THEORETICAL ELEMENTARY PARTICLE PHYSICS

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1. Introduction and recap

1.1 Do you recognize the following equations?

- $i\hbar \dot{\psi} = \hat{H} \psi$ ... Schrödinger equation
- $a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$ ..... creation operator (harmonic oscillator)
- $\partial_t^2 \psi - \vec{\nabla}^2 \psi - m^2 \psi = 0$ ... Klein-Gordon equation

  or: $\partial_\mu \partial^\mu \psi - m^2 \psi = 0$
- $i \not{\partial}_\mu \psi - m \psi = 0$ ... Dirac equation
- $\partial_\mu \frac{\delta \mathcal{L}(\phi, \partial_\mu \phi)}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0$ ..... Euler - Lagrange equations in field theory
- $\frac{d\sigma}{d\Omega} = |f(\Theta, \phi)|^2$ ..... Differential x-sec as function of scattering amplitude
- $j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta\phi - J^\mu$ ... Noether's theorem for a Lagrangian transforming under $\phi \to \phi + \alpha \Delta\phi$ as $\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu J^\mu$

1.2 Conventions and Units

Metric: $g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g^{\mu\nu}$

Natural units: $\hbar = c = 1 \quad \Rightarrow$ e.g. $E^2 - \vec{p}^2 = m^2$

$\hookrightarrow \hbar \cdot c = 6.58 \cdot 10^{-16}\, eV s \cdot 3 \cdot 10^8 \frac{m}{s} = 197 \cdot 10^{-9} \underbrace{eV \cdot m}_{= MeV\, fm}$

$\Rightarrow 1\,m = 5.066 \cdot 10^6\, eV^{-1}$

$1\,s = 1.52 \cdot 10^{15}\, eV^{-1}$

Cross sections measured in barn : $1\,\text{barn} = 10^{-28}\,m^2$

$1\,pb = 10^{-12}\,b = 10^{-40}\,m^2$

$1\,fb = 10^{-15}\,b = 10^{-43}\,m^2$

[Peskin sec. 2]

1.3 Canonical quantization of scalar field theory

Consider a real scalar field $\phi(x)$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

[ask whether notation is clear!]

$$\Rightarrow \text{Eq. of motion: } \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \Rightarrow \boxed{\partial_\mu \partial^\mu \phi + m^2 \phi = 0}$$

... Klein-Gordon-eq.

Solution $\phi(x) \equiv \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\vec{x}}\, \phi_{\vec{p}}(t)$

Plug into e.o.m.:

$$\left(\frac{\partial^2}{\partial t^2} + \vec{p}^{\,2} + m^2\right) \phi_{\vec{p}}(t) = 0 \qquad \text{for each } \vec{p}$$

This is the eq. of motion for a harmonic oscillator with frequency $\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$, Hamilton operator $\hat{H}_{\vec{p}} = \frac{1}{2}\underbrace{\hat{\dot{\phi}}_{\vec{p}}^2}_{\equiv \hat{\pi}_{\vec{p}}} + \frac{1}{2}\omega_{\vec{p}}^2 \underbrace{\hat{\phi}_{\vec{p}}^2}_{\text{"coordinate"}}$

We know how to quantize the harmonic oscillator: Ladder operators

operator

$$\hat{\phi}_{\vec{p}} \equiv \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(\hat{a}_{\vec{p}} + \underbrace{\hat{a}^{+}_{-\vec{p}}}_{\text{convention!}}\right) \quad ; \quad \underbrace{\hat{\pi}_{\vec{p}}}_{\text{canonical momentum operator}} = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(\hat{a}_{\vec{p}} - \hat{a}^{+}_{\vec{p}}\right)$$

$$\left[\hat{a}_{\vec{p}}, \hat{a}^{+}_{\vec{p}'}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \qquad \text{(note normalization convention)}$$

$$\hat{H} = \omega_{\vec{p}}\left(\hat{a}^{+}_{\vec{p}}\hat{a}_{\vec{p}} + \underbrace{\frac{1}{2}(2\pi)^3 \delta(0)}_{\substack{\text{infinite zero point energy can be removed by} \\ \text{shifting "zero" energy (experiments only sensitive to} \\ \text{energy differences)}}}\right)$$

Eigenstates: $|\vec{p}\rangle \equiv a^{+}_{\vec{p}} \cdot \sqrt{2\omega_{\vec{p}}}\; \underbrace{|0\rangle}_{\text{vacuum state (zero mode)}}$

$$|n_{\vec{p}}\rangle \equiv \frac{1}{\sqrt{n_{\vec{p}}!}} \sqrt{2\omega_{\vec{p}}}\,\left(a^{+}_{\vec{p}}\right)^{n_{\vec{p}}} |0\rangle$$

Normalization: $\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 (2\omega_{\vec{p}})\, \delta^{(3)}(\vec{p} - \vec{q})$

$\Rightarrow$ Field operator in coordinate space (one harmonic oscillator for each $\vec{p}$ mode)

$$\boxed{\begin{aligned} \hat{\phi}(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\vec{x}} \cdot \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}} + \hat{a}^+_{-\vec{p}}\right) \\ &\overset{\text{rearrange terms}}{=} \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}}\, e^{i\vec{p}\vec{x}} + \hat{a}^+_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right) \end{aligned}}$$

$\hat{a}_{\vec{p}}$: annihilates particle with momentum $\vec{p}$

$\hat{a}^+_{\vec{p}}$: creates particle with momentum $\vec{p}$

Hamiltonian (energy)

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left[(\partial_0\phi)^2 - \mathcal{L} \right]$$

$$= \int \left[\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] d^3x$$

$$\overset{\text{Exercise}}{=} \int \frac{d^3p}{(2\pi)^3} \sqrt{\vec{p}^2 + m^2} \left(a^+_{\vec{p}} a_{\vec{p}} + \underbrace{\frac{1}{2}\left[a_{\vec{p}}, a^+_{\vec{p}}\right]}_{\text{infinite zero point energy will again be neglected}}\right)$$

Field states: e.g. $|\vec{p}\rangle \equiv \sqrt{2\omega_{\vec{p}}}\, a^+_{\vec{p}} |0\rangle$; $|\psi\rangle = \frac{1}{\sqrt{2!}} \sqrt{2E_{\vec{p}_1}}\, a^+_{\vec{p}_1} \sqrt{2E_{\vec{p}_2}} \left(a^+_{\vec{p}_2}\right)^2 |0\rangle$

general: $\boxed{\frac{1}{\sqrt{n_1!\, n_2! \dots}} \sqrt{2E_{\vec{p}_1}} \left(a^+_{\vec{p}_1}\right)^{n_1} \sqrt{2E_{\vec{p}_2}} \left(a^+_{\vec{p}_2}\right)^{n_2} \cdots |0\rangle}$

Note on Lorentz invariance:

- $\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}$ is Lorentz invariant

Proof: $\underbrace{\int \frac{d^4p}{(2\pi)^4} (2\pi)\, \delta(p^2 - m^2)\, \Theta(p^0)}_{\text{manifestly Lorentz-invariant}}$

$$= \int \frac{d^3p}{(2\pi)^3}\, dp^0\, \delta\left((p^0)^2 - \vec{p}^2 - m^2\right) \Theta(p^0)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0}\, dp^0\, \delta\left(p^0 - \sqrt{\vec{p}^2 + m^2}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\vec{p}^2 + m^2}}$$

- Our normalization of states

$$\langle \vec{p} \,|\, \vec{q} \rangle = \langle 0 |\, a_{\vec{p}}\, a^\dagger_{\vec{q}} \,| 0 \rangle \sqrt{2\omega_{\vec{p}}}\sqrt{2\omega_{\vec{q}}}$$

$$= (2\pi)^3 (2\omega_{\vec{p}})\, \delta^{(3)}(\vec{p}-\vec{q})$$

is Lorentz-covariant if creation/annihilation operators transform as

$$\underbrace{\mathcal{U}(\Lambda)}_{\text{Lorentz trafo in Hilbert space}}\, a^\dagger_{\vec{p}}\, \mathcal{U}(\underbrace{\Lambda}_{\text{Lorentz trafo in coordinate / momentum space}})^{-1} = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\, a^\dagger_{\Lambda\vec{p}}$$

The propagator

Switch to Heisenberg picture (t-dependence in the operators) for $\hat{\phi}(\vec{x})$

$$\hookrightarrow \boxed{\hat{\phi}(\vec{x},t) = e^{i\hat{H}t}\,\hat{\phi}(\vec{x})\, e^{-i\hat{H}t} \overset{\uparrow}{=} \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left(\hat{a}_{\vec{p}}\, e^{-ipx} + \hat{a}^\dagger_{\vec{p}}\, e^{ipx} \right)}$$

$[\hat{H}, \hat{a}^\dagger_{\vec{p}}] = \omega_{\vec{p}}$

$[\hat{H}, \hat{a}_{\vec{p}}] = -\omega_{\vec{p}}$

Consider $D(x-y) \equiv \langle 0 |\, \hat{\phi}(x)\,\hat{\phi}(y) \,| 0 \rangle$, the amplitude for creating a particle at $y = (\vec{y}, t_y)$ and annihilating it at $x = (\vec{x}, t_x)$ $\Rightarrow$ propagation of particle

$$D(x-y) = \langle 0 | \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}} \left(\hat{a}_{\vec{p}}\, e^{-ipx} + \hat{a}^\dagger_{\vec{p}}\, e^{ipx} \right)$$

$$\cdot \left(\hat{a}_{\vec{q}}\, e^{-iqy} + \hat{a}^\dagger_{\vec{q}}\, e^{iqy} \right) | 0 \rangle$$

$\langle 0 | a_{\vec{p}}\, a^\dagger_{\vec{q}} | 0 \rangle = (2\pi)^3 \delta(\vec{p}-\vec{q})$

$\langle 0 | a^\dagger_{\vec{p}}\, a_{\vec{q}} | 0 \rangle = 0$

$$\overset{\downarrow}{=} \int \frac{d^3p}{(2\pi)^3 \cdot 2\omega_{\vec{p}}}\, e^{-ip(x-y)}$$

[Note that this is Lorentz-invariant!]

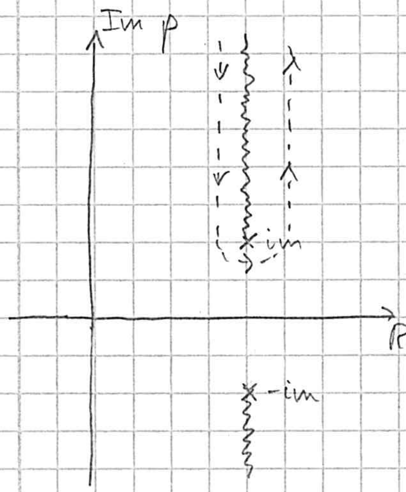
Purely timelike $x-y$: $x^0 - y^0 = t$; $\vec{x} = \vec{y} = 0$

$$\Rightarrow D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{\vec{p}^2+m^2}} e^{-i\sqrt{\vec{p}^2+m^2}t}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\overset{t\to\infty}{\sim} e^{-imt}$$

Purely spacelike $x-y$: $x^0 - y^0 = 0$; $\vec{x} - \vec{y} = \vec{r}$



$$\Rightarrow D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{i\vec{p}\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\omega_{\vec{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$

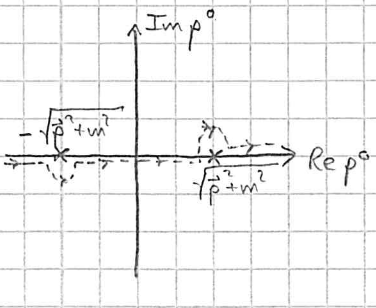
$$\overset{\rho = -ip}{=} \frac{1}{8\pi^2 r} \cdot 2 \int_m^\infty d\rho \frac{i\rho e^{-\rho r}}{\sqrt{-\rho^2+m^2}}$$

$$\overset{r\to\infty}{\sim} e^{-mr}$$ ..... spacelike propagation exponentially suppressed

We do not want to deal with negative energy states propagating backwards in time. Fortunately, they can always be replaced by positive energy antiparticles propagating forward in time. Here: real field $\Rightarrow$ particles are their own antiparticles

↳ More useful than $D(x-y)$ is the Feynman propagator

$$D_F(x-y) \equiv \langle 0 | \overset{\text{Time ordering}}{T} \phi(x) \phi(y) | 0 \rangle$$



$$= D(x-y)\,\Theta(x^0-y^0) + D(y-x)\,\Theta(y^0-x^0)$$

$$= -\int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \left[\frac{1}{p^0 - \sqrt{\vec{p}^2+m^2}+i\epsilon}\right]\left[\frac{1}{p^0 + \sqrt{\vec{p}^2+m^2}-i\epsilon}\right] e^{-ip(x-y)}$$

$$= \boxed{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-y)}}$$

Schwabl: Advanced QM

1.4 Interacting ϕ^4 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$$

$$H = \int d^3x \, \mathcal{H} = \int d^3x \left[(\partial_0 \phi)^2 - \mathcal{L} \right]$$

$$= \underbrace{\int d^3x \left[\frac{1}{2}(\partial_0 \phi)^2 + (\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 \right]}_{\equiv H_0(t)} + \underbrace{\int d^3x \, \frac{\lambda}{4!}\phi^4}_{\equiv H_I(t)}$$

Interaction picture: t-dependence corresponding to H_0 in the operators, t-dependence from H_I in the states
$\Rightarrow$ states and operators are identical to the Heisenberg picture states and operators of the free (non-interacting) theory considered above

e.g. $$\phi_I(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx} \right)$$

$$|\vec{p}\rangle = a_{\vec{p}}^\dagger \sqrt{2\omega_p} \, |0\rangle$$

Full time evolution in interacting theory

$$|\psi_I(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH t_0} |\psi_I(t_0)\rangle$$

back to interaction picture ; t evolution in Schrödinger picture ; interaction $\rightarrow$ Schrödinger representation

$$\equiv \underbrace{\mathcal{U}(t, t_0)}_{\text{time evolution operator}} |\psi_I(t_0)\rangle$$

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The time evolution operator $\mathcal{U}(t, t_0)$ satisfies

$$i \frac{\partial}{\partial t} \mathcal{U}(t, t_0) = H_I(t)\, \mathcal{U}(t, t_0)$$

$$\left[\frac{\partial}{\partial t}\mathcal{U} = iH_0 \mathcal{U} + e^{iH_0 t}(-iH) e^{-iH_0 t} \mathcal{U} = i e^{iH_0 t}(H_0 - H) e^{-iH_0 t} \mathcal{U} \right.$$

Solution (with initial condition $\mathcal{U}(t_0, t_0) = 1$):

$$\mathcal{U}(t, t_0) = 1 - i \int_{t_0}^{t} dt_1 \, H_I(t_1)\, \mathcal{U}(t_1, t_0)$$

$$= 1 - i\int_{t_0}^{t} dt_1\, H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) + \dots$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1) H_I(t_2) \dots H_I(t_n)$$

time ordering

$$= \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \dots \int_{t_0}^{t} dt_n\, T\, H_I(t_1) \dots H_I(t_n)$$

$$= T \exp\left[-i \int_{t_0}^{t} dt'\, H_I(t')\right]$$

$$= T \exp\left[-i \int d^4x\, \mathcal{H}_I(x)\right]$$

When computing matrix elements for scattering processes, we assume that at $t=-\infty$, interactions are irrelevant (wave packets of initial state particles do not overlap), then become relevant, and are again irrelevant at $t=+\infty$.

↳ At $t=\pm\infty$, states are eigenstates of free theory, which are evolved with

$$S \equiv \mathcal{U}(+\infty, -\infty)$$

<u>Example:</u> $2 \to 2$ scattering

$$\mathcal{M} = \langle \vec{p}_3, \vec{p}_4 |\, S\, \underbrace{| \vec{p}_1, \vec{p}_2 \rangle}_{= \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle}$$

$$= \langle \vec{p}_3, \vec{p}_4 | \vec{p}_1, \vec{p}_2 \rangle - i \langle \vec{p}_3, \vec{p}_4 | \int d^4x \frac{\lambda}{4!} \phi^4(x) | \vec{p}_1, \vec{p}_2 \rangle + \dots$$

assume $\vec{p}_1, \vec{p}_2 \neq \vec{p}_3, \vec{p}_4$

$$\overset{}{=} -i\frac{\lambda}{4!} \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\sqrt{2\omega_{\vec{p}_3}}\sqrt{2\omega_{\vec{p}_4}} \int d^4x$$

$$\cdot \langle 0 | a_{\vec{p}_3} a_{\vec{p}_4} \int \frac{d^3k_1}{(2\pi)^3\sqrt{2\omega_{\vec{k}_1}}} \int \frac{d^3k_2}{(2\pi)^3\sqrt{2\omega_{\vec{k}_2}}} \int \frac{d^3k_3}{(2\pi)^3\sqrt{2\omega_{\vec{k}_3}}} \int \frac{d^3k_4}{(2\pi)^3\sqrt{2\omega_{\vec{k}_4}}}$$

$$\left(a_{\vec{k}_1} e^{-ik_1x} + a^\dagger_{\vec{k}_1} e^{ik_1x}\right)\left(a_{\vec{k}_2} e^{-ik_2x} + a^\dagger_{\vec{k}_2} e^{ik_2x}\right)$$

$$\left(a_{\vec{k}_3} e^{-ik_3x} + a^\dagger_{\vec{k}_3} e^{ik_3x}\right)\left(a_{\vec{k}_4} e^{-ik_4x} + a^\dagger_{\vec{k}_4} e^{ik_4x}\right) a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle$$

- 4! possible "contractions" between $a_{\vec{p}_i}, a_{\vec{k}_j}$
- $a_{\vec{k}} a^\dagger_{\vec{p}} |0\rangle = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{k})$

$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

Srednicki sec. 6
Peskin sec. 9

2. Path Integrals

2.1 Path integrals in quantum mechanics

Consider non-relativistic system with

$$H(P,Q) = \frac{P^2}{2m} + V(Q)$$

[<u>Note</u>: No "hats" on operators from now on!

Amplitude for propagation from q' to q'':

$$\langle q''(t)| e^{-iHt} |q'(0)\rangle \underset{N\to\infty}{\overset{\delta t = \frac{t}{N+1}}{=}} \int \left(\prod_{j=1}^{N} dq_j\right) \langle q''| \underbrace{e^{-iH\delta t}}_{\cong 1 - iH\delta t} |q_N\rangle\langle q_N| e^{-iH\delta t} |q_{N-1}\rangle$$

$$\cdots \langle q_1| e^{-iH\delta t} |q'\rangle$$

$$\langle q_2(t_2)| e^{-iH\delta t} |q_1(t_1)\rangle \cong \int \frac{dp_1}{2\pi} \langle q_2(t_2)| \left(1 - i\frac{P^2}{2m}\delta t - iV(Q)\delta t\right|p_1\rangle\langle p_1|q_1(t_1)\rangle$$

$$\langle q|p\rangle = e^{ipq}]$$

$$\overset{(*)}{=} \int \frac{dp_1}{2\pi} \left[1 - i\frac{p_1^2}{2m}\delta t - iV\left(\frac{q_1+q_2}{2}\right)\delta t\right] e^{ip_1(q_2-q_1)}$$

for convenience later

$$\Rightarrow \langle q''(t)| e^{-iHt} |q'(0)\rangle \underset{q_{N+1}=q''}{\overset{q_0=q'}{=}} \int \underbrace{\prod_{\substack{j=0\ldots N\\k=1\ldots N}} dq_k \frac{dp_j}{2\pi}} \left(1 - i\frac{p_j^2}{2m}\delta t\right)\left(1 - i\delta t V\left(\frac{q_k+q_{k+1}}{2}\right)\right) e^{ip_j(q_{k+1}-q_k)}$$

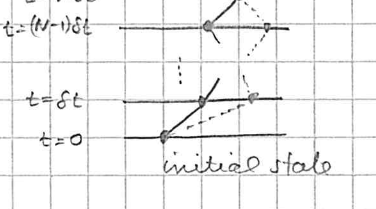
$$\simeq \int \mathcal{D}q\, \mathcal{D}p\, \exp\left[-i\int dt \left(\frac{p^2}{2m} + V(q) - p\dot{q}\right)\right]$$

$$q_{k+1} - q_k = \frac{q_{k+1}-q_k}{\delta t}\delta t \equiv \dot{q}\,\delta t$$

$$= \int \mathcal{D}q\, \mathcal{D}p\, \exp\left[i\int dt\, L(p,q)\right] \qquad (**)$$

Lagrange function

<u>Intuition</u>:



What is this good for?

↳ • Imagine we wish to compute $\langle q''(t) | Q(\tilde{t}) | q'(0) \rangle$ — particle's position at $t=\tilde{t}$

↳ $\langle q''(t) | e^{-iH(t-\tilde{t})} Q \, e^{-iH(\tilde{t}-0)} | q'(0) \rangle$

$$= \int \mathcal{D}q \, \mathcal{D}p \; q(\tilde{t}) \, e^{i\int dt \, L}$$

• Consider $\int \mathcal{D}q \, \mathcal{D}p \; q(t_1) \, q(t_2) \, e^{i\int dt \, L}$

$$= \begin{cases} \langle q''(t) | Q(t_1) Q(t_2) | q'(0) \rangle & \text{for } t_1 \geq t_2 \\ \langle q''(t) | Q(t_2) Q(t_1) | q'(0) \rangle & \text{for } t_1 < t_2 \end{cases}$$

$$= \langle q''(t) | T \, Q(t_1) \, Q(t_2) | q'(0) \rangle$$

=> A way of computing expectation values of time-ordered products!

Generalization to arbitrary Hamiltonian $H(P,Q)$:

Terms containing products of P and Q; equality ⊛ becomes more complicated.

Example: $\langle q_2(t_2) | Q^2 P^2 | q_1(t_1) \rangle = \int \frac{dp_1}{(2\pi)} \, q_2^2 \, p_1^2 \, e^{ip_1(q_2 - q_1)}$

but $\langle q_2(t_2) | P^2 Q^2 | q_1(t_1) \rangle = \int \frac{dp_1}{(2\pi)} \, q_1^2 \, p_1^2 \, e^{ip_1(q_2 - q_1)}$

=> cannot simply write $L(p,q)$ in (**)

=> from now on, we will only consider Hamiltonians for which

$$\langle q_2(t_2) | H(P,Q) | q_1(t_1) \rangle = \int \frac{dp_1}{2\pi} \, H\left(p_1, \frac{q_1+q_2}{2}\right) e^{ip_1(q_2-q_1)}$$

("Weyl-ordered" Hamiltonians). In the above example,

$$\frac{1}{4}\left(Q^2 P^2 + 2 Q P Q + P^2 Q^2\right)$$

would be Weyl-ordered.

