staudingerweg 7)

Beta Functions

1. **Beta Function of ϕ^4-theory (30 P.)**

   The ϕ^4-theory has the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \quad (1)$$
$$= \frac{1}{2}Z_\phi(\partial_\mu \phi)^2 - \frac{1}{2}Z_m m^2\phi^2 - \frac{Z_\lambda \lambda}{4!}\phi^4 \,. \quad (2)$$

   Write down the relations of the bare (indicated by the subscript 0) and renormalised quantities.

   In a former exercise we calculated the two-point function at one-loop level. To be able to calculate the beta-function we still need the four-point function to obtain the vertex correction. There are three diagrams contributing at one-loop level: the s-, t- and u-channel. Note that in order to obtain the beta function, we can focus on the divergent parts of the solutions! Use dimensional regularisation and the $\overline{MS}$ renormalisation scheme. From that calculate the beta function

$$\beta(\lambda) = \mu\frac{d\lambda}{d\mu}\,, \quad (3)$$

   where μ is the renormalisation scale. Now solve the equation for $\lambda(\mu)$. What do you observe as μ becomes very large?

2. **Beta Function of Yukawa Theory (30 P.)**

   The Lagrangian in terms of bare and renormalised quantities is given by:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - m_0^2\phi_0^2 + \bar{\psi}_0(i\partial\!\!\!/ - M_0)\psi_0 - ig_0\bar{\psi}_0\gamma^5\psi_0\phi_0 - \frac{\lambda_0}{4!}\phi_0^4 \quad (4)$$
$$= \frac{1}{2}Z_\phi(\partial_\mu \phi)^2 - Z_m m^2\phi^2 + Z_\psi\bar{\psi}(i\partial\!\!\!/)\psi - Z_M\bar{\psi}M\psi - iZ_g g\bar{\psi}\gamma^5\psi\phi - \frac{Z_\lambda\lambda}{4!}\phi^4 \,. (5)$$

   In the last exercises we calculated the divergent diagrams of the Yukawa theory at one-loop level in the zero-momentum substraction scheme. To calculate the beta function we only need the divergent parts of the amplitudes, setting the external momenta to zero is still valid. We obtain the same value for the $2/\varepsilon$ term, which is always standing next to the $\log(\mu^2)$ term. Use the obtained results to calculate the beta functions for the couplings g and λ:

$$\beta_g(g,\lambda) = \mu\frac{dg}{d\mu}\,, \quad (6)$$
$$\beta_\lambda(g,\lambda) = \mu\frac{d\lambda}{d\mu}\,. \quad (7)$$

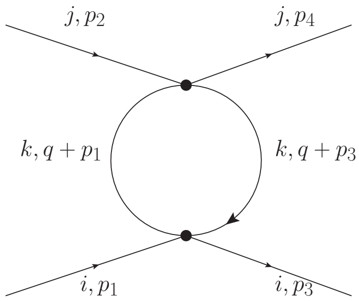
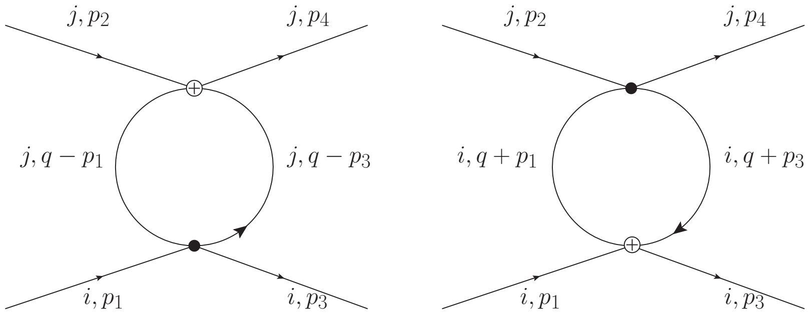


Figure 1: t-channel with ψ_k in the loop.



(a) t-channel with ψ_j in the loop. (b) t-channel with ψ_i in the loop.

Figure 2: t-channel diagrams with "open" loops. The black vertex symbolises that the spinor indices are contracted such, that there is a closed fermion loop, while at the crossed vertex the fermion loop is "open".

3. **Beta Function of the Gross-Neveu model (40 P.)**

   We want to compute the beta function of the Gross-Neveu model with the Lagrangian:

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i i \partial\!\!\!/ \psi_i + \frac{1}{2} g_0^2 \left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i \right)^2 . \tag{8}$$

   Calculate the four-point function $\psi_i(p_1)\psi_j(p_2) \to \psi_i(p_3)\psi_j(p_4)$ for $i \neq j$. In Figures 1 and 2 you see the contributing diagrams. The s- and u-channel are irrelevant for the beta function as their infinite parts cancel out. There are two types of vertices depicted in the figures, since in one case the spinor indices of the fields are contracted such, that the fermion loop closes, while in the other an internal line contracts to an external one. Note that in the latter case there is no additional minus sign and no trace in the amplitude. The Feynman rule for the vertex is given by ig^2.

   Compute the amplitudes of the diagrams using dimensional regularisation and use the $\overline{MS}$ scheme to renormalise the quantities (again concentrate on the divergent parts). Then calculate the beta function

$$\beta(g) = \mu \frac{dg}{d\mu} . \tag{9}$$

   Now solve the equation for $g(\mu)$. How does this coupling behave at large scales μ?