Srednicki sec. 9
Peskin sec. 9

2.2 The path integral for a free scalar field

- From sec. 1.3:

$$\mathcal{H} = \frac{1}{2}\underbrace{\Pi}_{=\partial_0\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

In analogy to sec. 2.1, with $q(t) \to \phi(\vec{x},t)$, $p(t) \to \Pi(\vec{x},t)$:

$$\langle \phi''(\vec{x},t) | e^{-iHt} | \phi'(\vec{x},0)\rangle = \int \underbrace{\mathcal{D}\phi}_{\equiv \prod_{i,j} d\phi(\vec{x}^i, t^j)} \mathcal{D}\Pi \, \exp\left[+i\int_0^T dt \int_{-\infty}^{\infty} d^3x \left(\Pi\dot{\phi} - \frac{1}{2}\Pi^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2\right)\right]$$

factor $\frac{1}{2}$ in exp

evaluate $\int \mathcal{D}\Pi \exp\left[+i\int d^4x \frac{1}{2}(-\Pi^2 + 2\Pi\dot{\phi} - \dot{\phi}^2)\right]$
absorb const. into redefinition of $\mathcal{D}\phi$

$$\stackrel{\Downarrow}{=} \int \mathcal{D}\phi \, \exp\left[i\int d^4x \left\{(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right\}\right]$$

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- In QFT, we are often interested in amplitudes where the i.s. and f.s. are the vacuum (in particular the propagator). Define

$$\langle 0(t=+\infty) | 0(t=-\infty)\rangle \equiv Z_0 \equiv \int \mathcal{D}\phi \, \exp\left[i\int_{-\infty}^{\infty} d^4x \, \mathcal{L}\right]$$

- External sources and functional derivatives

Extend Hamiltonian by a term $J(x)\cdot\phi(x)$
(analogy in QM: external force $H \to F(t)\cdot x(t)$, e.g. electric field)

$$\hookrightarrow \quad Z_0 \to Z_0(J) = \int \mathcal{D}\phi \, \exp\left[i\int_{-\infty}^{\infty} d^4x \, (\mathcal{L} + \phi J)\right]$$

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$

Define functional derivative

$$\frac{\delta}{\delta J(x_1)} J(x_2) \equiv \delta(x_1 - x_2)$$

$$\Rightarrow \langle 0|\phi(x_1)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)} Z_0[J]\Big|_{J=0}$$

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z_0[J]\Big|_{J=0}$$

and so on

2.3 Interacting scalar fields and the LSZ reduction formula

Consider theory given by

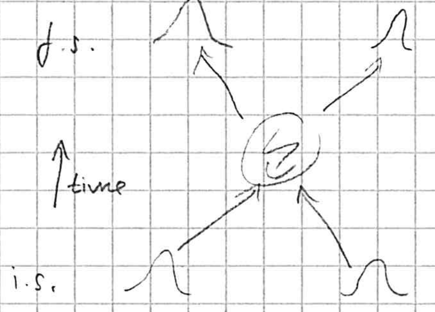
$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4}$$

Problems:

- $2 \to 2$ scattering; initial state $|i\rangle = \hat{a}^\dagger_{\vec{p}_1}\hat{a}^\dagger_{\vec{p}_2}|0\rangle$?

  $\hookrightarrow$ $|i\rangle$ is not eigenstate of Hamiltonian!

  Possible way out: wave packets instead of plane waves



  $\Rightarrow$ Particles don't know of each other at $t = \mp\infty$.

- How to compute higher order terms in

$$\mathcal{M} = \langle \vec{p}_3\, \vec{p}_4 | S | \vec{p}_1, \vec{p}_2 \rangle$$

$$= \langle \vec{p}_3, \vec{p}_4 | \vec{p}_1, \vec{p}_2 \rangle - i\frac{\lambda}{4!}\langle \vec{p}_3, \vec{p}_4 | \int d^4x\, \phi^4(x) | \vec{p}_1, \vec{p}_2 \rangle$$

$$+ \frac{1}{2}\left(-i\frac{\lambda}{4!}\right)^2 \underbrace{\langle \vec{p}_3, \vec{p}_4 | \int d^4x\, d^4y\, \phi^4(x)\phi^4(y) | \vec{p}_1, \vec{p}_2 \rangle}$$

more $\hat{a}$, $\hat{a}^\dagger$ operators from $\phi^4(x)\phi^4(y)$ than from external states;
$\rightarrow$ how to contract?

Srednicki sec. 5

We will now solve these problems.

Goal: Relate scattering amplitudes $\langle f|i\rangle$ to correlation functions $\langle 0|T\phi(x_1)\cdots|0\rangle$

Lagrangian

$\downarrow$

$Z_0[J]$

(compute using Feynman diagrams $\rightarrow$ see later)

$\downarrow$

correlation function $\langle 0|T\phi(x_1)\phi(x_2)\cdots|0\rangle$

$\downarrow$ LSZ formula

scattering amplitudes $\langle f|i\rangle$

$\downarrow$ see Theory 6

cross sections, decay rates

One-particle state in free theory (assume the same in interacting theory)

$$|k\rangle = a^\dagger_{\vec{k}}\,|0\rangle$$

with $a^\dagger_{\vec{k}}(t) = -\frac{i}{\sqrt{2k^0}}\int d^3x\; e^{-ikx}\overleftrightarrow{\partial_0}\,\phi(x) \;=\; \frac{-i}{\sqrt{2k^0}}\int d^3x\left(e^{-ikx}(\partial_0\phi) - (\partial_0 e^{-ikx})\,\phi\right)$, $\quad k^0 = \sqrt{m^2+\vec{k}^2}$

Property: $\quad a_{\vec{k}}(t)|0\rangle = 0 \quad \forall\,\vec{k}$

Normalization: $\quad \langle 0|0\rangle = 1$

$$\langle k|k'\rangle = (2\pi)^3\, 2k^0\, \delta^{(3)}(\vec{k}-\vec{k}'), \quad k^0 = \sqrt{m^2+\vec{k}^2}$$

$$e^{-ikx}\,\partial_0\phi(x) = \int\frac{d^3p}{(2\pi)^3\sqrt{2p^0}}\left(-ip^0\,\hat{a}_{\vec{p}}\,e^{-i(p+k)x} + ip^0\,\hat{a}^\dagger_{\vec{p}}\,e^{i(p-k)x}\right)$$

$$\left(\partial_0 e^{-ikx}\right)\phi(x) = \int\frac{d^3p}{(2\pi)^3\sqrt{2p^0}}\left(-ik^0\,\hat{a}_{\vec{p}}\,e^{-i(p+k)x} - ik^0\,\hat{a}^\dagger_{\vec{p}}\,e^{i(p-k)x}\right)$$

$$\Rightarrow -\frac{i}{\sqrt{2k^0}}\int d^3x\; e^{-ikx}\overleftrightarrow{\partial_0}\,\phi(x) = \frac{-i}{\sqrt{2k^0}}\int\frac{d^3p}{(2\pi)^3\sqrt{2p^0}}\Big[-i(p^0-k^0)\,\hat{a}_{\vec{p}}\,\delta^{(3)}(\vec{p}+\vec{k})\cdot(2\pi)^3\, e^{-i(p^0+k^0)x^0}$$
$$+\frac{-i}{\sqrt{2k^0}}\int\frac{d^3p}{(2\pi)^3\sqrt{2p^0}}\Big[i(p^0+k^0)\,\hat{a}^\dagger_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{k})\cdot(2\pi)^3\, e^{i(p^0-k^0)x^0}$$
$$= 0 + \hat{a}^\dagger_{\vec{p}}$$

Treat particles as wave packets localized at some $\vec{k}_0$

$$\hookrightarrow\; \hat{a}^\dagger_0 \equiv \int d^3k\; f_0(\vec{k})\,\hat{a}^\dagger_{\vec{k}}$$

with e.g. $f_0(\vec{k}) \propto \exp\left[-\frac{(\vec{k}-\vec{k}_0)^2}{4\sigma^2}\right]$

In coordinate space: $f_0(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} f_0(\vec{k})\, e^{-ikx}$

Using $k^0 = \sqrt{\vec{k}^2 + m^2} \simeq \sqrt{\vec{k}_0^2 + m^2} + \underbrace{\frac{\vec{k}_0}{\sqrt{\vec{k}_0^2+m^2}}}_{\equiv \vec{v}_0} \cdot (\vec{k}-\vec{k}_0)$:

$$f_0(\vec{x},t) \;\tilde{\propto}\; \exp\left[-\sigma^2 (\vec{x} - \vec{v}_0 t)^2\right]$$

negelecting dispersion which would require expanding k^0 to 2nd order

$\Rightarrow$ Two distinct particles 0,1 with $\vec{k}_1 \neq \vec{k}_0$ will be far apart at $t \to \pm\infty$
$\to$ interactions negligible at $t \to \pm\infty$. $\to$ can use above states even in interacting theory

Initial state $\quad |i\rangle \equiv \lim_{t\to -\infty} \hat{a}_0^+(t)\, \hat{a}_1^+(t)\, |0\rangle$

[Normalize f_0, f_1 such that $\langle i|i\rangle = 1$

Final state $\quad |f\rangle \equiv \lim_{t\to +\infty} \hat{a}_{0'}^+(t)\, \hat{a}_{1'}^+(t)\, |0\rangle$

To evaluate $\langle f|i\rangle$, we use

$$\hat{a}_0^+(+\infty) - \hat{a}_0^+(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 \hat{a}_0^+(t)$$

$$= \int d^3k\, f_0(\vec{k}) \int d^4x \frac{-i}{\sqrt{2k^0}} \underbrace{\partial_0\left(e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)\right)}$$

$$= -ik^0 \left(e^{-ikx} \overleftrightarrow{\partial_0} \phi(x)\right) + e^{-ikx} \overleftrightarrow{\partial_0} (\partial_0 \phi(x))$$

$$= e^{-ikx}\left[+(k^0)^2 + \partial_0^2\right]\phi(x)$$

$$= \int d^3k\, f_0(\vec{k}) \int d^4x \frac{-i}{\sqrt{2k^0}}\, e^{-ikx}\left[\partial_0^2 + \underbrace{\vec{k}^2}_{\hat{=} -\vec{\nabla}^2} + m^2\right]\phi(x)$$

integrate by parts twice $\quad \stackrel{\downarrow}{=} \int d^3k\, f_0(\vec{k}) \int d^4x \frac{-i}{\sqrt{2k^0}}\, e^{-ikx}\left[\partial^2 + m^2\right]\phi(x) \qquad (*)$

$$\Rightarrow \langle f|i\rangle = \langle 0|\, \hat{a}_{0'}(+\infty)\, \hat{a}_{1'}(+\infty)\, \hat{a}_0^+(-\infty)\, \hat{a}_1^+(-\infty)\, |0\rangle$$

$$= \langle 0|\, T\, \hat{a}_{0'}(+\infty)\, \hat{a}_{1'}(+\infty)\, \hat{a}_0^+(-\infty)\, \hat{a}_1^+(-\infty)\, |0\rangle$$

can add time ordering since operators are time-ordered anyway already rather: $\sigma \to 0$

Use $(*)$ (and its c.c.) with $f_i(\vec{k}) \equiv \sqrt{2k_i^0}\,\delta^{(3)}(\vec{k}-\vec{k}_i)$ to get rid of $\hat{a}_j, \hat{a}_j^\dagger$.
T ordering moves $\hat{a}_j(-\infty)$ to the right $\Rightarrow \hat{a}_j(-\infty)|0\rangle = 0$.
Similar for $\hat{a}_j^\dagger(+\infty)$

$$\hookrightarrow \langle f|i\rangle = \int d^4x_0\, d^4x_1\, d^4x_0'\, d^4x_1'\; e^{ik_0'x_0'}\left(\partial_{0'}^2+m^2\right)$$

$\partial_{0'} = \frac{\partial}{\partial (x_0')^\mu}$

$$\cdot\, e^{ik_1'x_1'}\left(\partial_{1'}^2+m^2\right)$$

$$\cdot\, e^{-ik_0x_0}\left(\partial_0^2+m^2\right)\; e^{-ik_1x_1}\left(\partial_0^2+m^2\right)$$

$$\cdot\, \langle 0|T\,\phi(x_0')\,\phi(x_1')\,\phi(x_0)\,\phi(x_1)|0\rangle$$

In general: n i.s. particles, n' final state particles:

$$\boxed{\begin{aligned}\langle f|i\rangle &= i^{n_i+n_f}\int d^4x_0\, e^{-ik_0x_0}\left(\partial_0^2+m^2\right)\cdots \\ &\quad\cdot \int d^4x_0'\, e^{ik_0'x_0'}\left(\partial_{0'}^2+m^2\right)\cdots \\ &\quad\cdot \langle 0|T\,\phi(x_0)\cdots\phi(x_0')\cdots|0\rangle\end{aligned}}$$

<u>Lehmann-Symanzik-Zimmermann (LSZ) reduction formula</u>

<u>Note:</u> This relies on the assumption that at $t\to\pm\infty$, one-particle states of the <u>interacting theory</u> are given by $\hat{a}_k^\dagger|0\rangle$, with the $\hat{a}_k^\dagger$ of the <u>free theory</u>.

This is true only if

otherwise, $\hat{a}_k^\dagger$ creates superposition of vacuum and 1-particle state

$$\boxed{\begin{aligned}\langle 0|\phi(x)|0\rangle &= 0 \qquad (1)\\ \langle k|\phi(x)|0\rangle &= e^{ikx} \qquad (2)\end{aligned}}$$

otherwise, incorrect normalizat[ion] of 1-particle states

[see Srednicki sec. 5 for details

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If (1), (2) are not satisfied, need to renormalize the theory.
(see later)

2.4 The Feynman propagator and Wick's theorem

Rewrite the action in Fourier space using

$$\tilde{\phi}(k) \equiv \int d^4x \, e^{ikx} \phi(x) \quad ; \quad \phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\phi}(k)$$

Then, with $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \phi J$ (free scalar field w/ external source)

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[\tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k)$$

$$+ \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k) \Big]$$

split into two terms and add factor $\frac{1}{2}$ for later convenience

Note: Fourier trafo is linear $\Rightarrow \int \mathcal{D}\phi$ can be replaced by $\int \mathcal{D}\tilde{\phi}$.

Change variables: $\tilde{\chi}(k) \equiv \tilde{\phi}(k) + \dfrac{\tilde{J}(k)}{k^2 - m^2 + i\varepsilon}$ ("complete the square")

(constant shift $\Rightarrow \int \mathcal{D}\tilde{\phi} \to \int \mathcal{D}\tilde{\chi}$)

$$\hookrightarrow S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k) - \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\varepsilon} \right]$$

$$\Rightarrow \langle 0|0\rangle_J = \int \mathcal{D}\tilde{\chi} \, \exp\left[\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{\chi}(k)(+k^2 - m^2)\tilde{\chi}(-k) \right]$$

$$\cdot \exp\left[-\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2 + i\varepsilon} \right]$$

First line is just $Z_0[J=0]$ with i.c., f.c. $= |0\rangle$. Since non-interacting system initially in ground state will remain there, $\langle 0|0\rangle_{J=0} = 1$

$$\Rightarrow Z_0[J] = \langle 0|0\rangle_J = \exp\left[-\frac{1}{2} \int d^4x \, d^4x' \, J(x) D_F(x-x') J(x') \right]$$

with $\boxed{D_F(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}}$ (looks familiar!)

$$\Rightarrow \quad \langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J(x_2)}\right) Z_0[J]\Big|_{J=0}$$

$$= \frac{\delta}{\delta J(x_1)} \left[\int d^4x' \, D_F(x_2 - x') \, J(x') \right] Z_0[J]\Big|_{J=0}$$

$$= \left[D_F(x_2 - x_1) + (\text{terms containing } J) \right] Z_0[J]\Big|_{J=0}$$

$$= D_F(x_2 - x_1)$$

For higher order correlation functions:

- even number of ϕ factors: e.g.

$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle$$

$$= D_F(x_1 - x_2) \, D_F(x_3 - x_4) + D_F(x_1 - x_3) \, D_F(x_2 - x_4)$$

$$+ D_F(x_1 - x_4) \, D_F(x_2 - x_3)$$

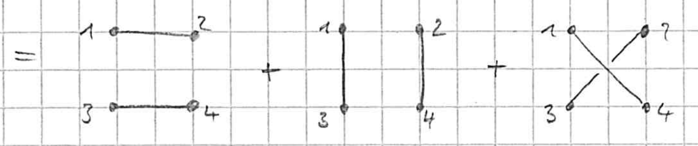
in general

$$\boxed{\langle 0 | T \phi(x_1) \phi(x_2) \cdots \phi(x_{2n}) | 0 \rangle = \sum_{\text{pairings}} D_F(x_{i_1} - x_{i_2}) \cdots D_F(x_{i_{2n-1}} - x_{i_{2n}})}$$

<u>Wick's theorem</u>

Diagrammatically:

$$\langle 0 | T \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle$$



- odd number of ϕ factors: $\boxed{\langle 0 | T \phi(x_1) \cdots \phi(x_{2n-1}) | 0 \rangle = 0}$

Srednicki sec. 9

2.5 Feynman diagrams for interacting scalar fields

Consider theory specified by

$$\boxed{\mathcal{L} = \frac{1}{2} Z_\phi (\partial_\mu \phi)^2 - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{6} Z_g g \phi^3 + Y\phi} \quad (*)$$

Normalization:

$\langle 0|\phi(x)|0\rangle = 0$

$\langle k|\phi(x)|0\rangle = e^{ikx}$

m is the physical mass of the particle

g defines the interaction strength

} required for LSZ formula!

States are normalized according to $\langle 0|0\rangle = 1$, $\langle k'|k\rangle = (2\pi)^3 \cdot 2k^0 \, \delta^{(3)}(\vec{k}-\vec{k}')$.

Note: This theory does not have a ground state (Hamiltonian unbounded from below)

We ignore this problem since it does not affect perturbation theory. But $(*)$ would not be a good theory for a realistic field. More realistic theory: ϕ^4 theory $\rightarrow$ exercises

Path integral:

$$Z[J] \equiv \langle 0|0\rangle_J = \int \mathcal{D}\phi \, \exp\left[i \int d^4x \left(\mathcal{L}_0 + \mathcal{L}_1 + J\phi\right)\right] \quad (1)$$

with

$$\mathcal{L}_0 \equiv \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L}_1 \equiv \frac{1}{6} Z_g g \phi^3 + \underbrace{\frac{1}{2}(Z_\phi - 1)(\partial_\mu \phi)^2 - \frac{1}{2}(Z_m - 1) m^2 \phi^2 + Y\phi}_{\equiv \mathcal{L}_{ct} \text{ ("counterterm Lagrangian")}}$$

we expect for $g \to 0$ that $Z_i \to 1$, $Y \to 0$

show intermediate steps! → Rewrite $Z[J]$ by replacing $\mathcal{L}_1[\phi(x)]$ by $\mathcal{L}_1\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]$

$$\hookrightarrow \quad Z[J] \underset{\substack{\uparrow\\ \text{subtle normalization}\\ \text{issue not discussed here}\\ \text{(related to definition of states at } t=\pm\infty)}}{\propto} \exp\left[i\int d^4x\, \mathcal{L}_1\left[\frac{1}{i}\frac{\delta}{\delta J(x)}\right]\right] \underbrace{\int \mathcal{D}\phi\, \exp\left[i\int d^4x\,(\mathcal{L}_0 + J\phi)\right]}_{=Z_0[J]} \qquad (1')$$

[To see equivalence of (1) and (1'), Taylor-expand first exponential in (1') and evaluate derivatives term by term, then resum.]

First, ignore $\mathcal{L}_{ct}$ and define

$$Z_1[J] \propto \exp\left[\frac{i}{6} Z_g g \int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right] Z_0[J]$$

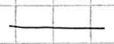
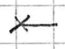
proportionality constant determined by requiring $Z_1[0]=1$.

Taylor-expand in g (weak interaction) and J

$$Z_1[J] \propto \sum_{V=0}^{\infty} \frac{1}{V!}\left[\frac{iZ_g g}{6}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right]^V \cdot \sum_{P=0}^{\infty}\frac{1}{P!}\left[-\frac{1}{2}\int d^4y\, d^4z\; J(y)\, D_F(y-z)\, J(z)\right]^P$$

"V" as in "vertex". "P" as in propagator

To organize terms, use Feynman diagrams, where

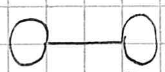
	$=$	$D_F(x-y)$
	$=$	$i\int d^4x\, J(x)$
	$=$	$i Z_g g \int d^4x$

} determine the term up to numerical prefactors

note: Srednicki uses ● — for this →

Examples:

- $V=2$, $P=3$, $E = 2P - 3V = 0$ ← number of "external" lines —×

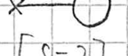


$[S=2^3]$

$[S = 2\cdot 3!]$

$\sim (iZ_g g)^2 \int d^4x_1\, d^4x_2\, D_F^3(x_1-x$

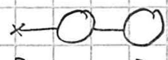
- $V = 1$; $E = 1$ $\quad (P = 2)$



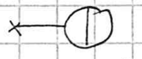
$\sim \; i Z_g g \int d^4x_1 \; i \int d^4x_2 \; J(x_2) \, D_F(x_1 - x_2) \, D_F(x_1 - x_1)$

[$S = 2$]

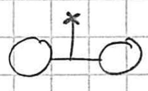
- $V = 3$; $E = 1$ $\quad (P = 4)$



[$S = 2^2$]



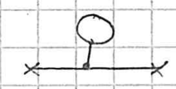
[$S = 2^2$]



[$S = 2^3$]

- $V = 2$; $E = 2$ $\quad (P = 2)$

[$S = 2^2$]



[$S = 2^2$]

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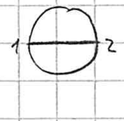
Multiplicity (prefactor) of diagrams

- Consider diagram with P propagators, V vertices
  - permute 3 functional derivatives for each vertex
    ↳ factor $(3!)^V$ $\quad \rightarrow$ cancels $\left(\frac{1}{6}\right)^V$ in $Z_1[J]$
  - permute vertices
    ↳ factor $V!$ $\quad \rightarrow$ cancels $\frac{1}{V!}$ from Taylor series
  - interchange $J(y)$ and $J(z)$ coming with each propagator in $Z_1[J]$
    ↳ factor $2!$ $\quad \rightarrow$ cancels $\frac{1}{2}$ in 2^{nd} line of $Z_1[J]$
  - interchange propagators
    ↳ factor $P!$ $\quad \rightarrow$ cancels $\frac{1}{P!}$ from Taylor series

However: Some of these permutations lead to <u>same</u> match-up of J-factors and derivatives
↳ to avoid double counting, divide by appropriate <u>symmetry factor S</u>.

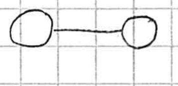
Determine by checking which permutations can be undone by other permutations

Examples:

- 

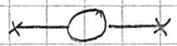
  - permuting propagators (3! possibilities) can be undone by permuting derivatives
  - interchanging vertices (2 possibilities) can be undone by swapping endpoints of each propagator

  ↳ $S = 2 \cdot 3!$

-  $\rightarrow \left(\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_1)}\right)\left(\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_2)}\right)\left[J(y_1)\,J(z_1)\right]\left[J(y_2)\,J(z_2)\right]\left[J(y_3)\,J(z_3)\right]$

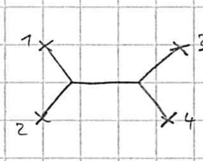
  swapping endpoints of propagators $\Longleftrightarrow$ swapping derivatives

  ↳ $S = 2^3$

- 

  - swap external sources $\longleftrightarrow$ swap vertices
  - swap propagators in loop $\longleftrightarrow$ swap derivatives

  ↳ $S = 2 \cdot 2$

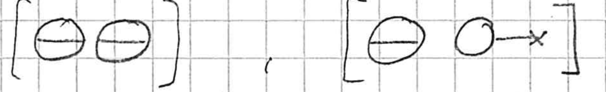
- 

  - swap $1 \leftrightarrow 2$ $\longleftrightarrow$ swap derivatives at LH vertex
  - swap $3 \leftrightarrow 4$ $\longleftrightarrow$ swap derivatives at RH vertex
  - swap $(1,2) \leftrightarrow (3,4)$ $\Longleftrightarrow$ swap vertices

  ↳ $S = 2 \cdot 2 \cdot 2$

Disconnected diagrams

$Z_1[J]$ yields also disconnected diagrams, e.g.



General expression: $\frac{1}{S_D} \cdot \prod_I (C_I)^{n_I}$

n_I: number of C_I factors in the disconnected diagram

S_D: addt'l symmetry factor

$\prod_I$: product over all connected diagrams

C_I: individual connected diagram

S_D arises from permuting identical G_I factors

$$\hookrightarrow \quad S_D = \prod_I n_I !$$

$$\Rightarrow \quad Z_1[J] \propto \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (G_I)^{n_I}$$

$$= \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (G_I)^{n_I}$$

$$= \prod_I \exp[G_I]$$

$$= \exp\left(\Sigma_I G_I\right)$$

$\Rightarrow$ Normalization $Z_1[0] = 1$ is ensured by omitting vacuum diagrams (diagrams without external vertices)
[equivalently: dividing by exponential of sum of all connected vacuum diagrams]

$$\Rightarrow \quad Z_1[J] = \exp\Big(\underbrace{\Sigma_{I \neq \{0\}}}_{\text{"only diagrams with external vertices"}} G_I\Big) \equiv \exp\left(i W_1[J]\right)$$

Srednicki sec 9

2.6 Renormalization and ultraviolet divergencies

Consider

$$\langle 0|\phi(x)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x)} Z_1[J]\Big|_{J=0}$$

$$= \frac{\delta}{\delta J(x)} W_1[J]\Big|_{J=0}$$

[sum of all diagrams with one external line, with the source removed

symmetry factor

$$= \underbrace{\tfrac{1}{2}(-i)\, i^2 g\int d^4y\; D_F(x-y)\, D_F(y-y)}_{=\frac{1}{i}\frac{\delta}{\delta J(x)}\left[\times\!\!-\!\!\bigcirc\right]} + \mathcal{O}(g^3)$$

[factors i from ×— and ⟩—, factor $(-i)$ from $W_1[J] = -i\sum_{I\neq\{0\}} C_I$

Z_g omitted since $Z_g = 1 + \mathcal{O}(g^2)$]

$$\neq 0$$

$\Rightarrow$ LSZ reduction formula not applicable ?!?

Remedy: Add counterterm $\mathcal{L}_{ct} \supset Y\phi$ to make $\langle 0|\phi(x)|0\rangle = 0$ "by hand".

Feynman rule: $\boxed{\otimes\!\!-\!\!- \;=\; iY\int d^4x}$

[Note: Srednicki uses ×— for this

$$\hookrightarrow \langle 0|\phi(x)|0\rangle = \Big(\underbrace{iY}_{\times\!-\!\otimes} + \underbrace{\tfrac{1}{2}ig\,D_F(0)}_{+\;\times\!-\!\bigcirc}\Big)\int d^4y\, D_F(x-y) + \mathcal{O}(g^3)\Big)$$

Choose $Y = -\frac{1}{2} g\, D_F(0) + \mathcal{O}(g^3)$

$$= -\frac{1}{2} g\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon} + \mathcal{O}(g^3$$

Problem: This is infinite !

Ultraviolet divergence for $k\to\infty$.

To deal with the UV divergence, we first need to regularize it. Several possibilities: we choose a UV cutoff Λ, introduced such that $D_F(x-y)$ remains Lorentz-invariant:

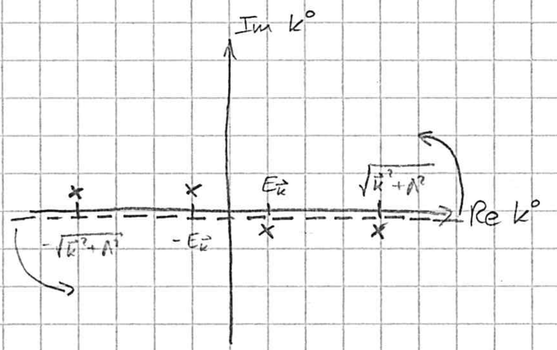
$$D_F(x-y) \longrightarrow \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon} \left(\frac{\Lambda^2}{k^2 - \Lambda^2 + i\varepsilon} \right)^2$$

$$D_F(0) \longrightarrow \frac{1}{(2\pi)^4} \int dk^0 \int d^3k \frac{i}{(k^0)^2 - \vec{k}^2 - m^2 + i\varepsilon} \left(\frac{\Lambda^2}{(k^0)^2 - \vec{k}^2 - \Lambda^2 + i\varepsilon} \right)^2$$

$$= \frac{1}{(2\pi)^4} \int dk^0 \int d^3k \frac{i}{(k^0 - E_{\vec{k}} + i\varepsilon)(k^0 + E_{\vec{k}} - i\varepsilon)}$$

$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

$$\cdot \left(\frac{\Lambda^2}{(k^0 - \sqrt{\vec{k}^2 + \Lambda^2} + i\varepsilon)(k^0 + \sqrt{\vec{k}^2 + \Lambda^2} - i\varepsilon)} \right)^2$$



- At $k^0 \to \pm\infty, \pm i\infty$, the integral vanishes $\to$ can close integration contour at infinity

Residue of outer poles $\pm\sqrt{\vec{k}^2+\Lambda^2} \mp \varepsilon$:

$$\propto \frac{1}{2!} \frac{d}{dk^0} \frac{i}{(k^0 - E_{\vec{k}})(k^0 + E_{\vec{k}})} \left(\frac{\Lambda^2}{k^0 \pm \sqrt{\vec{k}^2+\Lambda^2}} \right)^2 \Bigg|_{k^0 = \pm\sqrt{\vec{k}^2+\Lambda^2}}$$

$$\propto \frac{1}{\Lambda}$$

Residue of inner poles $\pm E_{\vec{k}} \mp \varepsilon$

$$\propto \frac{i}{2E_{\vec{k}}}$$

- Rotate contour by 90° counter-clockwise (now runs along imaginary k^0 axis): $k^0 \to ik^0$. Define 4-dim Euclidean vector $\varkappa \equiv (ik^0, \vec{k})$

$$\hookrightarrow D_F(0) = i \int \frac{d^4\varkappa}{(2\pi)^4} \frac{i}{-\varkappa^2 - m^2} \left(\frac{\Lambda^2}{-\varkappa^2 - \Lambda^2} \right)^2$$

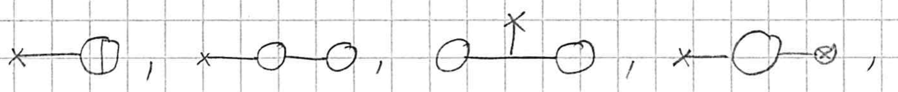
$$= +\frac{1}{(2\pi)^4}\int \underset{\substack{\uparrow\\ \text{surface area}\\ \text{of 3-sphere}}}{2\pi^2}\, \kappa^3\, d\kappa \quad \frac{1}{\kappa^2+m^2}\left(\frac{\Lambda^2}{\kappa^2+\Lambda^2}\right)^2$$

$$= \frac{\Lambda^2}{16\pi^2}$$

Solution: Ignore the problem, accept infinite parameters in the Lagrangian, as long as they do not affect observables.

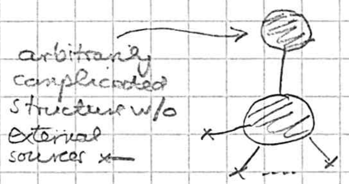
More formally: regularize $D_F(0)$ in the UV [see previous page], show that $\langle 0|\phi(x)|0\rangle$ and other observables remain finite, then send $\Lambda \to \infty$ (Λ = UV cutoff for regularization)

At higher orders in g, the $Y\phi$ counterterm cancels diagrams like



i.e. all diagrams with a single source $\times\!\!-\!\!-$.

$\Longrightarrow$ After renormalization, any diagram of the form



(tadpole diagrams)

vanishes!

end 12.11.2014

Remaining counterterms $\propto Z_\phi - 1$ and $Z_m - 1$ lead to additional vertex

$$\boxed{-\!\otimes\!- \;=\; i\int d^4x \left[-(Z_\phi - 1)\underbrace{\partial_x^2}_{} - (Z_m - 1)\, m^2\right]}$$

acts on one (and only one) of the propagators attached to the vertex

Z_ϕ, Z_m are adjusted to ensure $\langle k|\phi(x)|0\rangle = e^{ikx}$ and m is the physical mass of the particle.

<u>Summary</u>: • Start with theory that (at tree level) satisfies

$\langle 0 \| \phi(x) \| 0 \rangle = 0$
$\langle k \| \phi(x) \| 0 \rangle = e^{ikx}$
m is physical mass
g determined by measured x-sec

(*)

• Compute quantum corrections (loop diagrams)
→ these will spoil (*), often by infinite amounts

• Introduce <u>counterterms</u> to cancel problematic loop diagrams

Srednicki sec. 10

2.7 Scattering amplitudes

Lagrangian $\mathcal{L} = \mathcal{L}_0 + J\phi$ ✓

$\downarrow$

$Z[J]$
(compute using Feynman diagrams) ✓

$\downarrow$

Correlation functions $\langle 0|T\phi(x_1)\phi(x_2)\cdots|0\rangle$
(from functional derivatives of $Z[J]$)

$\downarrow$ LSZ formula

Scattering amplitudes $\langle f|i\rangle$

} this section

$\downarrow$ see Theo 6

Cross sections, decay rates ✓

Correlation functions

Two-point function

Use shorthand notation $\delta_i \equiv \frac{1}{i}\frac{\delta}{\delta J(x_i)}$

$$\hookrightarrow \langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \delta_1\delta_2 \underbrace{Z[J]}_{=\exp(iW[J])}\Big|_{J=0}$$

$$= i\,\delta_1\delta_2 W[J]\Big|_{J=0} - \left(\delta_1 W[J]\Big|_{J=0}\right)\cdot\left(\delta_2 W[J]\Big|_{J=0}\right)$$

$\langle 0|\phi(x)|0\rangle = 0$

$$\overset{\downarrow}{=} i\,\delta_1\delta_2 W[J]\Big|_{J=0}$$

$$= \delta_1\delta_2\left[\ \right]$$ 

sum of all diagrams with this structure

$=$  $= D_F(x_1-x_2) + O(g^2)$

This is called the <u>full propagator</u>.

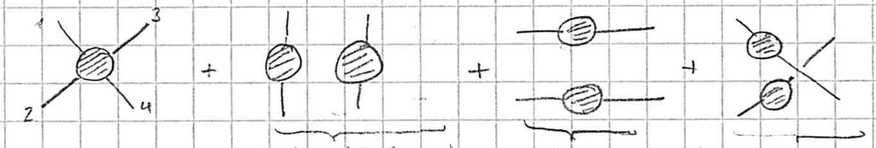
[Remember: In free theory $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D_F(x_1-x_2)$]

Four-point function

$$\langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \delta_1\delta_2\delta_3\delta_4 \underbrace{Z[J]}_{=\exp[iW[J]]}\Big|_{J=0}$$

$$= \Big[\delta_1\delta_2\delta_3\delta_4\, iW[J] + (\delta_1\delta_2\, iW[J])(\delta_3\delta_4\, iW[J])$$

$$+ (\delta_1\delta_3\, iW[J])(\delta_2\delta_4\, iW[J]) + (\delta_1\delta_4\, iW[J])(\delta_2\delta_3\, iW[J])\Big]_{J=0}$$

=  (*)

$D_F(x_1-x_2)D_F(x_3-x_4) + O(g^2)$ $\quad$ $D_F(x_1-x_3)D_F(x_2-x_4) + O(g^2)$ $\quad$ $D_F(x_1-x_4)D_F(x_2-x_3) + O(g^2)$

Apply LSZ formula:

$$\langle f | i\rangle = i^4 \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\; e^{-i(k_1x_1 + k_2x_2 - k_3x_3 - k_4x_4)}$$

particles 3,4 $\quad$ 1,2

$$\cdot\, (\partial_1^2 + m^2)(\partial_2^2 + m^2)(\partial_3^2 + m^2)(\partial_4^2 + m^2)$$

$$\cdot\, \langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle$$

Last 3 terms in (*): e.g.

$$\int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4\; e^{-i(k_1x_1 + k_2x_2 - k_3x_3 - k_4x_4)}$$

$$\cdot\, \underbrace{\left[(\partial_1^2 + m^2)(\partial_3^2 + m^2)D_F(x_1 - x_3)\right]}_{\equiv F(\underbrace{x_1 - x_3}_{\equiv x_{13}})}\left[(\partial_2^2 + m^2)(\partial_4^2 + m^2)D_F(x_2 - x_4)\right]$$

$$+ O(g^2)$$

$x_{ij} \equiv x_i - x_j$

$X_{ij} \equiv \frac{x_i + x_j}{2}$

$\Rightarrow x_i = X_{ij} + \frac{x_{ij}}{2}$

$x_j = X_{ij} - \frac{x_{ij}}{2}$

$$\stackrel{\downarrow}{=} \int d^4x_{13}\, d^4X_{13}\, d^4x_{24}\, d^4X_{24}\; e^{-i\left[(k_1 - k_3)X_{13} + \frac{k_1 + k_3}{2}x_{13} + (k_2 - k_4)X_{24} + \frac{k_2 + k_4}{2}x_{24}\right]}$$

$$\cdot\, F(x_{13})F(x_{24}) + O(g^2)$$

$$= (2\pi)^4\delta^{(4)}(k_1 - k_3)\cdot(2\pi)^4\delta^{(4)}(k_2 - k_4)\cdot\underbrace{\tilde{F}\left(\frac{k_1 - k_2}{2}\right)\tilde{F}\left(\frac{k_3 - k_4}{2}\right)}_{\text{Fourier transform}}$$

$$+ O(g^2)$$

no momentum exchange

↳ no scattering — uninteresting

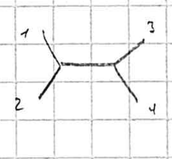
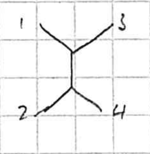
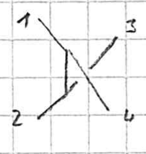
For real scattering processes (with momentum transfer):

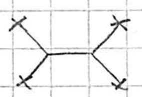
only <u>connected</u> diagrams matter

$$\boxed{\langle 0 | T\phi(x_1) \cdots \phi(x_n) | 0 \rangle_C \equiv \delta_1 \cdots \delta_n \, iW[J] \big|_{J=0}}$$

In our example

$$\langle 0 | T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle_C = \delta_1 \delta_2 \delta_3 \delta_4 \, iW[J] \big|_{J=0}$$

=  +  +  $+ \; O(g^4)$

Start from  $[S=8]$

4! ways of contracting δ_i's with J's.

↳ 3 sets of 8 identical diagrams
divide by symmetry factor 8 → 3 diagrams

$$= (ig)^2 \int d^4y \, d^4z \; D_F(y-z)$$

$$\cdot \Big[D_F(x_1-y) D_F(x_2-y) D_F(x_3-z) D_F(x_4-z)$$

$$+ D_F(x_1-y) D_F(x_3-y) D_F(x_2-z) D_F(x_4-z)$$

$$+ D_F(x_1-y) D_F(x_4-y) D_F(x_2-z) D_F(x_3-z) \Big] + O(g^4)$$

Apply LSZ formula:

Use $$(\partial_i^2 + m^2) \underbrace{D_F(x_i - y)}_{= \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x_i-y)}}{k^2 - m^2 + i\varepsilon}} = -i \, \delta^{(4)}(x_i - y)$$

$$\hookrightarrow \langle f | i \rangle = (ig)^2 \int d^4y \, d^4z \; D_F(y-z) \Big[e^{-i(k_1 y + k_2 y - k_3 z - k_4 z)}$$

$$+ e^{-i(k_1 y + k_2 z - k_3 y - k_4 z)} + e^{-i(k_1 y + k_2 z - k_3 z - k_4 y)} \Big]$$

$$+ O(g^4)$$

$$D_F(y-z) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(y-z)}}{k^2-m^2+i\varepsilon}$$

$$= (ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \cdot \Big[(2\pi)^4 \delta^{(4)}(k_1+k_2+k)\, (2\pi)^4 \delta^{(4)}(k_3+k_4+k)$$

$$+ (2\pi)^4 \delta^{(4)}(k_1-k_3+k) \cdot (2\pi)^4 \delta^{(4)}(-k_2+k_4+k)$$

$$+ (2\pi)^4 \delta^{(4)}(k_1-k_4+k)\, (2\pi)^4 \delta^{(4)}(-k_2+k_3+k) \Big] + \mathcal{O}(g^4)$$

$$= \underbrace{(2\pi)^4 \delta^{(4)}(k_1+k_2-k_3-k_4)}_{\text{overall 4-momentum conservation}}$$

$$\underbrace{(ig)^2 \left[\frac{i}{(k_1+k_2)^2-m^2+i\varepsilon} + \frac{i}{(k_1-k_3)^2-m^2+i\varepsilon} + \frac{i}{(k_1-k_4)^2-m^2+i\varepsilon} \right] + \mathcal{O}(g^4)}_{\equiv\, i\mathcal{T} \text{ (scattering matrix element)}}$$

end 18.11.2014

This procedure works in general!

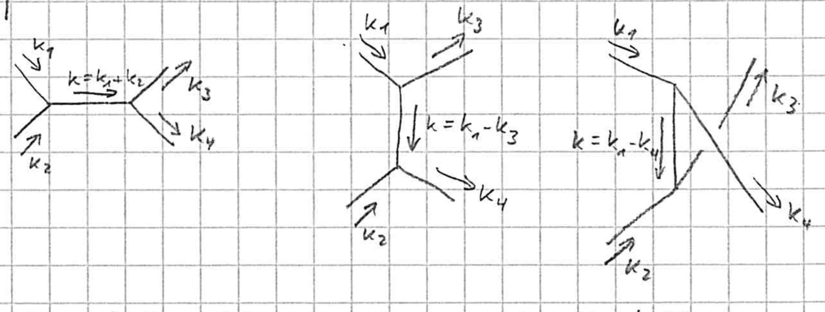
↳ **Feynman rules** for computing $\langle f | i \rangle$

1. Draw a line for each external (initial or final state particle)

2. Connect inner end of each line to a vertex $\succ$, include extra internal lines to achieve this

3. Assign 4-momenta to the lines, conserve 4-momentum at each vertex. (Direction of momentum flow on internal lines is arbitrary.) Drawing little arrows may help

   Example:



4. The value of the diagram is given by the following factors:
   - External lines : 1
   - Internal line with 4-momentum k : $\frac{i}{k^2-m^2+i\varepsilon}$
   - Vertex : $i\, Z_g\, g$

5. In loop diagrams, some internal momenta l_i are not fixed by 4-momentum conservation. Integrate over them: $\int d^4 l_i \,/(2\pi)^4$.

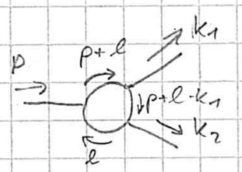
6. For loop diagrams: divide by symmetry factor, given by exchanges of internal propagators that leave the diagram invariant.

7. Include diagrams with counterterm vertices

$$-\otimes- = i\left[(Z_\phi - 1)k^2 - (Z_m - 1)m^2\right] \quad (\text{is of } O(g^2))$$

8. Sum over all diagrams

<u>Example:</u>



$$\sim \underbrace{\frac{1}{3!}}_{\text{symmetry factor}} (iZ_g g)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m^2 + i\varepsilon} \cdot \frac{i}{(p+l)^2 - m^2 + i\varepsilon} \frac{i}{(p+l-k_1)^2 - m^2 + i\varepsilon}$$

Peskin, sec. 9.5
Srednicki sec. 43, 44

3. Yukawa theory

Introduce fermions

<u>Goal:</u> Path integral formalism for (anticommuting) spinor fields

3.1 Mathematical tool: Anticommuting numbers (Grassmann numbers)

For $\theta, \eta \in G$: ← set of Grassmann numbers

$$\boxed{\theta \eta = - \eta \theta}$$

$$\Rightarrow \boxed{\theta^2 = 0}$$

[For finite sets of Grassmann numbers, one can construct matrix representations.

$\Rightarrow$ For any real or complex-valued $f(\theta)$:

$$f(\theta) = \underbrace{A}_{\text{ordinary number}} + \underbrace{B}_{\text{Grassmann number}} \theta$$

[Taylor series is finite since $\theta^2 = 0$

Need to define $\int d\theta \, f(\theta)$ such that

$$\int d\theta \, \underbrace{c}_{\text{ordinary number}} \cdot f(\theta) = c \cdot \int d\theta \, f(\theta)$$

$$\int d\theta \, f(\theta + \eta) = \int d\theta \, f(\theta)$$

$\Rightarrow \int d\theta \, 1 = 0$]

$$\Rightarrow \boxed{\int d\theta \, f(\theta) = \int d\theta \, (A + B\theta) \equiv B}$$

Convention for multiple integrals:

$$\boxed{\int d\theta \, d\eta \, \eta \theta = +1}$$

Complex Grassmann numbers

real Grassmann numbers

$$\boxed{\begin{aligned} \theta &\equiv \frac{\theta_r + i\theta_i}{\sqrt{2}} \\ \theta^* &\equiv \frac{\theta_r - i\theta_i}{\sqrt{2}} \end{aligned}}$$

convention

Define: $(\theta\eta)^* \equiv \eta^* \theta^* = -\theta^* \eta^*$

$$\int d\theta^* d\theta \;\; \theta\theta^* = 1$$

Examples:

- $$\int d\theta^* d\theta \;\; e^{-\theta^* b \theta} = \int d\theta^* d\theta \left(1 - \theta^* b \theta\right)$$
  $$= b .$$

  Note. Compare to $\int dx\, dx^* \; e^{-x^* b x} = \frac{\pi}{b}$

- $$\int d\theta^* \, d\theta \;\; \theta\theta^* \, e^{-\theta^* b \theta} = 1$$

- In multiple dimensions: Coordinates $\theta_1, \cdots, \theta_n$

  $$I \equiv \int d\theta_1^* d\theta_1 \cdots d\theta_n^* d\theta_n \;\; e^{-\theta_i^* \underset{\text{Hermitian matrix}}{B_{ij}} \theta_j} = ?$$

  Would like to transform to eigenbasis of B, but what about the integration measure

  ↳ Unitary transformation $\theta_i \to \theta_i' = \underset{\text{unitary}}{U_{ij}} \theta_j$:

  $$\prod_i \theta_i' = \frac{1}{n!} \varepsilon^{i'j' \dots \ell'} \theta_{i'}' \theta_{j'}' \cdots \theta_{\ell'}'$$

  $$= \frac{1}{n!} \varepsilon^{i'j' \dots \ell'} U_{i'i} \theta_i \, U_{j'j} \theta_j \cdots U_{\ell'\ell} \theta_\ell$$

  $$= \frac{1}{n!} \varepsilon^{i'j' \dots \ell'} U_{i'i} U_{j'j} \cdots U_{\ell'\ell} \, \varepsilon^{ij \dots \ell} \left(\prod_k \theta_k\right)$$

  $$= \underbrace{\left(\det U\right)}_{= 1 \text{ for unitary } U} \left(\prod_k \theta_k\right)$$

eigenvalues of B

$$\Rightarrow I = \int d\theta_1^{*\prime} d\theta_1' \cdots d\theta_n^{*\prime} d\theta_n' \, e^{-\sum_k \theta_k^* b_k \theta_k}$$

$$= \prod_k b_k$$

$$= \underline{\underline{\det B}}$$

- Exercise: Show that $\left(\prod_i \int d\theta_i^* d\theta_i\right) \theta_k \theta_l^* e^{-i\theta_i^* B_{ij} \theta_j} = (\det B)(B^{-1})_{kl}$

end 20.11.2014

3.2 The fermionic path integral

Consider free Dirac field:

$$\boxed{\mathcal{L}_0 = \bar{\psi}(i\not{\partial} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta}$$

$\bar{\psi}$, ψ: complex Grassmann numbers (4-component spinor)

$\bar{\eta}$, η: external sources (spinor) (complex Grassmann numbers)

Define:

$$\boxed{Z_0[\eta, \bar{\eta}] = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \exp\left[i \int d^4x \, \mathcal{L}_0\right]}$$

Remember for scalar field:

$$Z_0[J] = \exp\left[-\frac{1}{2}\int d^4x \, d^4x' \, J(x') \underbrace{D_F(x-x')}_{= \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}} J(x)\right]$$

Analogously:

$$\boxed{Z_0[\eta, \bar{\eta}] = \exp\left[-\int d^4x \, d^4x' \, \bar{\eta}(x) \, S_F(x-x') \, \eta(x')\right]}$$

with the Feynman propagator for Dirac fermions

$$\boxed{S_F(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{(\not{k} + m) \, i e^{-ik(x-x')}}{k^2 - m^2 + i\varepsilon}}$$

Correlation functions from functional derivatives as before

Define $\frac{d}{d\eta}\theta\eta = -\frac{d}{d\eta}\eta\theta = -\theta$ [θ, η ... Grassmann numbers

$\Rightarrow$ e.g. $\langle 0 | T\,\psi(x_1)\,\bar{\psi}(x_2) | 0 \rangle = \frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x_1)}\,\frac{-1}{i}\frac{\delta}{\delta\eta(x_2)}\,Z[\eta,\bar{\eta}]\Big|_{\eta,\bar{\eta}=0}$

minus sign for anticommuting $\delta/\delta\eta$ and $\bar{\psi}$ in $\bar{\psi}\eta$ term of $\mathcal{L}_0$

Srednicki sec. 41

3.3 LSZ reduction for fermions

The LSZ reduction formula can be generalized for the case of Dirac fermions:

$$\boxed{\begin{aligned}\langle f|i\rangle = i^{n_i+n_f}\int d^4x_1 \cdots d^4x_1' \cdots e^{+ip_1'x_1'}\left[\bar{u}_{s_1'}(\vec{p}_1')(i\not{\partial}_{1'}+m)\right]_{\alpha_1'}\cdots \\ \cdot \langle 0|T\,\Psi_{\alpha_{n_f}}(x_{n_f})\cdots\Psi_{\alpha_1'}(x_1')\,\bar{\Psi}_{\alpha_1}(x_1)\cdots\bar{\Psi}_{\alpha_{n_i}}(x_{n_i})|0\rangle \\ \cdot\left[(-i\overleftarrow{\not{\partial}}_1+m)\,u_{s_1}(\vec{p}_1)\right]_{\alpha_1} e^{-ip_1x_1}\cdots\end{aligned}}$$

where α_i are spinor indices, s_i are the spin orientations of the fermions and p_i are their 4-momenta.

For incoming [outgoing] anti-fermions, annihilated [created] by $\bar{\Psi}$ [Ψ], replace

$$i\left[(-i\overleftarrow{\not{\partial}}_j+m)\,u_{s_j}(\vec{p}_j)\right]_{\alpha_j}e^{-ip_jx_j}\rightarrow -ie^{-ip_jx_j}\left[\bar{v}_{s_j}(\vec{p}_j)(i\not{\partial}_j+m)\right]_{\alpha_j}$$

$$\left[\,ie^{+ip_{j'}x_{j'}}\left[\bar{u}_{s_{j'}}(\vec{p}_{j'})(i\not{\partial}_{j'}+m)\right]_{\alpha_{j'}}\rightarrow -i\left[(-i\overleftarrow{\not{\partial}}_{j'}+m)\,v_{s_{j'}}(\vec{p}_{j'})\right]_{\alpha_{j'}}e^{+ip_{j'}x_{j'}}\,\right]$$

Srednicki sec. 45

3.4. Feynman rules for Dirac fermions

Consider Yukawa theory:

$$\boxed{\mathcal{L} = \bar{\Psi}\left(Z_\psi i\partial\!\!\!/ - Z_m m\right)\Psi + \frac{1}{2}Z_\phi\left(\partial_\mu\phi\right)^2 - \frac{1}{2}Z_M M^2\phi^2 + Z_g g\,\phi\bar{\Psi}\Psi + \phi J + \bar{\eta}\Psi + \bar{\Psi}\eta}$$

(We will not consider loop processes here, therefore counterterms are omitted in the following)

E.g. ϕ = Higgs boson, Ψ = electron, muon or τ lepton

$$Z[\bar{\eta},\eta,J] \propto \exp\left[ig\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right] Z_0[\bar{\eta},\eta,J]$$

minus sign

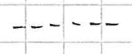
with

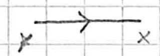
$$Z_0[\bar{\eta},\eta,J] = \exp\left[-\int d^4x\, d^4x'\;\bar{\eta}(x)\,S_F(x-x')\,\eta(x')\right] \cdot \exp\left[-\frac{1}{2}\int d^4x\,d^4x'\;J(x)\,D_F(x-x')\,J(x')\right]$$

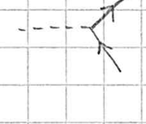
$$S_F(x-x') = \int\frac{d^4k}{(2\pi)^4}\;\frac{(k\!\!\!/+m)\,i e^{-ik(x-x')}}{k^2-m^2+i\varepsilon}$$

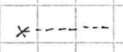
$$D_F(x-x') = \int\frac{d^4k}{(2\pi)^4}\;\frac{i e^{-ik(x-x')}}{k^2-m^2+i\varepsilon}$$

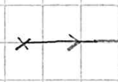
As before, Taylor-expand $Z[\bar{\eta},\eta,J]$ and write individual terms as Feynman diagrams, where

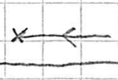
 $= D_F(x-y)$

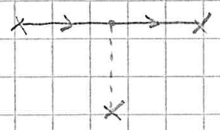
 $= S_F(x-y)$

 $= ig\int d^4x$

 $= i\int d^4x\; J(x)$

 $= i\int d^4x\; \eta(x)$

 $= i\int d^4x\; \bar{\eta}(x)$

Example:  $= i^4 g \int d^4x\, d^4y\, d^4z\, d^4w$

$$\cdot \left[\bar{\eta}(x)\, S_F(x-y)\, S_F(y-z)\, \eta(z)\right]$$

$$\cdot\, D_F(y-w)\, J(w)$$

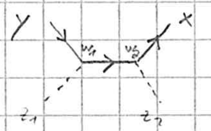
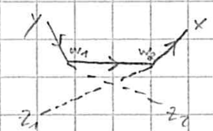
Scattering amplitudes

Consider the process $e^-\phi \longrightarrow e^-\phi$

$\hookrightarrow \langle 0 | T\, \Psi_\alpha(x)\, \bar{\Psi}_\beta(y)\, \phi(z_1)\, \phi(z_2) | 0 \rangle_c$

only connected diagrams involve momentum transfer! (see also sec. 2.7)

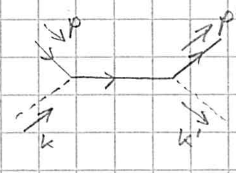
$$= \frac{1}{i}\frac{\delta}{\delta \bar{\eta}_\alpha(x)}\, \frac{-1}{i}\frac{\delta}{\delta \eta_\beta(y)}\, \frac{1}{i}\frac{\delta}{\delta J(z_1)}\, \frac{1}{i}\frac{\delta}{\delta J(z_2)}\, iW(\bar{\eta}, \eta, J)\Big|_{\bar{\eta}=\eta=J=0}$$

$=$  $+$  $+ \mathcal{O}(g^4)$

$$= (ig)^2 \int d^4w_1\, d^4w_2 \left[S_F(x-w_2)\, S_F(w_2-w_1)\, S_F(w_1-y)\right]_{\alpha\beta}$$

$$\cdot\, D_F(z_1-w_1)\, D_F(z_2-w_2) \quad + \quad (z_1 \leftrightarrow z_2) + \mathcal{O}(g^4)$$

Momentum assignments:



$\hookrightarrow \langle f | i \rangle \left[= \langle 0 | T\, a(\vec{k}')\, b_{s'}(\vec{p}')\, b_s^\dagger(\vec{p})\, a^\dagger(\vec{k}) | 0 \rangle \right.$

spin index

LSZ reduction
+ use $(i\not{\partial}_x + m) S_F(x-y) = -i\delta^{(4)}(x-y)$

$$\stackrel{\Downarrow}{=} (ig)^2\, \bar{u}_{s'}(\vec{p}') \left[\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2} + \frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}\right] u_s(\vec{p})$$

This procedure leads to the general

Feynman rules for Yukawa theory

1. Draw a solid line for each external fermion; for incoming particles / outgoing antiparticles, draw arrow towards vertex; for outgoing particles / incoming antiparticles, draw arrow away from vertex.

2. Draw a dashed line for each external scalar

3. Connect external lines by adding vertices and internal propagators

4. Assign 4-momenta to the lines. For fermion lines, momentum flow is always along the arrows. Therefore, for anti-particles replace $p \to -p$.

5. The value of the diagram is given by the following factors:
   - external scalar: 1
   - incoming fermion: $u_{s_i}(\vec{p}_i)$
   - outgoing fermion: $\bar{u}_{s_i}(\vec{p}_i)$
   - incoming antifermion: $\bar{v}_{s_i}(\vec{p}_i)$
   - outgoing antifermion: $v_{s_i}(\vec{p}_i)$
   - vertex: ig
   - internal scalar: $i / (k^2 - M^2 + i\varepsilon)$
   - internal fermion: $i(\not{p} + m) / (p^2 - m^2 + i\varepsilon)$

   For fermion lines, write factors starting at the external line with an arrow pointing out of the diagram, then follow arrows backwards.

6. Figure out the overall sign of the diagram by counting, in the relevant term of the taylor expansion of $Z[\eta, \bar{\eta}, J]$, the number of fermion anticommutations necessary to put $\delta/\delta\eta$ and $\delta/\delta\bar{\eta}$ next to the $\eta, \bar{\eta}$ factors on which they act.

   Note: In practice, only relative sign of diagrams are of interest

   Note: Fermion loops always come with an extra minus sign.

7. Treat loops and counterterms as for ϕ^3 theory.
   Note: fermion loops yield a trace over Dirac indices

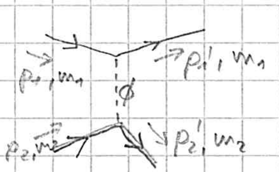
8. Sum over all diagrams

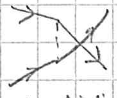
end 26.11.2014

Peskin sec. 4

3.5 The Yukawa potential

Consider scattering of two distinguishable fermions by exchange of a scalar in the non-relativistic limit



[Note: second diagram  would only contribute for indistinguishable fermions]

E.g. neutron — proton scattering

$$\langle f | i \rangle = (ig)^2 \, \bar{u}_{s'}(p_1') u_s(p_1) \cdot \bar{u}_{r'}(p_2') u_r(p_2) \frac{i}{(p_1'-p_1)^2 - M^2 + i\varepsilon}$$

Use non-rel. spinor $u_s(p) = \sqrt{m} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}$

spin index — Pauli-Spinor

$$\Rightarrow \bar{u}_{s'}(p_1') u_s(p_1) = 2m \, \xi_{s'}^{+} \xi_s = 2m \, \delta_{s's}$$

Moreover $(p_1'-p_1)^2 = (E_1'-E_1)^2 - (\vec{p}_1'-\vec{p}_1)^2$

$$\simeq \frac{1}{4m_1^2}(\vec{p}_1'^2 - \vec{p}_1^2) - (\vec{p}_1'-\vec{p}_1)^2$$

$$\simeq - |\vec{p}_1' - \vec{p}_1|^2$$

$$\Rightarrow \langle f | i \rangle = \frac{ig^2}{|\vec{p}_1'-\vec{p}_1|^2 + M^2} \cdot 2m\,\delta_{s's} \cdot 2m\,\delta_{r'r} \qquad (*)$$

Compare to scattering amplitude in QM for scattering on a potential V (Born approximation):

$$\langle p_1' | iT | p_1 \rangle = -i \tilde{V}(\vec{p}_1' - \vec{p}_1)\,(2\pi)\,\delta(E_{\vec{p}_1'} - E_{\vec{p}_1}) \qquad (**)$$

Fourier trafo of $V(\vec{x})$

$$\Rightarrow \tilde{V}(\vec{p}_1'-\vec{p}_1) \underset{\uparrow}{=} \frac{-g^2}{|\vec{p}_1'-\vec{p}_1|^2 + M^2}$$

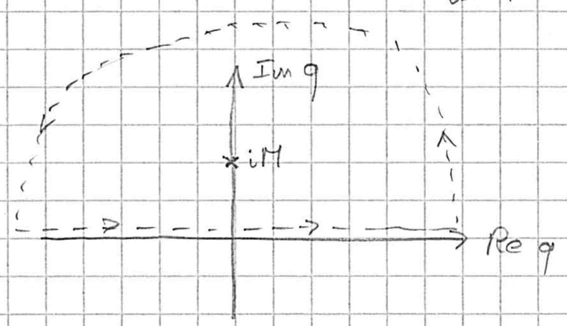
compare (*) and (**)

(Factors 2m come from relativistic normalization of $|p_1\rangle$, $|p_1'\rangle$ in QFT, must be dropped when comparing to QM, where states are normalized to 1.)

$$\Rightarrow \quad V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{\vec{q}^2 + M^2} e^{i\vec{q}\vec{x}}$$

$$= \frac{-g^2}{4\pi^2} \int_0^\infty dq \, q^2 \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2+M^2}$$

$$= \frac{-g^2}{4\pi^2 i r} \int_{-\infty}^{\infty} \frac{q e^{iqr}}{q^2+M^2}$$



$$= \frac{-g^2}{4\pi^2 i r} \cdot 2\pi i \frac{iM e^{-rM}}{2iM}$$

$$\boxed{V(\vec{x}) = -\frac{g^2}{4\pi} \frac{1}{r} e^{-rM}}$$

$\Rightarrow$ attractive force with range $\sim \frac{1}{M}$.

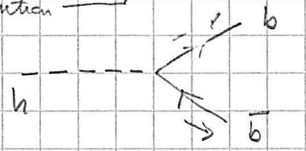
E.g. nuclear forces: range ~ 1 fm, Pion mass ~ 100 MeV ✓

3.6 Higgs boson decays

Let ϕ = Higgs boson, ψ = b quark ; y_b = b-h coupling

Goal: Compute rate for dominant Higgs decay
$h \to b\bar{b}$

$$\mathcal{L} \supset \underbrace{-\tfrac{1}{\sqrt{2}} y_b}_{\text{convention}} \, \bar{b} b h$$



$$p_h = (m_h, 0, 0, 0)$$
$$p_b = \left(\frac{m_h}{2} ; 0 ; 0 ; \sqrt{\frac{m_h^2}{4} - m_b^2}\right)$$
$$p_{\bar{b}} = \left(\frac{m_h}{2} ; 0 ; 0 ; -\sqrt{\frac{m_h^2}{4} - m_b^2}\right)$$

$$i\mathcal{M} = \frac{-i y_b}{\sqrt{2}} \bar{u}(p_b) v(p_{\bar{b}})$$

$$\sum_{\text{spins}} \overline{|\mathcal{M}|^2} = \frac{1}{2} y_b^2 \, \mathrm{tr}\left[(\not{p}_b + m_b)(\not{p}_{\bar{b}} - m_b)\right]$$
$$= \frac{1}{2} y_b^2 \left[\mathrm{tr}\, \not{p}_b \not{p}_{\bar{b}} - 4 m_b^2\right]$$
$$= \frac{1}{2} y_b^2 \left(4\, p_b p_{\bar{b}} - 4 m_b^2\right)$$
$$= 2 y_b^2 \left(\frac{m_h^2}{4} + \frac{m_h^2}{4} - m_b^2 - m_b^2\right)$$
$$= y_b^2 m_h^2 \left(1 - \frac{4 m_b^2}{m_h^2}\right)$$

$$\Gamma(h \to b\bar{b}) = \underbrace{3}_{\text{red, green, blue b-quarks}} \cdot \frac{1}{2 m_h} |\mathcal{M}|^2 \int d^3 p_b \frac{1}{(2\pi)^3} \frac{1}{(2E_b)} \int d^3 p_{\bar{b}} \frac{1}{(2\pi)^3} \frac{1}{2E_{\bar{b}}} (2\pi)^4 \delta^{(4)}(p_b + p_{\bar{b}} - p_h)$$

$$= 3 \frac{y_b^2 m_h^2}{8\pi m_h \left(\frac{m_h}{2}\right)^2} \left(1 - \frac{4 m_b^2}{m_h^2}\right) \int \underbrace{d p_b}_{} \, p_b^2 \, \overbrace{\delta(2E_b - m_h)}^{= \frac{1}{2}\delta\left(E_b - \frac{m_h}{2}\right)}$$

$$= dE_b \cdot \frac{dp_b}{dE_b} = dE_b \frac{d\sqrt{E_b^2 - m_b^2}}{dE_b}$$
$$= dE_b \cdot \frac{E_b}{p_b}$$

$$= 3 \frac{y_b^2}{4\pi m_h} \left(1 - \frac{4 m_b^2}{m_h^2}\right) \frac{m_h}{2} \sqrt{\frac{m_h^2}{4} - m_b^2}$$
$$= \frac{3 y_b^2 m_h}{16\pi} \left(1 - \frac{4 m_b^2}{m_h^2}\right)^{3/2}$$

Peskin eq. on p. 776:

$$\Gamma = 3 \frac{\alpha m_h}{8 s_w^2} \frac{m_b^2}{m_W^2} \left(1 - \frac{4 m_b^2}{m_h^2}\right)^{3/2}$$
$$= \frac{3 e^2 m_h}{32\pi s_w^2} \frac{m_b^2}{\underbrace{g^2}_{= e^2/s_w^2} \frac{v^2}{4}} \left(1 - \frac{4 m_b^2}{m_h^2}\right)^{3/2}$$
$$\overset{y_b = \frac{\sqrt{2} m_b}{v}}{=} \frac{3 m_h}{16\pi} y_b^2 \left(1 - \frac{4 m_b^2}{m_h^2}\right)^{3/2}$$

Plugging in numbers:

$$m_b = 4.18 \text{ GeV}$$

$$m_h = 125.7 \text{ GeV}$$

$$y_b = 0.024$$

$$\Rightarrow \Gamma(h \to b\bar{b}) = 4.3 \text{ MeV}$$

Peskin sec. 4, sec. 9

4. Quantum electrodynamics

4.1 Lagrangian

Free fermion: $\mathcal{L}_{Dirac} = \overline{\Psi}(i\not{\partial} - m)\Psi$

Free photon: $\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

↳ Euler-Lagrange eq. $\partial_\mu \frac{\delta \mathcal{L}}{\partial^\mu A^\nu} = 0$

$$\Leftrightarrow \partial_\mu \frac{\delta}{\partial^\mu A^\nu} \left(\partial^\rho A^\sigma - \partial^\sigma A^\rho\right)\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) = 0$$

$$\Leftrightarrow \partial_\mu \frac{\delta}{\delta^\mu A^\nu} \left(2\,(\partial^\rho A^\sigma)(\partial_\rho A_\sigma) - 2(\partial^\rho A^\sigma)(\partial_\sigma A_\rho)\right) = 0$$

$$\Leftrightarrow \partial_\mu \left[2 \cdot 2\,\partial^\mu A^\nu - 2 \cdot 2\,\partial^\nu A^\mu\right] = 0$$

$$\Leftrightarrow \boxed{\partial_\mu F^{\mu\nu} = 0} \quad \Rightarrow \text{Maxwell eq. in vacuum}$$

Note: 2^{nd} Maxwell eq. $\partial_\mu \left(\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}\right) = 0$
is automatically fulfilled:

$$\partial^\mu F^{\rho\sigma} + \partial^\rho F^{\sigma\mu} + \partial^\sigma F^{\mu\rho} = 0$$

Minimal coupling: $p_\mu \longrightarrow p_\mu - eA_\mu$

$$\Leftrightarrow i\partial_\mu \rightarrow i\partial_\mu - eA_\mu$$

$$\hookrightarrow \boxed{\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\Psi}(i\not{\partial} - m)\Psi - e\overline{\Psi}\gamma^\mu A_\mu \Psi}$$

4.2 Gauge invariance

$\mathcal{L}_{QED}$ is invariant under the transformation

$$\boxed{\begin{aligned} \psi(x) &\to e^{i\alpha(x)}\psi(x) \\ A_\mu(x) &\to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \end{aligned}} \qquad (*)$$

Since, at each spacetime point, the factor $e^{i\alpha(x)}$ is an element of the unitary group

$$U(1) \equiv \left\{ e^{i\alpha} \mid \alpha \in [0, 2\pi) \right\},$$

we say that QED is $U(1)$ gauge invariant.

Define covariant derivative

$$\boxed{D_\mu \equiv \partial_\mu + ieA_\mu}$$

Property: $D_\mu \psi(x) \to \left(\partial_\mu + ieA_\mu - i(\partial_\mu \alpha)\right) e^{i\alpha}\psi$

$$= i(\partial_\mu \alpha)e^{i\alpha}\psi + e^{i\alpha}\partial_\mu \psi + ieA_\mu e^{i\alpha}\psi - i(\partial_\mu\alpha)e^{i\alpha}\psi$$

$$= e^{i\alpha} D_\mu \psi$$

$$\Rightarrow \boxed{\mathcal{L}_{QED} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

Apply Noether's theorem: $\partial_\mu \mathcal{J}^\mu(x) = 0$ (current conservation)

for the current

change of ψ under infinitesimal gauge trafo: $\Delta\psi = i\alpha\psi$

$$\mathcal{J}^\mu(x) = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)} \Delta\psi = -\alpha\bar{\psi}\gamma^\mu\psi$$

$\Rightarrow$ Also $\boxed{j^\mu \equiv \bar{\psi}\gamma^\mu\psi}$ is conserved: $\partial_\mu j^\mu = 0$

And: The electric charge $Q \equiv \int d^3x \; j^0$ is a constant:

$$\dot{Q} = \int d^3x \, \frac{d}{dt} j^0 \underset{\partial_\mu j^\mu = 0}{=} \int d^3x \, \vec{\nabla} \vec{j} \underset{\text{Gauss theorem}}{= 0}$$

Srednicki sec. 57
Peskin sec. 9.4

4.3 Quantization of the photon field

Consider $\mathcal{L}_{Maxwell} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

Path integral:

$$Z_0[J] = \int \underbrace{\mathcal{D}A}_{= \mathcal{D}A^1 \mathcal{D}A^2 \mathcal{D}A^3 \mathcal{D}A^4} \exp\left[i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right) \right]$$

$$\left[\text{Use: } -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \int d^4x \, 2\left[(\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu) \right] \right.$$

$$\overset{\text{integrate by parts}}{=} \frac{1}{2} \int d^4x \left[A^\nu \partial^2 A_\nu - A^\nu (\partial^\mu \partial_\nu A_\mu) \right]$$

$$\Rightarrow Z_0[J] = \int \mathcal{D}\tilde{A} \exp\left[i \int \frac{d^4k}{(2\pi)^4} \left(-\frac{1}{2} \tilde{A}^\nu(k) \, k^2 \tilde{A}_\nu(-k) + \frac{1}{2} \tilde{A}^\nu(k) k^\mu k_\nu \tilde{A}_\mu(-k) \right. \right.$$

Fourier trafo of A

$$\left. \left. + \tilde{J}^\mu(k) \tilde{A}_\mu(-k) \right) \right]$$

$$= \int \mathcal{D}\tilde{A} \exp\left[-\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \left(\tilde{A}_\nu(k) (k^2 g^{\mu\nu} - k^\mu k^\nu) \tilde{A}_\mu(-k) \right. \right.$$

$$\left. \left. - \tilde{J}^\mu(k) \tilde{A}_\mu(-k) - \tilde{J}^\mu(-k) \tilde{A}_\mu(k) \right) \right]$$

As in sec. 2.4 we next would like to complete the square to write $Z_0[J] = Z_0[0] \times$ (term with J and with photon propagator)

↳ Requires inverting 4×4 matrix $k^2 g^{\mu\nu} - k^\mu k^\nu \equiv k^2 P^{\mu\nu}(k)$
Then, we could define

$$\tilde{B}^\mu(k) \equiv \tilde{A}^\mu(k) + \left((k^2 P)^{-1}(k) \right)^{\mu\nu} \tilde{J}_\nu(k)$$

Problem: $P^{\mu\nu}$ is singular! [Proof: $P^{\mu\nu} k_\nu = 0 \Rightarrow$ zero eigenvalue]

end 03.12.2014

Reason: Gauge invariance!

Remember: $\mathcal{L}_{Maxwell}$ is invariant under $A_\mu(x) \to A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$

$i\partial_\mu \leftrightarrow k_\mu$

$$\Leftrightarrow \tilde{A}_\mu(k) \to \tilde{A}_\mu(k) + \frac{i}{e} k_\mu \tilde{\alpha}(k)$$

$\Rightarrow$ The component of A_μ proportional to k_μ is unphysical.
[It also does not contribute to $\tilde{J}^\mu(k) \tilde{A}_\mu(-k)$ since $k_\mu \tilde{J}^\mu(k) = 0$.

$\Rightarrow$ Redefine $\mathcal{D}\tilde{A}$ to mean integration only over components of $\tilde{A}$ orthogonal to k, i.e. $k_\mu \tilde{A}^\mu(k) = 0 \Leftrightarrow \partial_\mu A^\mu(x) = 0$ (<u>Lorenz - gauge</u>)

Within this subspace $P^{\mu\nu}$ is the identity matrix since

- $$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\left(g_\nu{}^\lambda - \frac{k_\nu k^\lambda}{k^2}\right)$$
$$= g^{\mu\lambda} - \frac{k^\mu k^\lambda}{k^2} = P^{\mu\lambda}(k) .$$

$\Rightarrow$ $P^{\mu\nu}(k)$ is a projection operator, its eigenvalues can only be zero or one.

- $$g^{\mu\nu} P_{\mu\nu}(k) = 3$$

$\hookrightarrow$ 3 eigenvalues are 1, we already know that the 4-th one is 0 (and the latter corresponds to eigenvector k^μ)

$\Rightarrow$ In subspace $\perp k^\mu$: $\left(k^2 P^{\mu\nu}\right)^{-1} = \frac{1}{k^2} P^{\mu\nu}$

$\hookrightarrow$ $$\tilde{B}^\mu(k) \equiv \tilde{A}^\mu(k) - \frac{P^{\mu\nu}\tilde{J}_\nu(k)}{k^2}$$

$\hookrightarrow$ $$Z_0[J] = \int \underset{\substack{\uparrow \\ \text{now means only} \\ A_\mu \perp k^\mu \Leftrightarrow B_\mu \perp k^\mu}}{\mathcal{D}B} \exp\left[-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\left(\underbrace{\tilde{B}_\nu(k)\, k^2 P^{\mu\nu} \tilde{B}_\mu(-k)}_{\to Z_0[0]=1} + \tilde{J}_\mu(k) \frac{P^{\mu\nu}}{k^2+i\varepsilon} \tilde{J}_\nu(-k)\right)\right]$$

$$\overset{\text{use } Z_0[0]=1}{=} \exp\left[\frac{i}{2}\int d^4x\, d^4x'\, J_\mu(x)\, \Delta^{\mu\nu}(x-x')\, J_\nu(x')\right]$$

with the photon propagator

$$\boxed{\Delta^{\mu\nu}(x-x') \equiv \int\frac{d^4k}{(2\pi)^4} \frac{-i P^{\mu\nu}}{k^2+i\varepsilon} e^{-i(x-x')k}}$$

Since $k^\mu J_\mu(x) = 0$, this is equivalent to simply using

$$\boxed{\Delta^{\mu\nu}(x-x') \equiv \int\frac{d^4k}{(2\pi)^4} \frac{-i g^{\mu\nu}}{k^2+i\varepsilon} e^{-i(x-x')k}}$$

4.4 QED Feynman rules

Similar to Yukawa theory, with scalars (- - - -) replaced by photons (∿∿∿).

$\boxed{\text{The QED vertex } \text{∿∿∿}\!\!< \text{ is } -ie\gamma^\mu}$

$$\boxed{\text{The photon propagator is } \frac{-ig_{\mu\nu}}{p^2+i\varepsilon}}$$

(Lorentz indices are contracted with the γ^μ from the vertices to which the photon is attached.)

External photon lines are represented by a polarization vector $\varepsilon^\mu(p)$ (incoming) or $\varepsilon^{\mu *}$ (outgoing) with

$$\varepsilon^\mu(p) = \tfrac{1}{\sqrt{2}}(0,\ 1,\ \pm i,\ 0) \quad (\text{for } p \parallel \hat{e}_z)$$

for circularly polarized photons

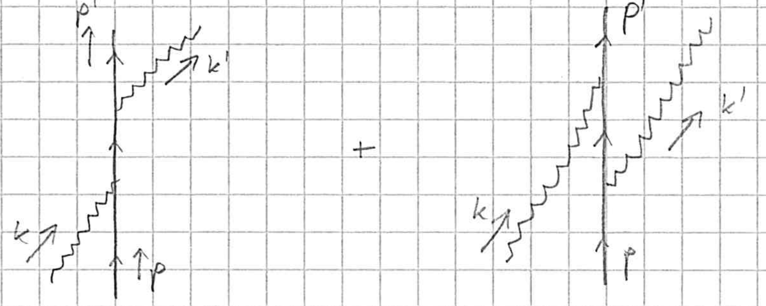
Easiest way of seeing this: write photon field as an expansion in classical solutions of wave eq. times creat./ann. operators

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_r \left(a^r_{\vec p}\, \varepsilon^r_\mu(p)\, e^{-ipx} + a^{r\dagger}_{\vec p}\, \varepsilon^{r*}_\mu(p)\, e^{ipx} \right)$$

Peskin sec. 5.5
Srednicki sec. 59

4.5 Example: Compton Scattering

$$\boxed{e^- \gamma \to e^- \gamma}$$



$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\frac{i(\not p + \not k + m)}{(p+k)^2 - m^2}(-ie\gamma^\nu)u(p) \cdot \varepsilon^*_\mu(k')\,\varepsilon_\nu(k)$$

$$+ \bar{u}(p')(-ie\gamma^\nu)\frac{i(\not p - \not k' + m)}{(p-k')^2 - m^2}(-ie\gamma^\mu)u(p) \cdot \varepsilon^*_\mu(k')\,\varepsilon_\nu(k)$$

$$= -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu(\not p + \not k + m)\gamma^\nu}{(p+k)^2 - m^2} + \frac{\gamma^\nu(\not p - \not k' + m)\gamma^\mu}{(p-k')^2 - m^2}\right]u(p)$$

<u>Use:</u> $k^2 = 0$; $k'^2 = 0$

$p^2 = m^2$; $p'^2 = m^2$

$\not p\gamma^\nu u(p) = (2p^\nu - \gamma^\nu \not p)u(p) \overset{\text{e.o.m.}}{=} (2p^\nu - \gamma^\nu m)u(p)$

$$= -ie^2\,\varepsilon^*_\mu(k')\,\varepsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu \not k \gamma^\nu + 2p^\nu\gamma^\mu}{2pk} + \frac{-\gamma^\nu \not k' \gamma^\mu + 2p^\mu\gamma^\nu}{-2pk'}\right]u(p)$$

Next, compute $\overline{|\mathcal{M}|^2}$ (averaged over fermion spins and photon polarizations).

<u>Use:</u>

- $\sum_{\text{spins}} u(p)\,\bar{u}(p) = \not p + m$

- $\sum_{\text{polarizations}} \varepsilon^*_\mu(k)\,\varepsilon_\nu(k) \Longleftrightarrow -g_{\mu\nu}$

<u>Proof:</u> With $\varepsilon^\pm_\mu(k) = (0, 1, \pm i, 0)$ ($k \parallel \hat{e}_z$, i.e. $(k^\mu) = (|\vec{k}|, 0, 0, |\vec{k}|)$)

$$\sum_{r=+,-} \varepsilon^{r*}_\mu(k)\,\varepsilon^r_\nu(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= -g_{\mu\nu} + \hat{e}_{t\mu}\hat{e}_{t\nu} - \hat{e}_{z\mu}\hat{e}_{z\nu}$$

$e_t \cdot k = k^0$

$k = (k^0, 0, 0, k^0)$

with $(\hat{e}_t^\mu) = (1,0,0,0)^T$ and $(\hat{e}_z^\mu) = (0,0,0,1)^T$.

Note that $\hat{e}_z^{\ \mu} = \dfrac{k^\mu - (\hat{e}_t \cdot k)\hat{e}_t^\mu}{\hat{e}_t \cdot k}$.

$\mathcal{M}$ has to be gauge-invariant under $A^\mu(x) \to A^\mu(x) - \frac{1}{e}\partial^\mu \alpha(x)$ or, in momentum space

$$\tilde{A}^\mu(k) \longrightarrow \tilde{A}^\mu(k) + \frac{i}{e} k^\mu \tilde{\alpha}(k)$$

Note: This argument assumes that for $\mathcal{M} = \mathcal{M}^\mu A_\mu$, $\mathcal{M}^\mu$ does not change under gauge transformations

But for 1-particle states, $\tilde{A}^\mu(k) = \varepsilon^\mu(k)$

↳ contributions $\propto k^\mu$ cannot change observables

$$\Rightarrow \sum_{r=+,-} \varepsilon_\mu^{r*}(k)\, \varepsilon_\nu^r(k) \Leftrightarrow -g_{\mu\nu} + \hat{e}_t^\mu \hat{e}_t^\nu - \hat{e}_t^\mu \hat{e}_t^\nu$$

$$= -g_{\mu\nu}$$

$\square$

$$\Rightarrow \underbrace{\frac{1}{4}}_{\text{average over initial state spins}} \overline{|\mathcal{M}|^2} = \frac{e^4}{4} g_{\mu\sigma} g_{\nu\sigma} \operatorname{tr}\left\{ (\not{p}'+m)\left[\frac{\gamma^\mu \not{k} \gamma^\nu + 2p^\nu \gamma^\mu}{2pk} + \frac{-\gamma^\nu \not{k}' \gamma^\mu + 2p^\mu \gamma^\nu}{-2pk'}\right] \cdot (\not{p}+m)\left[\frac{\gamma^\sigma \not{k} \gamma^\rho + 2p^\sigma \gamma^\rho}{2pk} + \frac{-\gamma^\rho \not{k} \gamma^\sigma + 2p^\rho \gamma^\sigma}{-2pk'}\right]\right\}$$

$$= \frac{e^4}{4} \left[\frac{T_1}{(2pk)^2} + \frac{T_2}{(2pk)(2pk')} + \frac{T_3}{(2pk')(2pk)} + \frac{T_4}{(2pk')^2}\right]$$

Example: $T_2 = \operatorname{tr}\left[(\not{p}'+m)(\gamma^\mu \not{k} \gamma^\nu + 2p^\nu \gamma^\mu)(\not{p}+m)(\gamma_\mu \not{k}' \gamma_\nu - 2p_\mu \gamma_\nu)\right]$

$$= \operatorname{tr}\left[\not{p}'(\gamma^\mu \not{k} \gamma^\nu + 2p^\nu \gamma^\mu)\not{p}(\gamma_\mu \not{k}' \gamma_\nu - 2p_\mu \gamma_\nu)\right]$$

$$+ m^2 \operatorname{tr}\left[(\gamma^\mu \not{k} \gamma^\nu + 2p^\nu \gamma^\mu)(\gamma_\mu \not{k}' \gamma_\nu - 2p_\mu \gamma_\nu)\right]$$

$$= \operatorname{tr}\left[+\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\mu \not{k}' \gamma_\nu + 2\not{p}' \gamma^\mu \not{p} \gamma_\mu \not{k}' \not{p} - 2\not{p}' \not{p} \not{k} \gamma^\nu \not{p} \gamma_\nu - 4\not{p}' \not{p} \not{p} \not{p}\right]$$

$$+ m^2 \operatorname{tr}\left[+\gamma^\mu \not{k} \gamma^\nu \gamma_\mu \not{k}' \gamma_\nu + 2\gamma^\mu \gamma_\mu \not{k}' \not{p} - 2\not{p} \not{k} \gamma^\nu \gamma_\nu - 4\not{p}\not{p}\right]$$

<u>Use:</u>

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$

- $tr\left[\not p'\gamma^\mu \not k \gamma^\nu \not p \gamma_\mu \not k' \gamma_\nu\right] = -2tr\left[\not p' \not p \gamma^\nu \not k \not k' \gamma_\nu\right]$

$$= -2tr\left[\not p' \not p\right] \cdot 4kk' = -32(kk')(pp')$$

- $2\,tr\left[\not p'\gamma^\mu \not p \gamma_\mu \not k' \not p\right] = -4\,tr\left[\not p' \not p \not k' \not p\right]$

$$= -4tr\left[\not p' \not p\, 2(pk') - \not p' \not k' \not p \not p\right]$$

$$= -32(pk')(p'p) + 16\, m^2\, p'\cdot k'$$

- $-2tr\left[\not p' \not p\, \not k \gamma^\nu \not p \gamma_\nu\right] = -2tr\left[\not p' \not p \not k (-2\not p)\right]$

$tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]$

$= 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

$$= +16\,(p'p\; kp - p'k\; m^2 + p'p\; kp)$$

$$= 32(p'p)(pk) - 16m^2\, p'k$$

- $tr\left[-4\not p' \not p \not p \not p\right] = -4m^2\, tr\left[\not p' \not p\right] = -16m^2\, pp'$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

- $m^2\, tr\left[+\gamma^\mu \not k \gamma^\nu \gamma_\mu \not k' \gamma_\nu\right] = +4m^2\, tr\left[k^\nu \not k' \gamma_\nu\right] = +16m^2\, kk'$

- $m^2\, tr\left[2\gamma^\mu\gamma_\mu \not k' \not p\right] = 32\, p\cdot k'\, m^2$

- $m^2\, tr\left[-2\not p \not k \gamma^\nu\gamma_\nu\right] = -32\, p\cdot k\, m^2$

- $m^2\, tr\left[-4\not p \not p\right] = -16\, m^4$

$$\Rightarrow T_2 = -32(kk')(pp') - 32(pk')(p'p) + 16m^2(p'k') + 32(p'p)(pk)$$

$$- 16m^2 p'k - 16m^2 pp' + 16m^2 kk' + 32m^2 pk' - 32pk m^2 - 16m^4$$

$$\boxed{\begin{aligned} &\underline{\text{Mandelstam variables}} \\ s &= (p+k)^2 = 2pk + m^2 \\ &= (p'+k')^2 = 2p'k' + m^2 \\ t &= (p'-p)^2 = 2m^2 - 2pp' = -2kk' \\ u &= (k'-p)^2 = m^2 - 2pk' = m^2 - 2kp' \end{aligned}}$$

$$\Rightarrow T_2 = -32\left(-\frac{t}{2}\right)\left(m^2 - \frac{t}{2}\right) - 32\left(\frac{m^2}{2} - \frac{u}{2}\right)\left(m^2 - \frac{t}{2}\right)$$

$$+ 16m^2\left(\frac{s}{2} - \frac{m^2}{2}\right) + 32\left(m^2 - \frac{t}{2}\right)\left(\frac{s}{2} - \frac{m^2}{2}\right)$$

$$- 16m^2\left(\frac{m^2}{2} - \frac{u}{2}\right) - 16m^2\left(m^2 - \frac{t}{2}\right) + 16m^2\left(-\frac{t}{2}\right)$$

$$+ 32m^2\left(\frac{m^2}{2} - \frac{u}{2}\right) - 32m^2\left(\frac{s}{2} - \frac{m^2}{2}\right) - 16m^4$$

<u>Note:</u> $s + t + u = 2m^2$

$$= 8t\underbrace{(2m^2 - t)}_{=s+u} - 8(m^2 - u)\underbrace{(2m^2 - t)}_{=s+u}$$

$$+ 8m^2(s - m^2) + 8\underbrace{(2m^2 - t)}_{=s+u}(s - m^2)$$

$$- 8m^2(m^2 - u) - 8m^2\underbrace{(2m^2 - t)}_{=s+u} - 8m^2 t$$

$$+ 8m^2(2m^2 - 2u) - 8m^2(2s - 2m^2) - 8 \cdot 2m^4$$

$$= 8\Big[\underbrace{t(s+u) + u(s+u) + s(s+u)}_{=2m^2(s+u)} - m^2\cancel{(s+u)}$$

$$+ m^2(s - m^2) - m^2\cancel{(s+u)} - m^2(m^2 - u) - m^2(\cancel{s} + \cancel{u})$$

$$- m^2(2m^2 - \cancel{s} - \cancel{u}) + m^2(2m^2 - 2u) - m^2(2s - 2m^2)$$

$$- 2m^4\Big]$$

$$= 8\left[m^2(s-m^2) + m^2(u-m^2) - 2m^2(u-m^2) - 2m^2(s-m^2) - 4m^4\right]$$

$$= -8\left[m^2(s-m^2) + m^2(u-m^2) + 4m^4\right]$$

Similarly:
- $T_3 = T_2$
- $T_1 = 16\left(2m^4 + m^2(s-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right)$
- $T_4 = 16\left(2m^4 + m^2(u-m^2) - \frac{1}{2}(s-m^2)(u-m^2)\right)$

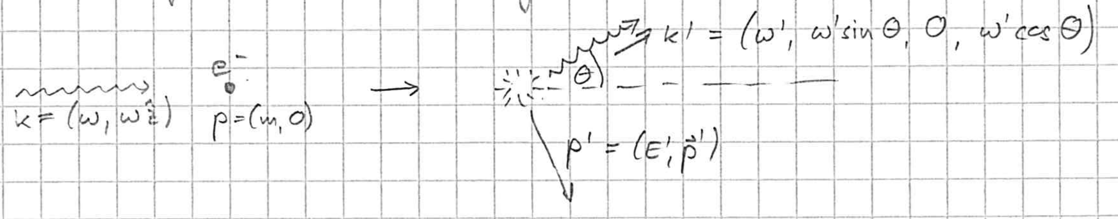
$$\Rightarrow \frac{1}{4}\overline{|\mathcal{M}|^2} = \frac{e^4}{4}\left[\frac{T_1}{(s-m^2)^2} - \frac{T_2}{(s-m^2)(u-m^2)} - \frac{T_3}{(s-m^2)(u-m^2)} + \frac{T_4}{(u-m^2)^2}\right]$$

$$= 4e^4\left[2m^4\left(\frac{1}{(s-m^2)^2} + \frac{1}{(u-m^2)^2}\right) + m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\right.$$
$$\left. - \frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right) + m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right) + 4m^4\frac{1}{(s-m^2)(u-m^2)}\right]$$

$$= 4e^4\left[2m^4\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)^2 + 2m^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)\right.$$
$$\left. - \frac{1}{2}\left(\frac{u-m^2}{s-m^2} + \frac{s-m^2}{u-m^2}\right)\right]$$

$$= 2e^4\left[m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2 + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + \frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'}\right]$$

- Consider lab frame (e^- initially at rest):



$$\Rightarrow p\cdot k = m\omega$$
$$p\cdot k' = m\omega'$$

Determine ω': $m^2 = p'^2 = (p + k - k')^2$

$$= p^2 + 2p(k-k') + \underbrace{k^2}_{0} + \underbrace{k'^2}_{0} - 2kk'$$
$$= m^2 + 2m(\omega - \omega') - 2\omega\omega'(1-\cos\theta)$$

$$\Rightarrow \quad \frac{1}{\omega'} = \frac{1}{\omega} + \frac{1}{m}(1-\cos\theta)$$

$$\omega' = \frac{\omega m}{m+\omega(1-\cos\theta)} \qquad \Rightarrow \quad \boxed{\frac{\omega}{\omega'} = 1+\frac{\omega}{m}(1-\cos\theta)}$$

$$d\sigma = \frac{1}{2\omega}\,\frac{1}{2m}\,\underbrace{\frac{1}{|\vec{v}_k-\vec{v}_p|}}_{=1}\cdot\frac{1}{4}\overline{|\mathcal{M}|^2}\cdot\underbrace{d\Pi_2}_{\text{2-particle phase space}}$$

Phase space integral:

$$\int d\Pi_2 = \int \frac{d^3k'}{(2\pi)^3 2\omega'}\frac{d^3p'}{(2\pi)^3 2E'}\delta^{(4)}(p+k-p'-k')\,(2\pi)^4$$

$$= \int \frac{d\omega'\,\omega'^2\,d\Omega}{(2\pi)^2}\,\frac{1}{4\omega' E'}\,\delta(m+\omega-E'-\omega')$$

$$E' = \sqrt{\vec{p}'^2+m^2} = \sqrt{(\vec{k}-\vec{k}')^2+m^2} = \sqrt{\omega^2+\omega'^2+m^2-2\omega\omega'\cos\theta}$$

$$\frac{d(E'+\omega')}{d\omega'} = \frac{2\omega'-2\omega\cos\theta}{2E'}+1$$

$$= \int \frac{d\cos\theta}{2\pi}\,\frac{\omega'}{4E'}\,\frac{1}{\left|1+\frac{\omega'-\omega\cos\theta}{E'}\right|}$$

$$= \frac{1}{8\pi}\int d\cos\theta\,\frac{\omega'}{\underbrace{E'+\omega'}_{=m+\omega}-\omega\cos\theta}$$

$$= \frac{1}{8\pi}\int d\cos\theta\,\frac{\omega'}{m+\omega(1-\cos\theta)}$$

$$= \frac{1}{8\pi}\int d\cos\theta\,\frac{\omega'^2}{m\omega}$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{4\omega m}\cdot 2e^4\left[m^4\left(\frac{1}{m\omega}-\frac{1}{m\omega'}\right)^2+2m^2\left(\frac{1}{m\omega}-\frac{1}{m\omega'}\right)+\frac{m\omega'}{m\omega}+\frac{m\omega}{m\omega'}\right]$$

$$\cdot\frac{1}{8\pi}\frac{\omega'^2}{m\omega}$$

$$= \frac{e^4}{16\pi}\,\frac{\omega'^2}{m^2\omega^2}\left[m^2\frac{(\omega'-\omega)^2}{\omega^2\omega'^2}+2m\frac{\omega'-\omega}{\omega\omega'}+\frac{\omega'}{\omega}+\frac{\omega}{\omega'}\right]$$

$$= \frac{\pi\alpha}{m^2}\left[\frac{m^2}{\omega^2}\left(\frac{\omega'}{\omega}-1\right)^2+2\frac{m}{\omega}\left(\frac{\omega'^2}{\omega^2}-\frac{\omega'}{\omega}\right)+\frac{\omega'^3}{\omega^3}+\frac{\omega'}{\omega}\right]$$

$$= \frac{\pi\alpha}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{m^2}{\omega^2}\left(1-\frac{\omega}{\omega'}\right)^2+2\frac{m}{\omega}\left(1-\frac{\omega}{\omega'}\right)+\frac{\omega'}{\omega}+\frac{\omega}{\omega'}\right]$$

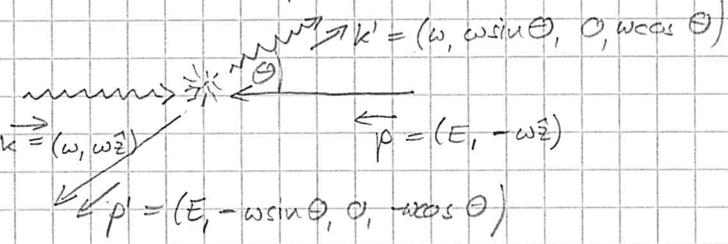
$$= \frac{\pi\alpha}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\underbrace{(1-\cos\theta)^2-2(1-\cos\theta)}_{=1-2\cos\theta+\cos^2\theta-2+2\cos\theta}+\frac{\omega'}{\omega}+\frac{\omega}{\omega'}\right]$$

$4-12 \qquad = -\sin^2\theta$

$$= \frac{\pi\alpha}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]$$

- Compton scattering at high energy ($E \gg m$) and large angle ($\theta \approx \pi$)

↳ center of mass frame



$$\Rightarrow \left.\begin{aligned} p\cdot k &= E\omega + \omega^2 \\ p\cdot k' &= E\omega + \omega^2\cos\theta \end{aligned}\right\} \text{at } E \gg m \ (E \sim \omega);\ \theta \approx \pi: \ pk' \to 0$$

$$\left[\begin{aligned}\frac{1}{4}\overline{|\mathcal{M}|^2} = 2e^4\Big[& m^4\left(\frac{1}{E\omega+\omega^2}-\frac{1}{E\omega+\omega^2\cos\theta}\right)^2 \\ &+ 2m^2\left(\frac{1}{E\omega+\omega^2}-\frac{1}{E\omega+\omega^2\cos\theta}\right) \\ &+ \frac{E\omega+\omega^2\cos\theta}{E\omega+\omega^2}+\frac{E\omega+\omega^2}{E\omega+\omega^2\cos\theta}\Big]\end{aligned}\right.$$

$$\frac{1}{4}\overline{|\mathcal{M}|^2} \approx 2e^4\frac{p\cdot k}{p\cdot k'} = 2e^4\frac{E+\omega}{E+\omega\cos\theta}$$

Phase space integral:

$$\int d\Pi_2 = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{2E}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega}\,(2\pi)^4\delta^{(4)}(p+k-p'-k')$$

$$= \int\frac{d\omega'\,\omega'^2\,d\Omega}{(2\pi)^2\cdot 4E\omega}\,\delta(E+\omega-E'-\omega')$$

$$\left[\begin{aligned} E' &= \sqrt{m^2+\omega'^2} \\ \frac{d(E'+\omega')}{d\omega'} &= \frac{1}{2E'}\cdot 2\omega' + 1\end{aligned}\right]$$

$$= \int d\cos\theta\,\frac{1}{2\pi}\frac{\omega}{4E}\frac{1}{\frac{\omega}{E}+1}$$

$$= \int d\cos\theta\,\frac{1}{2\pi}\frac{\omega}{4(\omega+E)}$$

$$\Rightarrow \frac{d\sigma}{d\cos\theta} = \frac{1}{2E\cdot\cancel{2\omega}}\cdot\cancel{\frac{1}{2}}\cdot\frac{\cancel{2}e^4\cancel{(E+\omega)}}{E+\omega\cos\theta}\cdot\frac{1}{2\pi}\frac{\cancel{\omega}}{\cancel{4(\omega+E)}}$$

$$\left.E = \sqrt{m^2+\omega^2} \approx \omega + \frac{m^2}{2\omega}\right] \qquad = \frac{\pi\alpha^2}{2E(E+\omega\cos\theta)} = \frac{\pi\alpha^2}{2\left(\omega+\frac{m^2}{2\omega}\right)\left(\omega+\frac{m^2}{2\omega}+\omega\cos\theta\right)}$$

end 04.12.2014

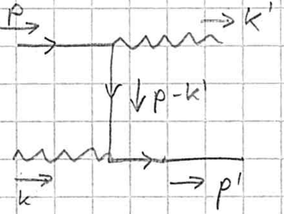
$$= \frac{\pi\alpha^2}{2\left[\omega^2+m^2+\omega^2\cos\theta+\frac{m^2}{2}\cos\theta\right]}$$

$$= \frac{\pi\alpha^2}{2\left[\frac{m^2}{2}+\left(\omega^2+\frac{m^2}{2}\right)(1+\cos\theta)\right]}$$

$$\begin{aligned} s &= (E+\omega)^2 \\ &\simeq \left(2\omega+\frac{m^2}{2\omega}\right)^2 \\ &\cong 4\omega^2+2m^2 \\ &= 4\left(\omega^2+\frac{m^2}{2}\right) \end{aligned} \longrightarrow \;\simeq \frac{2\pi\alpha^2}{2m^2+s(1+\cos\theta)}$$

<u>Note</u>: This becomes very large at $\cos\theta = -1$
(violating validity of perturbative expansion)

Reason: Internal propagator in



is nearly on-shell:

$$(p-k')^2-m^2 = -2pk' = 2\omega(E+\omega\cos\theta) \simeq 0$$

$\Rightarrow$ <u>"Collinear divergence"</u>

Total x-sec: $\sigma = \int d\cos\theta \, \frac{d\sigma}{d\cos\theta}$

$$\cong \frac{2\pi^2\alpha}{s} \underline{\log\frac{s}{m^2}}$$

typical structure
"large log" for $s \gg m^2$

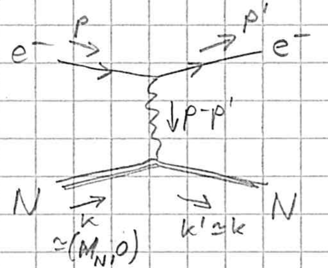
Peskin sec. 6.2

4.6 Loop corrections: The electron vertex function

e^- scattering from external em field

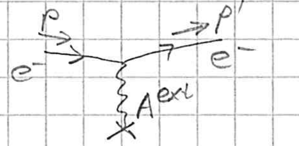
$$\mathcal{L} \supset -e\bar{\Psi}\gamma^\mu\Psi \cdot \underbrace{A_\mu^{ext}}_{\text{treated as classical field}} \qquad (1)$$

E.g. e^- scattering on heavy nucleus

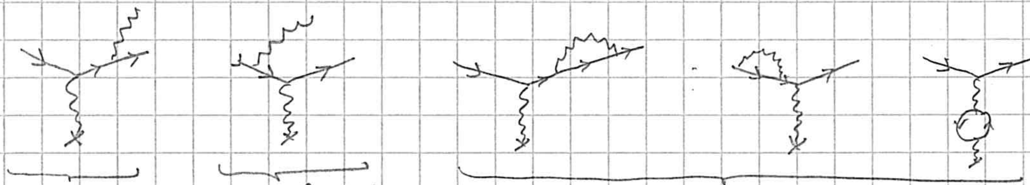


$$= \bar{u}_e(p')\,(-ie\gamma^\mu)\,u_e(p)\,\underbrace{\frac{-ig_{\mu\nu}}{(p-p')^2}\,\bar{u}_N(k)\,(-ie\gamma^\nu)\,u_N(k)}_{\equiv\,\tilde{A}_\mu^{ext}(p'-p)}$$

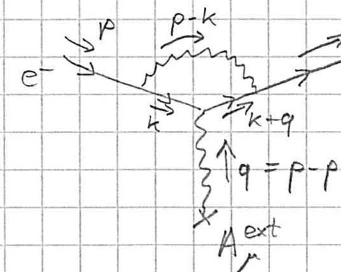
To see that the same expression is obtained from the Lagrangian (1), go back to sec. 2.7 (scattering amplitudes from applying LSZ reduction to a correlation function) and introduce a vertex with an x-dependent "coupling" $eA_\mu^{ext}(x)$.
The Fourier trafo corresponding to the d^4y integral then yields $\tilde{A}_\mu^{ext}(p-p')$.

$\Rightarrow$ 

Now: consider next order in perturbation theory



final state radiation (FSR) | initial state radiation (ISR) | external leg corrections



$$= \int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\,(-ie\gamma^\nu)\,\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\varepsilon}\,(-ie\gamma^\mu)$$

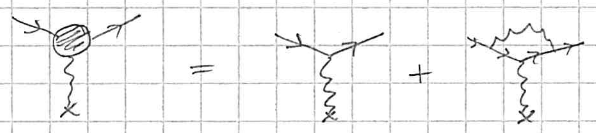
$$\cdot\,\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}\,(-ie\gamma^\rho)\,u(p)\cdot\frac{-ig_{\nu\rho}}{(p-k)^2+i\varepsilon}\cdot\tilde{A}_\mu^{ext}(q)$$

$$\equiv \bar{u}(p')\,(-ie\,\delta\Gamma^\mu)\,u(p)\cdot\tilde{A}_\mu^{ext}(p'-p)$$

4.6.1 Preliminary considerations

"Never do a complicated calculation before you know the result"

↳ first: what structure do we expect for

 + higher orders

$$= \bar{u}(p')(-ie\Gamma^\mu)u(p) \cdot \tilde{A}^{ext}_\mu(p'-p) + \text{higher orders}$$

$$\uparrow$$

$$\Gamma^\mu = \gamma^\mu + \varepsilon\Gamma'^\mu + \ldots$$

- Γ^μ can only contain p, p', γ^μ, m, e (nothing else appears in Feynman rules). Only $p^\mu, p'^\mu, \gamma^\mu$ transform als Lorentz vectors

$$\Rightarrow \Gamma^\mu = \gamma^\mu \cdot A + (p'^\mu + p^\mu) \cdot B + (p'^\mu - p^\mu)\, C$$

- A, B, C can contain Dirac matrices (γ^μ) contracted with vectors (p^μ, p'^μ) $\Rightarrow \not{p}, \not{p}'$. But:

$$\not{p}\, u(p) = m \cdot u(p)$$
$$\bar{u}(p')\not{p}' = \bar{u}(p') \cdot m$$

$\Rightarrow$ can treat A, B, C as ordinary numbers

- Possible Lorentz-scalar combination of p, p':

$$p^2 = p'^2 = m \quad ; \quad p \cdot p' = \tfrac{1}{2}\left[(p'-p)^2 - 2m^2\right]$$
$$= \tfrac{1}{2}q^2 - m^2$$

$$\Rightarrow A = A(q^2) \; ; \; B = B(q^2) \; ; \; C = C(q^2)$$

[+ dependence on constants m, e

- Gauge invariance

$$\partial_\mu j^\mu = 0 \Rightarrow q_\mu\left[\bar{u}(p')(-ie\Gamma^\mu)u(p)\right] = 0$$

see e.g. Peskin/Schröder sec. 7.4

[We know this at the classical level. The fact that it holds after quantum corrections are included, is called <u>Ward identity</u> and will not be proven here.

Note that $\bar{u}(p')\gamma^\mu u(p)\, q_\mu = \bar{u}(p')(\not{p}' - \not{p})u(p) = 0$

$$\bar{u}(p')\, u(p)\, \underbrace{(p'^\mu + p^\mu)\cdot q_\mu}_{= p'^2 - p^2 = 0} = 0$$

But $\bar{u}(p')\, u(p)\, (p'^\mu - p^\mu) q_\mu$ can be $\neq 0$

$\Rightarrow \boxed{C = 0}$

$\Rightarrow$ Using Gordon identity $\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$

$$\boxed{\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$

"form factors"

Interpretation:

- F_1 : corrections to electrostatic coupling of electrons
  At very low E $(q \to 0)$, $\sigma^{\mu\nu}q_\nu$ irrelevant
  and expect $F_1(0) = 1$ $\Rightarrow$ electric charge e

Perkin sec. 6.2

- F_2 : relevant for magnetic interactions
  By taking non-rel. limit of $u, \bar{u}$ and
  A_μ^{ext} for a static $\vec{B}$-field, we can show that

$$\bar{u}(p')\,(-ie\Gamma^\mu)\, u(p)\, A_\mu^{ext}$$

yields Born approximation for scattering from potential

$$V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}(\vec{x})$$

with

$$\langle\vec{\mu}\rangle = \frac{e}{m}\Big[\underbrace{F_1(0)}_{=1} + F_2(0)\Big]\, \xi'^{+}\,\frac{\vec{\sigma}}{2}\,\xi$$

Pauli spinors for $\bar{u}(p')$, $u(p)$

$$\equiv g\left(\frac{e}{2m}\right)\vec{S}$$

$\uparrow$ Landé g-factor

with $g = 2 + 2F_2(0)$

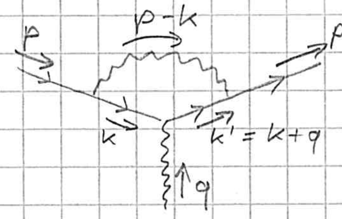
$\Rightarrow F_2(0)$ is called <u>anomalous magnetic moment</u>

Precision of electron "g - 2" allow testing quantum corrections to the QED vertex.

4.6.2 Simplifying the integrand

Peskin sec. 6.3

Let us now compute $F_1(q^2)$ and $F_2(q^2)$ at order α:



$$= \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') (-ie\gamma^\nu) \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\varepsilon} (-ie\gamma^\mu) \cdot \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon} (-ie\gamma^\rho) u(p) \frac{-ig_{\nu\rho}}{(p-k)^2+i\varepsilon}$$

$$= +2e^3 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \frac{\not{k}\gamma^\mu(\not{k}+\not{q}) - 2(2k+q)^\mu m + m^2\gamma^\mu}{[(k+q)^2-m^2+i\varepsilon][k^2-m^2+i\varepsilon][(p-k)^2+i\varepsilon]} u(p)$$

- $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu$
  $= -2\gamma^\sigma\gamma^\rho\gamma^\nu$
- $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$
- $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

Dealing with denominators: Feynman parameters

Goal: Bring denominator to form

$$\frac{1}{[(k-a)^2+b^2]^3}$$

end 10.12.2014

then shift integration variable $k \to \ell \equiv k-a$
$\Rightarrow$ spherically symmetric integral

Use

$$\boxed{\frac{1}{A_1 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1A_1 + \cdots + x_nA_n]^n}}$$

$x_1 \ldots x_n$ are called Feynman parameters

Proof: By induction

$$\Rightarrow \frac{1}{[(k+q)^2-m^2+i\varepsilon][k^2-m^2+i\varepsilon][(p-k)^2+i\varepsilon]}$$

$$= \int dx\, dy\, dz\, \delta(1-x-y-z) \cdot \frac{2}{D^3}$$

with

$$D = x[(k+q)^2-m^2] + y[k^2-m^2] + z[(p-k)^2-m^2] + \underbrace{(x+y+z)}_{=1} i\varepsilon$$

$$= k^2 + 2xkq - 2zpk + xq^2 + zp^2 - (x+y)m^2 + i\varepsilon$$

$\ell \equiv k + xq - zp$
$$= \ell^2 - x^2q^2 - z^2\underbrace{p^2}_{=m^2} + 2xzqp + xq^2 + zp^2 - \underbrace{(x+y)}_{=1-z}m^2 + i\varepsilon$$

$q\cdot p = p'p - p^2$
$= -\frac{1}{2}(p'-p)^2$
$= -\frac{1}{2}q^2$

$$= \ell^2 + q^2\underbrace{(-x^2 + x - xz)}_{=x(1-x-z)=xy} - m^2(1-2z+z^2) + i\varepsilon$$

$$= \ell^2 - \underbrace{\left(-xyq^2 + (1-z)^2m^2\right)}_{\equiv \Delta} + i\varepsilon$$

<u>Now, for the numerator:</u>

Shift integration variable: $k \to \ell$; $d^4k = d^4\ell$

Note that: 1) $\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0$

depends only on ℓ^2!

2) $\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu\ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{4}g^{\mu\nu}\ell^2}{D^3}$

For $\mu \neq \nu \to$ similar to 1)

For $\mu = \nu$ Lorentz invariance requires proportionality to $g^{\mu\nu}$

Prefactor: Contract both sides with $g_{\mu\nu}$

$$\Rightarrow \text{Numerator} = \bar{u}(p')\left[\not{k}\gamma^\mu(\not{k}+\not{q}) - 2(2k+q)^\mu m + m^2\gamma^\mu\right]u(p)$$

$$= \bar{u}(p')\Big[(\not{\ell} - x\not{q} + z\not{p})\gamma^\mu(\not{\ell} + (1-x)\not{q} + z\not{p})$$
$$- 2(2\ell + (1-2x)q + 2zp)^\mu m + m^2\gamma^\mu\Big]u(p)$$

$$\overset{1)}{\to} \bar{u}(p')\Big[\not{\ell}\gamma^\mu\not{\ell} + (-x\not{q} + z\not{p})\gamma^\mu((1-x)\not{q} + z\not{p})$$
$$- 2((1-2x)q + 2zp)^\mu m + m^2\gamma^\mu\Big]u(p)$$

$$\bullet\ \not{\ell}\gamma^\mu\not{\ell} = \ell_\nu \gamma^\nu\gamma^\mu\gamma^\rho \ell_\rho$$

$$\to \frac{1}{4} \ell^2 \gamma^\nu\gamma^\mu\gamma_\nu = -\frac{1}{2} \ell^2\gamma^\mu$$

$$\bullet\ \bar u(p')(-x\not{q} + z\not{p})\gamma^\mu[(1-x)\not{q} + z\not{p}]\,u(p)$$

$$= \bar u(p')\Big(-x\underbrace{\not{p}'}_{\to m \text{ by e.o.m}} + (x+z)\not{p}\Big)\gamma^\mu\Big[(1-x)\not{p}' + \underbrace{(x+z-1)}_{=-y}\underbrace{\not{p}}_{\to m}\Big]u(p)$$

$$= \bar u(p')\Big[xy m^2\gamma^\mu - x(1-x)\, m \underbrace{\gamma^\mu\not{p}'}_{=2p'^\mu - \not{p}'\gamma^\mu} - y\overbrace{(x+z)}^{=1-y} m\ \underbrace{\not{p}\gamma^\mu}_{=2p^\mu-\gamma^\mu\not{p}}$$

$$+ \underbrace{(x+z)}_{=1-y}\not{p}\gamma^\mu\not{p}'(1-x)\Big]u(p)$$

$$\stackrel{\text{e.o.m.}}{=} \bar u(p')\Big[xym^2\gamma^\mu - 2x(1-x)mp'^\mu + x(1-x)m^2\gamma^\mu$$

$$-2y(1-y)mp^\mu + y(1-y)m^2\gamma^\mu$$

$$+ (1-y)(1-x)\left(2mp^\mu + 2mp'^\mu - 2pp'\gamma^\mu - m^2\gamma^\mu\right)\Big]u(p)$$

$\not{p}\gamma^\mu\not{p}'$

$= 2p^\mu\not{p}' - \gamma^\mu\not{p}\not{p}'$

$= 2p^\mu\not{p}' - 2pp'\gamma^\mu + \gamma^\mu\not{p}'\not{p}$

$= 2p^\mu\not{p}' - 2pp'\gamma^\mu + 2p'^\mu\not{p} - \not{p}'\gamma^\mu\not{p}$

$$= \bar u(p')\Big[\gamma^\mu m^2\big(xy + x(1-x) + y(1-y) - (1-x)(1-y)\big)$$

$$- \underbrace{2pp'}_{=-q^2+2m^2}\gamma^\mu(1-x)(1-y)$$

$$+ p'^\mu\big(-2x(1-x)m + 2m(1-x)(1-y)\big)$$

$$+ p^\mu\big(-2y(1-y)m + 2m(1-x)(1-y)\big)\Big]u(p)$$

Denominator symmetric under $x \Leftrightarrow y$

$$\longrightarrow \bar u(p')\Big[\gamma^\mu m^2\Big((1-x)\underbrace{(x-1+y)}_{-z} + (1-y)\underbrace{(y-1+x)}_{-z} + xy - (1-x)(1-y)\Big)$$

$$+\gamma^\mu q^2(1-x)(1-y)$$

$$+ m(p^\mu+p'^\mu)\big(-x(1-x) - y(1-y) + 2(1-x)(1-y)\big)\Big]u(p)$$

$$= \bar u(p')\Big[\gamma^\mu m^2\Big(-z\underbrace{(1-x-y)}_{z} - z\underbrace{-1+x+y}_{-z}\Big)$$

$$+\gamma^\mu q^2(1-x)(1-y)$$

$$+\,m(p^\mu+p'^\mu)\Big((1-x)\underbrace{(1-y-x)}_{z}+(1-y)\underbrace{(1-y-x)}_{z}\Big)\Big]\,u(p)$$

$$=\bar u(p')\Big[\gamma^\mu m^2\left(-z^2-2z\right)+\gamma^\mu q^2(1-x)(1-y)$$

$$+\,2m(p^\mu+p'^\mu)\left(z^2+z\right)\Big]\,u(p)$$

- $-2\Big(\underbrace{(1-2x)}_{\to 1-x-y}q+2zp\Big)^\mu \;\rightarrow\; -2\left(zp'^\mu-zp^\mu+2zp^\mu\right)$

since D symmetric under $x \Leftrightarrow y$

$$=-2z\left(p'^\mu+p^\mu\right)$$

$\Rightarrow$ Numerator $\longrightarrow$

$$\bar u(p')\Big[-\tfrac{1}{2}\gamma^\mu \ell^2+m^2\gamma^\mu\left(1-2z-z^2\right)$$

$$+\gamma^\mu q^2(1-x)(1-y)+m(p^\mu+p'^\mu)\left(z^2-z\right)\Big]\,u(p)$$

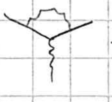
Gordon identity

$$\overset{\downarrow}{=}\ \bar u(p')\Big[-\tfrac{1}{2}\gamma^\mu\ell^2+\gamma^\mu q^2(1-x)(1-y)$$

$$+\,m^2\gamma^\mu\Big(\underbrace{1-2z-z^2+2z^2-2z}_{=1-4z+z^2}\Big)$$

$$-\,i\sigma^{\mu\nu}q_\nu\, m\, z(z-1)\Big]\,u(p)$$

<u>Overall expression:</u>



$$=2e^3\int dx\,dy\,dz\,\delta(1-x-y-z)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{2}{D^3}$$

$\leftarrow = \ell^2-\Delta$

$$\cdot\,\bar u(p')\Big[\gamma^\mu\Big(-\tfrac{1}{2}\ell^2+q^2(1-x)(1-y)+m^2\left(1-4z+z^2\right)\Big)$$

$$-\,i\sigma^{\mu\nu}q_\nu\, m\, z(z-1)\Big]\,u(p)$$

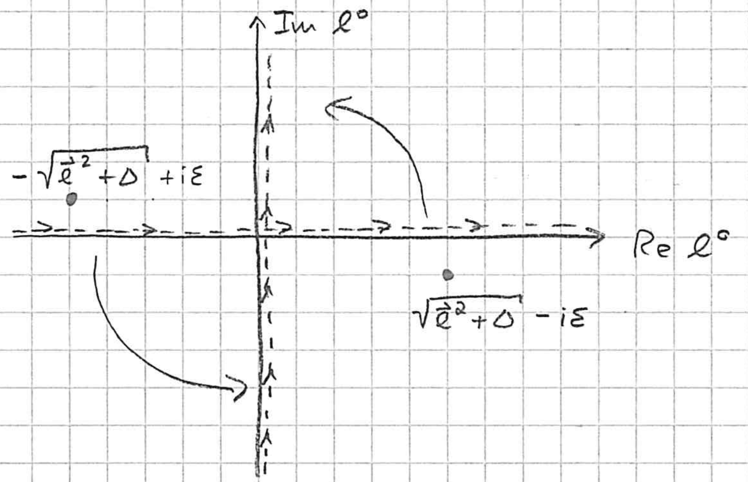
Peskin sec. 6.3
and sec. 7.5

4.6.3 The 4-momentum integral

Brute-force method:

- ℓ^0-integral via complex contour integration (taking into account that poles are shifted by $\pm i\varepsilon$)
- $\vec{\ell}$ integral in spherical coordinates

More elegant method: <u>Wick rotation</u>



Rotate contour by 90° counterclockwise

$$\hookrightarrow \int_{-\infty}^{\infty} d\ell^0 \longrightarrow \int_{-i\infty}^{+i\infty} d\ell^0 = i\int_{-\infty}^{\infty} d\ell^0_E$$

define $\ell^0_E = -i\ell^0$
(E ... Euclidean)

(works only if integrand goes to zero faster than $\left(\frac{1}{\ell^0_E}\right)^4$ at ∞!)

$$\Rightarrow \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} = i\int\frac{d^4\ell_E}{(2\pi)^4}\frac{1}{[-\ell_E^2-\Delta]^m}$$

$\vec{\ell}_E \equiv \vec{\ell}$

$\ell_E^2 \equiv (\ell^0_E)^2 + \vec{\ell}_E^{\,2}$

$\Rightarrow \ell_E$ is a vector in 4d <u>Euclidean</u> space

$$= \frac{(-1)^m i}{(2\pi)^4}\int_0^\infty \underbrace{2\pi^2}_{\text{surface "area" of 3-sphere}} d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m}$$

$$= \frac{(-1)^m i}{8\pi^2}\,\frac{1}{2}\,\frac{1}{(m-1)(m-2)}\,\frac{1}{\Delta^{m-2}}$$

$$\boxed{\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} = \frac{(-1)^m i}{16\pi^2}\frac{\Gamma(m-2)}{\Gamma(m)}\left(\frac{1}{\Delta}\right)^{m-2}} \quad \text{for } m>2 \qquad (*)$$

Similarly: $$\boxed{\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{[l^2-\Delta]^m} = \frac{(-1)^{m-1} i}{16\pi^2} \frac{2\,\Gamma(m-3)}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-3}}$$ for $m>3$ (**)

[see Peskin, appendix A.4 for more formulas of this type

<u>Problem</u>: For $m=3$, Wick rotation in (**) not possible (cannot close contour at ∞ since integral does not vanish there.

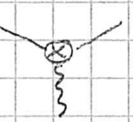
This <u>UV divergence</u> (ultraviolet divergence) can be seen from the expression

$$\int d^4 l \frac{l^2}{[l^2-\Delta]^3} \longrightarrow \log l \Big|_{l=0}^{\infty} \Rightarrow \text{"logarithmic divergence"}$$

At large l, the integrand goes to $\frac{1}{(l^2)^2}$

↳ $\int d^4 l \frac{1}{(l^2)^2}$ "$\sim$" $\int dl \frac{l^3}{l^4} = \int dx \frac{1}{x} \longrightarrow \infty$

<u>Solution</u>: <u>Renormalize</u> the theory by adding an infinite counterterm

 $= -ie(Z_e-1)\gamma^\mu$

to cancel the divergence

To define $Z_e - 1$, we need to first <u>regularize</u> the divergence.

Several regularization schemes:

- UV cutoff $\int_{-\infty}^{\infty} d^4 l \longrightarrow \int_{-\Lambda}^{\Lambda} d^4 l$, then take limit $\Lambda \to \infty$ in the very end
- Pauli-Villars scheme: Add fictitious heavy particle
  ↳ photon propagator $\frac{1}{(p-k)^2+i\varepsilon} \longrightarrow \frac{1}{(p-k)^2+i\varepsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\varepsilon}$
- Here: <u>Dimensional regularization</u>

end 17.12.2014

↳ Make # of spacetime dimensions d a variable

Idea: $\int d^d l \frac{l^2}{[l^2-\Delta]^3}$ is finite at $d = 4-\varepsilon$

Take limit $\varepsilon \to 0$ in the very end.

mass dimension changes — include factor μ^{ε} to fix. (μ = renormalization scale)

In $d > 4$ dimensions, Wick rotation is allowed

$$-\int \frac{d^4 l}{(2\pi)^4} \frac{\gamma^\nu \not{l} \gamma^\mu \not{l} \gamma_\nu}{[l^2-\Delta]^m} \rightsquigarrow -\int \frac{d^d l}{(2\pi)^d} \frac{\frac{(2-\varepsilon)^2}{d} l^2 \gamma^\mu}{[l^2-\Delta]^m} \to (-1)^{m-1} i \int \frac{d^d l_E}{(2\pi)^d} \frac{-\frac{(2-\varepsilon)^2}{d} l_E^2 \gamma^\mu}{[l_E^2+\Delta]^m}$$

$$= (-1)^{m-1} i \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dl_E \frac{l_E^2 \cdot l_E^{d-1}}{[l_E^2+\Delta]^m} \cdot \left(-\frac{(2-\varepsilon)^2}{d} \gamma^\mu\right) \qquad (*)$$

<u>Note</u>: in d-dim:

$$\int \frac{d^d l}{(2\pi)^4} \frac{l^\mu l^\nu}{D^3} = \int \frac{d^d l}{(2\pi)^4} \frac{\frac{1}{d} g^{\mu\nu} l^2}{D^3}$$

and $\gamma^\nu \gamma^\mu \gamma_\nu = -(2-\varepsilon)\gamma^\mu$

- Surface "area" of $(d-1)$-sphere:

$$(\sqrt{\pi})^d = \left(\int dx\, e^{-x^2}\right)^d = \int d^d x \exp\left(-\sum_{i=1}^{d} x_i^2\right)$$

$$= \int d\Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2} = \left(\int d\Omega_d\right) \cdot \frac{1}{2} \Gamma\left(\frac{d}{2}\right)$$

$$\Rightarrow \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

- $$\int_0^\infty dl_E \frac{l_E^{d+1}}{[l_E^2+\Delta]^m} = \frac{1}{2} \int_0^\infty d(l_E^2) \frac{(l_E^2)^{\frac{d}{2}}}{[l_E^2+\Delta]^m}$$

$$= \frac{1}{2} \int_0^1 dx \frac{1}{\Delta} \left(\Delta \frac{1-x}{x} + \Delta\right)^2 \cdot \frac{\Delta^{\frac{d}{2}} \left(\frac{1-x}{x}\right)^{\frac{d}{2}}}{\left[\Delta \frac{1-x}{x} + \Delta\right]^m}$$

$$x \equiv \frac{\Delta}{l_E^2+\Delta}, \quad dx = -\frac{\Delta \cdot d(l_E^2)}{[l_E^2+\Delta]^2}, \quad l_E^2 = \Delta \frac{1-x}{x}$$

$$= \frac{1}{2} \left(\frac{1}{\Delta}\right)^{m-\frac{d}{2}-1} \int_0^1 dx \frac{1}{x^2} \frac{x^m (1-x)^{d/2}}{x^{d/2}}$$

$$= \frac{1}{2} \left(\frac{1}{\Delta}\right)^{m-1-\frac{d}{2}} \frac{\Gamma\left(1+\frac{d}{2}\right) \Gamma\left(m-1-\frac{d}{2}\right)}{\Gamma(m)}$$

$$\Rightarrow \boxed{\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{[l^2-\Delta]^m} \longrightarrow (-1)^{m-1} i \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(m-1-\frac{d}{2}\right)}{\Gamma(m)} \left(\frac{1}{\Delta}\right)^{m-1-\frac{d}{2}}}$$

At $\varepsilon = 4-d$, this can be rewritten for $m=3$ by noting that

$$\Gamma(x) \overset{x \simeq 0}{=} \frac{1}{x} - \gamma + O(x)$$

Euler-Mascheroni constant $\simeq 0.5772$

$$\left(\frac{1}{\Delta}\right)^x \overset{x \simeq 0}{=} 1 - x \log \Delta + O(x^2)$$

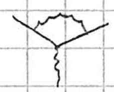
$2-\frac{d}{2}=\frac{\varepsilon}{2}\Big]$

from (*) (previous page) $\Longrightarrow \frac{(-1)^2\, i}{(4\pi)^2}\left(1+\frac{\varepsilon}{2}\log 4\pi\right)\left(2-\frac{\varepsilon}{2}\right)\underbrace{\frac{1}{\Gamma(3)}}_{=\frac{1}{2}}\left(\frac{2}{\varepsilon}-\gamma\right)\left(1-\frac{\varepsilon}{2}\log\Delta\right)$

$$= +\frac{i}{(4\pi)^2}\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log\Delta-\frac{1}{2}+O(\varepsilon)\right)$$

<u>Putting everything together</u>

from Feynman parameter formula

 $= e^3\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot 2\cdot\frac{i}{16\pi^2}$

$$\cdot\,\bar u(p')\Bigg[\underbrace{\left(-\frac{(2-\varepsilon)^2}{4-\varepsilon}\right)}_{\simeq -(1-\frac{3}{4}\varepsilon)}\gamma^\mu\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log\Delta-\frac{1}{2}\right)$$

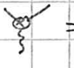
$$+\left\{-2\gamma^\mu\left(q^2(1-x)(1-y)+m^2(1-4z+z^2)\right)+2i\sigma^{\mu\nu}q_\nu\, m\, z(z-1)\right\}\left(-\frac{1}{2\Delta}\right)\Bigg]u(p)$$

?

$$= -ie\frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)$$

$$\cdot\,\bar u(p')\Bigg[+\gamma^\mu\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log\Delta-2\right)$$

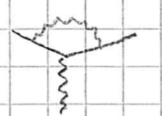
$$+\gamma^\mu\frac{1}{\Delta}\left(q^2(1-x)(1-y)+m^2(1-4z+z^2)\right)-i\sigma^{\mu\nu}q_\nu\frac{m}{\Delta}z(z-1)\Bigg]u(p)$$

Defining $Z_e-1=-\frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot\frac{2}{\varepsilon}$, the counterterm  $=-ie(Z_e-1)\gamma^\mu$ exactly cancels the divergence (<u>minimal subtraction</u> or MS)

It is often more convenient to define

$$Z_e-1=-\frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log\mu^2\right)$$

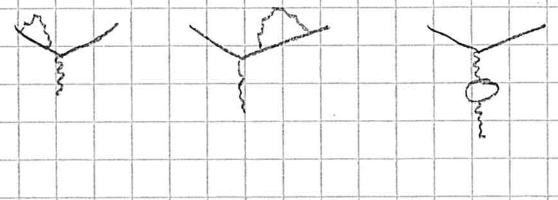
with an (unphysical) <u>renormalization scale μ</u> ($\overline{MS}$ renormalization)

$\Rightarrow$  $\rightarrow -ie\frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)$

$$\bar u(p')\Bigg[\gamma^\mu\left(-\log\frac{\Delta}{\mu^2}-2+\frac{q^2}{\Delta}(1-x)(1-y)+\frac{m^2}{\Delta}(1-4z+z^2)\right)$$

$$-i\sigma^{\mu\nu}q_\nu\frac{m}{\Delta}z(z-1)\Bigg]u(p)$$

Note: • $Z_e - 1$ also receives contributions from other loop diagrams, in particular

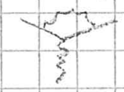
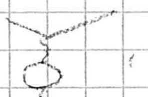


• Results depend on the renormalization scheme (cf. MS vs. $\overline{MS}$ schemes).

↳ ultimately, finite parts of counterterms are fixed by requiring physical observables to match experiment (+ field vev and normalization to satisfy conditions for LSZ reduction formula: $\langle \phi \rangle = 0$ and $\langle k | \phi | 0 \rangle = e^{ikx}$, see sec. 2.3)

• Impose renormalization condition that  matches experimental result for e^- scattering from fixed potential

↳ all loop corrections cancelled by counterterm

But:  ,  , etc. depend on q^2

$\Rightarrow$ If renormalization condition is imposed based on a different experiment (different q^2), different e must be used

$\Rightarrow$ $\boxed{e \longrightarrow e(q^2)}$

Renormalization group evolution or "running" of e.

• In dim. reg., be careful:

$$g^\mu{}_\mu = d$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(2-\varepsilon)\gamma^\nu$$

But still

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

(but with $\mu, \nu = 0 \ldots d$ and $g^{\mu\nu} = \mathrm{diag}(1, -1, \cdots -1)$)

end 18.12.2014

4.6.4 Summary: Recipe for loop calculations

1. Write down amplitude from Feynman rules
2. Introduce Feynman parameters to combine denominators
3. Complete the square (of the loop momentum k) in the denominators by shifting the integration variable appropriately ($k \to \ell$)
4. Rewrite numerator in terms of ℓ, drop odd powers
5. Do momentum integral using Wick rotation and dimensional regularization (or use formulary, e.g. Peskin/Schröder sec. A.4)

4.6.5 Lepton magnetic moments

Consider $F_2(q^2) = \frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)\,\frac{2m^2}{\Delta}\,z(z-1)$

Rember that $F_2(0) = \frac{g-2}{2}$ is a measurable quantity

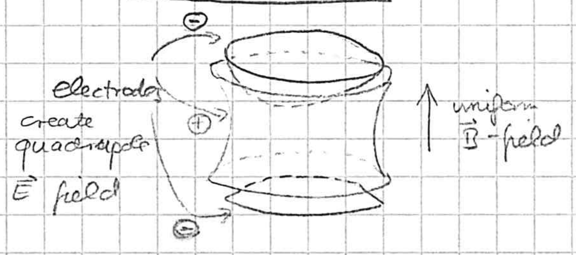
$$F_2(0) = -\frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)\,\frac{2m^2}{(1-z)^2 m^2}\,z(z-1)$$

$$= \frac{\alpha}{2\pi}\int dz\,(1-z)\cdot\frac{2z(1-z)}{(1-z)^2}$$

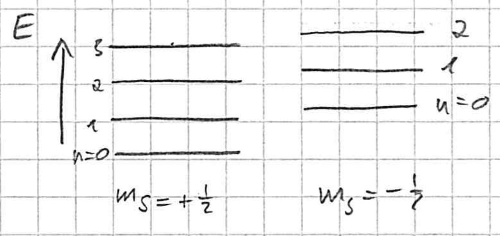
$$\boxed{\frac{g-2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)}$$

Measuring $g-2$:

- Of the electron: e^- in Penning trap



Due to potential $V = -\vec{\mu}\cdot\vec{B}$, energy levels depend on μ and thus on g.

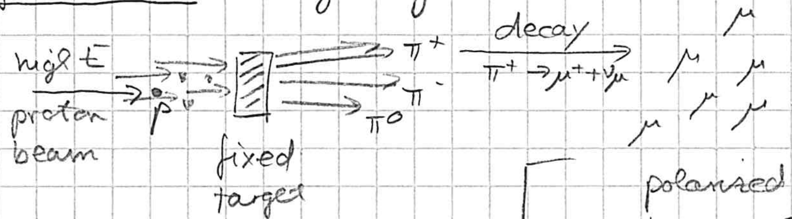


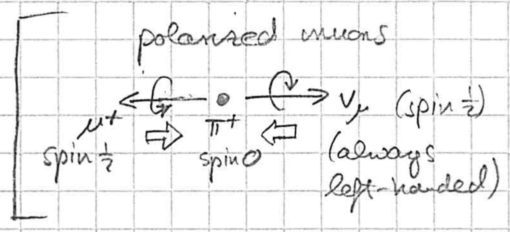
Apply oscillating e.m. fields to induce transitions. Polarization of oscillating fields determines if spin + orbital or orbital only transitions are induced

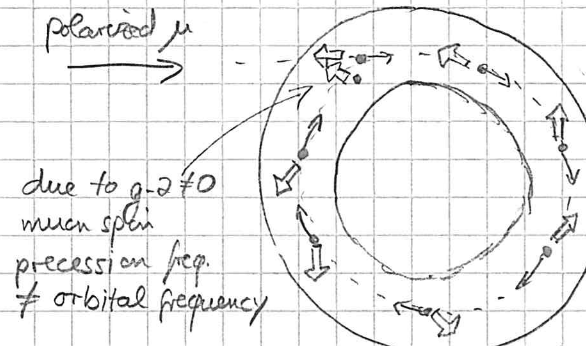
Measure change in e^- cyclotron frequency

Frequency of driving fields at resonance determines energy levels and thus $g-2$.

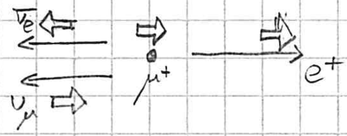
• <u>Of the muon:</u> Storage ring







Measure muon spin from decay $\mu^+ \longrightarrow e^+ \nu_e \overline{\nu_\mu}$



$\overline{\nu_e}$ always RH, ν_μ always LH, e^+ preferentially RH

↳ e^+ emitted in direction of muon spin

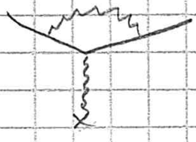
<u>Results:</u>

		Data	$\frac{\alpha}{2\pi}$	Most accurate prediction available
Hanneke Fogwell Gabrielse 2008 and Kinoshita (talk at Riken)	$\left(\frac{g-2}{2}\right)_e$	0.001 159 652 180 73(28)	0.001 161	0.001 159 652 182 79 (772
Bennett et al. 2006 and PDG Review on $(g-2)_\mu$	$\left(\frac{g-2}{2}\right)_\mu$	0.001 165 920 80 (63)	0.001 161	0.001 165 918 03 (49)

3.5σ discrepancy !

4.7 IR divergencies

Consider again 1-loop vertex corrections



$$= \bar{u}(p')\left(-ie\,\Gamma^{\mu,(1)}\right)u(p)\cdot\tilde{A}^{ext}_{\mu}(p'-p)$$

"1-loop order"

$$\Gamma^{\mu,(1)} = \gamma^\mu F_1^{(1)}(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2^{(1)}(q^2)$$

$$F_1^{(1)}(q^2) = \frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)$$

$$\cdot\left[-\log\frac{\Delta}{\mu^2} - 2 + \frac{q^2}{\Delta}(1-x)(1-y) + \frac{m^2}{\Delta}(1-4z+z^2)\right]$$

$$F_2^{(2)}(q^2) = \frac{\alpha}{2\pi}\int dx\,dy\,dz\;\delta(1-x-y-z)\cdot\frac{2m^2}{\Delta}z(z-1)$$

$$\Delta = m^2(1-z)^2 - xyq^2$$

Note that at $z\to 1$, $x,y\to 0$, $\Delta\to 0$

$\hookrightarrow$ the terms $\propto\frac{1}{\Delta}$ in $F_1(q^2)$ are infinite!

<u>Reason:</u> photon propagator as well as e^- propagators in loop can go on-shell simultaneously

<u>Discussion in the limit $q^2\gg m^2$</u> (see Peskin for full discussion)

$$F_1^{(1)}(q^2) \xrightarrow{q^2\gg m^2} \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z}dy\;\frac{q^2\overbrace{(y+z)(1-y)}^{\simeq 1\text{ in divergent region}}}{-y(1-y-z)q^2 + m^2(1-z)^2}$$

$$\left[\frac{d(w^2,\xi)}{d(y,z)} = \begin{pmatrix} 0 & -2(1-z) \\ \frac{1}{1-z} & -\frac{y}{(1-z)^2}\end{pmatrix}\right]$$

$$\overset{w\equiv 1-z}{\underset{\xi\equiv\frac{y}{1-z}}{=}} \frac{\alpha}{2\pi}\underbrace{\frac{1}{2}}_{\text{Jacobian}}\int_0^1 d\xi\int_0^1 d(w^2)\;\frac{q^2}{m^2w^2 - q^2\xi(1-\xi)w^2}$$

$$\left[\xi(1-\xi)w^2 = \frac{y}{1-z}\left(\frac{1-z-y}{1-z}\right)(1-z)^2\right]$$

We regularise the divergence by pretending the photon has a small mass M

$\hookrightarrow \Delta \to \Delta + M^2 \cdot z \simeq \Delta + M^2$ in the divergent region

$$\Rightarrow F_1^{(1)}(q^2) \overset{q^2 \gg m^2}{\simeq} \frac{\alpha}{4\pi} \int_0^1 d\xi \int_0^1 d(w^2) \frac{q^2}{w^2[m^2 - q^2\xi(1-\xi)] + M^2}$$

$$\overset{M^2\to 0}{\simeq} \frac{\alpha}{4\pi} \int_0^1 d\xi \frac{q^2}{m^2 - q^2\xi(1-\xi)} \log \frac{m^2 - q^2\xi(1-\xi)}{M^2}$$

numerator not very important for $M \to 0$

$$\underset{q^2 \gg m^2}{\overset{M^2\to 0}{\simeq}} \frac{\alpha}{4\pi} \log\left(-\frac{q^2}{M^2}\right) \left[\int_0^{0+\varepsilon} d\xi \frac{q^2}{m^2 - q^2 \xi \underbrace{(1-\xi)}_{\simeq 1}} + \int_{1-\varepsilon}^1 d\xi \frac{q^2}{m^2 - q^2 \underbrace{\xi}_{\simeq 1} (1-\xi)} \right]$$

$$= \frac{\alpha}{4\pi} \log\left(-\frac{q^2}{M^2}\right) \cdot (-2) \log\left(-\frac{q^2}{m^2}\right)$$

strictly speaking: $\log\left(\frac{m^2 - q^2\varepsilon}{m^2}\right)$
but this is $\simeq \log\left(\frac{-q^2}{m^2}\right)$ for $q^2 \gg m^2$

$\Rightarrow$ overall, including tree level and IR-divergent terms @ 1-loop

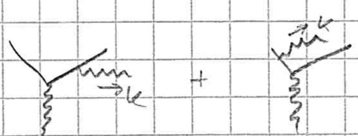
$$F_1(q^2) \underset{M^2 \to 0}{\overset{q^2 \gg m^2}{\simeq}} 1 - \frac{\alpha}{2\pi} \log\left(-\frac{q^2}{M^2}\right) \log\left(-\frac{q^2}{m^2}\right) + \dots$$

Note that F_1 is a correction to the electric charge

$$\Rightarrow \boxed{ \left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{1\text{-loop}} = \left.\frac{d\sigma(p\to p')}{d\Omega}\right|_{\text{tree}} \left(1 - \frac{\alpha}{\pi} \log\left(-\frac{q^2}{M^2}\right) \log\left(-\frac{q^2}{m^2}\right)\right) }$$

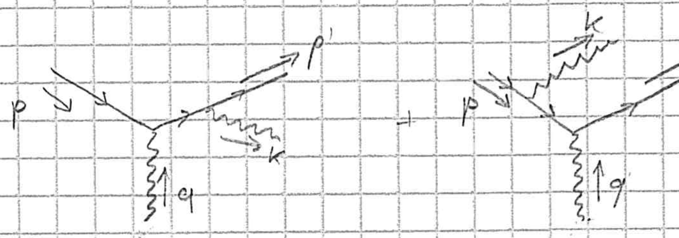
Cut!

We next consider the seemingly unrelated process



(with k very soft)

and show that its inclusion cancels the divergence



$$= \varepsilon^*_\nu(k)\,\bar{u}(p')\left[(-ie\gamma^\nu)\frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2+i\epsilon}(-ie\gamma^\mu) + (-ie\gamma^\mu)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}(-ie\gamma^\nu)\right]u(p)\,\underbrace{\quad}_{\tilde{A}^{ext}_\mu}$$

$$\overset{|\vec{k}|\ll|\vec{p}|,|\vec{p}'|,|\vec{q}|}{\simeq} -ie^2\,\bar{u}(p')\left[\gamma^\nu\frac{\not{p}'+m}{2p'\cdot k+i\epsilon}\gamma^\mu + \gamma^\mu\frac{\not{p}+m}{-2p\cdot k+i\epsilon}\gamma^\nu\right]u(p)\,\varepsilon^*_\nu(k)\,\tilde{A}^{ext}_\mu$$

anticommute $\not{p}'\gamma^\nu$ and $\not{p}\gamma^\nu$, use e.o.m.

$$= -ie^2\,\bar{u}(p')\left[\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right]\gamma^\mu u(p)\,\tilde{A}^{ext}_\mu$$

$$= i\mathcal{M}(p\to p')\; e\left[\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right]$$

matrix element for the LO process

14.01.2015

$$\Rightarrow \frac{d\sigma(p\to p'+\gamma)}{d\Omega} = \frac{d\sigma(p\to p')}{d\Omega}\bigg|_{LO}\cdot\sum_{pol}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2|\vec{k}|}e^2\left|\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right|^2$$

$$= \frac{d\sigma(p\to p')}{d\Omega}\bigg|_{LO}\int\frac{dk\,d\Omega_k}{(2\pi)^3 2k}e^2k^2\Bigg(\overbrace{\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)}}^{\simeq q^2 \text{ at } q^2\gg m^2} - \underbrace{\frac{p'^2}{(p'\cdot k)^2} - \frac{p^2}{(p\cdot k)^2}}_{\text{negligible at lc } q^2}$$

Angular integral: go to frame where $E_{\vec{p}} = E_{\vec{p}'} = E$

$$\hookrightarrow\; p = E(1,\vec{v})\;;\quad p' = E(1,\vec{v}')\;;\quad k = (|\vec{k}|,\vec{k})$$

$$\Rightarrow \int d\Omega_k\,\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)} = \int d\Omega_k\,\frac{2\cancel{E^2}(1-\vec{v}\cdot\vec{v}')}{\cancel{E^2}k^2(1-\hat{k}\cdot\vec{v})(1-\hat{k}\cdot\vec{v}')}$$

$$\overset{v,v'\simeq 1}{\simeq} 2\cdot 2\pi\,\frac{1}{k^2}\left[\int_{\hat{k}\cdot\vec{v}=\vec{v}'\cdot\vec{v}}^{\cos\theta=1} d\cos\theta\,\frac{1}{1-\cos\theta\cdot|\vec{v}|} + \int_{\hat{k}\cdot\vec{v}'=\vec{v}'\cdot\vec{v}}^{\cos\theta=1} d\cos\theta\,\frac{1}{1-\cos\theta\cdot|\vec{v}'|}\right]$$

at $v,v'\simeq 1$, most important contributions from $\vec{k}\parallel\vec{v}$ and $\vec{k}\parallel\vec{v}'$

lower boundaries not crucial — result will only depend on them logarithmically

$$= \frac{4\pi}{k^2}\left[\log\frac{1-\vec{v}\cdot\vec{v}'}{1-v} + \log\frac{1-\vec{v}\cdot\vec{v}'}{1-v'}\right]$$

$$v = \frac{p}{E} = \frac{\sqrt{E^2-m^2}}{E} \simeq 1-\frac{m^2}{2E^2}$$

$$= \frac{4\pi}{k^2}\log\frac{(p\cdot p')^2}{E^4\cdot\left(\frac{m^2}{2E^2}\right)^2}$$

$$\simeq \frac{8\pi}{k^2} \log \frac{-q^2}{m^2}$$

upper boundary is of this order; result will only depend logarithmically on exact value

$$\Rightarrow \frac{d\sigma(p \to p' + \gamma)}{d\Omega} \simeq \left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{LO} \cdot \int_0^{|\vec{q}|} \frac{dk}{(2\pi)^3 \cdot 2k} \, e^2 \cdot 8\pi \log \frac{-q^2}{m^2}$$

$$= \left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{LO} \cdot \frac{2\alpha}{\pi} \log \frac{-q^2}{m^2} \int_0^{|\vec{q}|} \frac{dk}{k}$$

.... another infinite integral

Regularize by pretending photon has a small mass μ

↳ lower cutoff to the integral

$$\frac{d\sigma(p \to p' + \gamma)}{d\Omega} \simeq \left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{LO} \cdot \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)$$

Remember: Virtual correction gave

$$\left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{1\text{-loop}} \simeq \left.\frac{d\sigma(p \to p')}{d\Omega}\right|_{LO} \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)\right]$$

$\Rightarrow$ The divergence cancels when both virtual corrections and soft photon emission are considered.

This is indeed the rigest thing to do since infinitely soft photons cannot be separately detected experimentally

4.8 Renormalization group evolution

"Bare" QED Lagrangian:

$$\mathcal{L}_{QED} = \bar{\psi}_0 (i\not{\partial}) \psi_0 \left[- \bar{\psi}_0 m_0 \psi_0 \right] - e_0 \bar{\psi}_0 \gamma^\mu A_{\mu 0} \psi_0 - \frac{1}{4} F_{\mu\nu 0} F^{\mu\nu}{}_0$$

Identical to renormalised Lagrangian

$$\mathcal{L}_{QED} = Z_\psi \bar{\psi} i\not{\partial} \psi \left[- Z_m \bar{\psi} m \psi \right] - Z_e e \bar{\psi} \gamma^\mu A_\mu \psi - \frac{1}{4} Z_A F_{\mu\nu} F^{\mu\nu}$$

where

$$\psi_0(x) = Z_\psi^{1/2} \psi(x)$$

$$\left[m_0 = Z_m Z_\psi^{-1} m \right]$$

$$e_0 = Z_e Z_\psi^{-1} Z_A^{-\frac{1}{2}} e$$

$$A_{\mu 0}(x) = Z_A^{\frac{1}{2}} A(x)$$

The renormalised and bare theory must be equivalent, therefore in particular

$$\langle 0 | T \psi(x_1) \cdots \psi(x_n) | 0 \rangle = \langle 0 | T \psi_0(x_1) \cdots \psi_0(x_n) | 0 \rangle Z_\psi^{-\frac{n}{2}}$$

and similar for correlation functions involving $\bar{\psi}$ and A_μ and combinations of them. Also, physical observables must be identical in both theories.

Renormalised theory depends on renormalization scale μ. Shifting μ must leave physics invariant. This requires a corresponding shift in the renormalized coupling constants. For massless QED

$$\mu \to \mu + \delta\mu \qquad\qquad e \to e + \delta e$$

$$\psi \to (1 + \delta\eta)\, \psi$$

$$A_\mu \to (1 + \delta\xi)\, A_\mu$$

$\Rightarrow$ For $G^{(n_\psi, n_{\bar\psi}, n_A)} \equiv \langle 0 | \psi(x_1) \cdots \psi(x_{n_\psi}) \bar\psi(y_1) \cdots \bar\psi(y_{n_{\bar\psi}}) A_\mu(z_1) \cdots A_\rho(z_{n_A}) | 0 \rangle$

$$\delta G^{(n_\psi, n_{\bar\psi}, n_A)} = (n_\psi \delta\eta + n_{\bar\psi} \delta\eta^* + n_A \delta\xi) \, G^{(n_\psi, n_{\bar\psi}, n_A)}$$

$$= \frac{\partial G^{(n_\psi, n_{\bar\psi}, n_A)}}{\partial\mu} \delta\mu + \frac{\partial G^{(n_\psi, n_{\bar\psi}, n_A)}}{\partial e} \delta e$$

Define $\boxed{\beta \equiv \frac{\mu}{\delta\mu} \delta e}$; $\boxed{\gamma_\psi = -\frac{\mu}{\delta\mu} \delta\eta}$; $\boxed{\gamma_A = -\frac{\mu}{\delta\mu} \delta\xi}$

$$\Rightarrow \boxed{\left[\mu \frac{\partial}{\partial\mu} + \beta \frac{\partial}{\partial e} + n_\psi \gamma_\psi + n_{\bar\psi} \gamma_\psi^* + n_A \gamma_A\right] G^{(n_\psi, n_{\bar\psi}, n_A)} = 0}$$

Callan - Symanzik Equation

Of particular interest is β, which describes the running of the coupling constant and hence the strength of the interaction at different scales.

Computation of β

$$\mu \frac{\delta e}{\delta\mu} = \mu \frac{\delta}{\delta\mu} \left(Z_\psi Z_A^{\frac{1}{2}} Z_e^{-1} \right) e_0$$

$$\overset{1}{=} \mu e_0 \underbrace{\left(\frac{\delta(Z_\psi - 1)}{\delta\mu} + \frac{1}{2} \frac{\delta(Z_A - 1)}{\delta\mu} - \frac{\delta(Z_e - 1)}{\delta\mu} \right)}_{\text{derivatives of the counterterms!}}$$

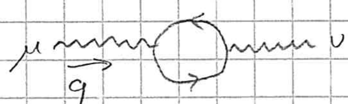
1st order expansion
$Z_\psi = 1 + O(e^2)$
$Z_A = 1 + O(e^2)$
$\vdots$

We have seen: $\quad Z_e - 1 = -\frac{\alpha}{2\pi} \int dx\, dy\, dz\, \delta(1-x-y-z) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log \mu^2 \right.$

$$\frac{\partial(Z_e - 1)}{\partial\mu} = \frac{\alpha}{\pi} \cdot \frac{1}{\mu} \int dx\, dy\, dz\, \delta(1-x-y-z)$$

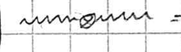
$$= \frac{\alpha}{2\pi} \frac{1}{\mu} \qquad ?$$

Similarly, we can compute the diagrams

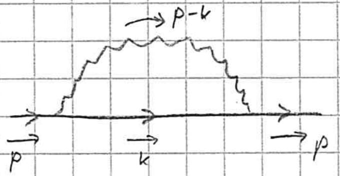


$$\overset{\text{Peskin sec. 7}}{=} i(q^2 g^{\mu\nu} - q^\mu q^\nu)\left(\frac{2\alpha}{\pi}\right)\int_0^1 dx\; x(1-x)\left(\frac{2}{\varepsilon} - \log\tilde{\Delta} - \gamma + \log 4\pi\right)$$

$$\Rightarrow \quad Z_A - 1 = \frac{2\alpha}{6\pi}\log\mu^2 + (\text{terms not containing } \mu)$$

$\boxed{\quad}$  $= i(q^2 g^{\mu\nu} - q^\mu q^\nu)(Z_A - 1)$

$$\frac{\partial(Z_A - 1)}{\partial\mu} = \frac{2\alpha}{3\pi}\frac{1}{\mu}$$



$$\overset{\text{Peskin 10.41}}{=} i\frac{\alpha}{4\pi}\int_0^1 dx\left(\frac{2}{\varepsilon} - \log\hat{\Delta} - \gamma + \log 4\pi\right)(2-\varepsilon)\times\not{p}$$

$$\Rightarrow \quad Z_\psi - 1 = \frac{\alpha}{4\pi}\log\mu^2 + (\text{terms not containing } \mu)$$

$\boxed{\quad}$  $= i\not{p}(Z_\psi - 1)$

$$\frac{\partial(Z_\psi - 1)}{\partial\mu} = \frac{\alpha}{2\pi}\frac{1}{\mu}$$

$$\Rightarrow \quad \beta = \mu e_0\left(\frac{\alpha}{2\pi}\frac{1}{\mu} + \frac{1}{2}\left(\frac{2\alpha}{3\pi}\frac{1}{\mu}\right) - \frac{\alpha}{2\pi}\frac{1}{\mu}\right)$$

$$\boxed{\beta = \frac{e^3}{12\pi^2}}$$

(note $e_0 = e + O(e^2)$ $\Rightarrow$ at lowest order $e = e_0$)

$\Rightarrow$ Coupling becomes stronger ($\beta > 0$) for larger μ = higher energy

$$\mu\frac{\partial e(\mu)}{\partial\mu} = \frac{e^3}{12\pi^2} \quad \Rightarrow \quad \frac{1}{e^3}de = \frac{1}{12\pi^2}\frac{1}{\mu}d\mu$$

$$-\frac{1}{2}\frac{1}{e^2}\Big|_{e_1}^{e_2} = \frac{1}{12\pi^2}\ln\mu\Big|_{\mu_1}^{\mu_2}$$

$$\frac{1}{e_2^2} - \frac{1}{e_1^2} = -\frac{1}{6\pi^2}\ln\frac{\mu_2}{\mu_1}$$

$$\boxed{e_2^2 = \frac{e_1^2}{1 - \frac{e_1^2}{6\pi^2}\ln\frac{\mu_2}{\mu_1}}}$$

Peskin sec. 15
Srednicki sec. 69

5. Non-Abelian Gauge Theories

evtl. weglassen

5.1 Gauge interactions from symmetry arguments

Goal: Show how (gauge) interactions follow from symmetry

Consider Dirac field $\psi(x)$ and impose symmetry of $\mathcal{L}$ under

$$\boxed{\psi \longrightarrow e^{i\alpha(x)}\psi(x) \equiv \mathcal{U}(x)\,\psi(x)}$$

Mass term $-m\bar{\psi}\psi$ is invariant, but kinetic term $\bar{\psi}\, i\partial\!\!\!/\, \psi$ is not!

$$n^\mu \partial_\mu \psi(x) = \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$

Define comparator $W(y,x)$ such that

$$W(y,x) \longrightarrow e^{i\alpha(y)}\, W(y,x)\, e^{-i\alpha(x)}$$

and covariant derivative D_μ:

$$n^\mu D_\mu \psi(x) \equiv \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - W(x+\varepsilon n, x)\,\psi(x)}{\varepsilon}$$

$$\Rightarrow \boxed{D_\mu \psi(x) \longrightarrow e^{i\alpha(x)}\,\psi(x)}$$

and $\bar{\psi}\, i D\!\!\!\!/\, \psi$ is gauge invariant.

Properties of $W(y,x)$: we define without loss of generality

- $W(y,y) \equiv 1$
- $|W(y,x)| = 1$

Infinitesimally:

arbitrary vector field "gauge connection"

$$W(x+\varepsilon n, x) \equiv 1 - i\varepsilon e\, n^\mu A_\mu(x)$$

arbitrarily pulled out

At finite separation:

$$W(y,x) = \exp\left[-ie\oint_x^y dx^\mu A_\mu(x)\right] \qquad \text{"Wilson line"}$$

$$\Rightarrow \boxed{D_\mu \psi(x) = \partial_\mu \psi(x) + i e A_\mu(x) \psi(x)}$$

Transformation of $W(y,x)$ implies

$$W(x+\varepsilon n, x) \longrightarrow e^{i\alpha(x+\varepsilon n)} W(x+\varepsilon n, x) e^{-i\alpha(x)}$$

$$= e^{i\alpha(x)} \left(1 + i\varepsilon n^\mu \partial_\mu \alpha(x)\right)\left(1 - i\varepsilon e n^\mu A_\mu(x)\right) e^{-i\alpha(x)}$$

$$= 1 + i\varepsilon e n^\mu \left(A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)\right)$$

$$\Rightarrow \boxed{A_\mu(x) \longrightarrow A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$$

Existence of photon field follows from gauge symmetry.

Construction of photon kinetic term:

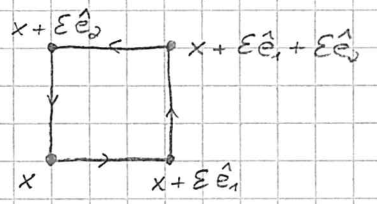
↳ look for gauge invariant term depending on A_μ, but not ψ. Should be quadratic in A_μ to yield propagator $\langle A_\mu(x) A_\nu(y) \rangle$

Note that

$$W(x_1, x_2)\, W(x_2, x_3) \cdots W(x_{n-1}, x_n)\, W(x_n, x_1)$$

is gauge invariant. Since terms in $\mathcal{L}$ should be local (depend on only one x), consider limit $x_j \to x \ \forall j$.

Infinitesimal loop:



$$\hookrightarrow W(x, x+\varepsilon\hat{e}_2)\, W(x+\varepsilon\hat{e}_2, x+\varepsilon\hat{e}_1+\varepsilon\hat{e}_2)\, W(x+\varepsilon\hat{e}_1+\varepsilon\hat{e}_2, x+\varepsilon\hat{e}_1)$$

$$\cdot W(x+\varepsilon\hat{e}_1, x) \quad =$$

$$= \exp\left[-ie\varepsilon\left(-\hat{e}_2^{\mu} A_\mu\left(x+\frac{\varepsilon}{2}\hat{e}_2\right) - \hat{e}_1^{\mu} A_\mu\left(x+\frac{\varepsilon}{2}\hat{e}_1 + \varepsilon\hat{e}_2\right) + \hat{e}_2^{\mu} A_\mu\left(x+\varepsilon\hat{e}_1+\frac{\varepsilon}{2}\hat{e}_2\right) + \hat{e}_1^{\mu} A_\mu\left(x+\frac{\varepsilon}{2}\hat{e}_1\right)\right)\right]$$

expand to $\mathcal{O}(\varepsilon^2)$

$$\simeq 1 - ie\varepsilon\Big(-\underbrace{\hat{e}_2^{\mu}\,\partial^\nu A_\mu(x)\cdot\frac{\varepsilon}{2}\hat{e}_{2\nu}}_{=-\frac{\varepsilon}{2}\partial^2 A_2(x)} - \underbrace{\hat{e}_1^{\mu}\,\partial^\nu A_\mu(x)\left(\frac{\varepsilon}{2}\hat{e}_{1\nu}+\varepsilon\hat{e}_{2\nu}\right)}_{-\frac{\varepsilon}{2}\partial^1 A_1(x)+\varepsilon\partial^2 A_1(x)}$$

from contracting spatial indices

$$- \frac{\varepsilon}{2}\partial^2 A_2(x) + \varepsilon\partial^1 A_2(x) - \frac{\varepsilon}{2}\partial^1 A_1(x)\Big)$$

$$= 1 - ie\varepsilon^2\left(\partial_1 A_2(x) - \partial_2 A_1(x)\right)$$

Similarly for other spacetime directions

$$\hookrightarrow \quad \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \equiv F_{\mu\nu}(x)$$

is gauge invariant.

Postulate <u>gauge kinetic term</u>

$$\boxed{\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

$\Rightarrow$ QED Lagrangian completely follows from symmetry

... including Maxwell's equations!

5.2 Non-Abelian gauge symmetry

Transformations $e^{i\alpha}$ form group: $U(1)$
("unitary trafos in 1D")

Consider now theory with N fermions $\psi = (\psi_1, \psi_2, \dots \psi_N)$ and require invariance under

$$\psi(x) \to U(x)\, \psi(x)$$

where $U(x)$ is an $N \times N$ transformation matrix with $|\det U| = 1$ (to preserve particle number).

We write

$$U(x) = \exp\left[i \alpha^a(x)\, t^a\right]$$

where t^a is a set of matrices ("generators") that defines the allowed transformations.

Example:

- $U(x) \in SU(2)$ ("special unitary transformations (i.e. unitary trafos that have $\det U = 1$, not just $|\det U| = 1$) in 2D")

  $\hookrightarrow t^a = \frac{\sigma^a}{2}$ Pauli matrices

- $U(x) \in SU(3)$

  $\hookrightarrow t^a = \frac{\lambda^a}{2}$ Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}; \quad \lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}; \quad \lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}; \quad \lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}; \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

To define $\mathcal{L}$ invariant under $\psi \to \mathcal{U}\psi$, define $W(y,x)$ such that

$$W(y,x) \longrightarrow e^{i\alpha^a(y)t^a}\, W(y,x)\, e^{-i\alpha^a(x)t^a}$$

Write

for now arbitrary constant

$$W(x+\varepsilon n, x) \equiv 1 + i\varepsilon g\, t^a n^\mu A^a_\mu(x)$$

set of vector fields

$$\Rightarrow W(y,x) = P\exp\left[ig \oint dx^\mu\, t^a A^a_\mu(x)\right]$$

path ordering, implies that, in the terms of the exp series, factors $t^a A^a_\mu(x)$ are ordered along the integration contour:

$$t^a A^a_\mu(x_n) \cdots t^b A^b_\mu(x_1)$$

end of path

beginning of path

$\Rightarrow$ Covariant derivative

$$n^\mu D_\mu \psi = \lim_{\varepsilon\to 0} \frac{\psi(x+\varepsilon n) - W(x+\varepsilon n, x)\psi(x)}{\varepsilon}$$

$$= n^\mu \partial_\mu \psi(x) - i g n^\mu t^a A^a_\mu \psi(x)$$

$$\boxed{D_\mu = \partial_\mu - i g t^a A^a_\mu(x)} \quad \Rightarrow \mathcal{L} \supset \bar\psi(i\not{D} - m)\psi$$

Transformation of $A^a_\mu(x)$:

$$W(x+\varepsilon n, x) \to \underbrace{\mathcal{U}(x+\varepsilon n)}_{\simeq (1+\varepsilon n^\mu \partial_\mu)\,\mathcal{U}(x)}\left(1 + i\varepsilon g\, t^a n^\mu A^a_\mu(x)\right)\mathcal{U}^{-1}(x)$$

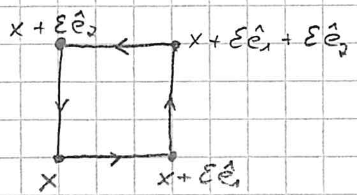
$$= 1 + \varepsilon n^\mu (\partial_\mu \mathcal{U})\mathcal{U}^{-1} + i\varepsilon g \cdot \mathcal{U} t^a n^\mu A^a_\mu \mathcal{U}^{-1}$$

$$= 1 + i\varepsilon g n^\mu \left(\mathcal{U} t^a A^a_\mu \mathcal{U}^{-1} - \frac{i}{g}\underbrace{(\partial_\mu \mathcal{U})\mathcal{U}^{-1}}_{= -\mathcal{U}\partial_\mu \mathcal{U}^+}\right)$$

since $\partial_\mu(\mathcal{U}\mathcal{U}^+) =$

$$\Rightarrow \boxed{A_\mu^a(x)t^a \longrightarrow \mathcal{U}(x)\left(A_\mu^a(x)t^a + \frac{i}{g}\partial_\mu\right)\mathcal{U}^{-1}(x)}$$

Gauge kinetic term:



$$\hookrightarrow W(x, x+\varepsilon\hat e_2)\, W(x+\varepsilon\hat e_2, x+\varepsilon\hat e_1+\varepsilon\hat e_2)\, W(x+\varepsilon\hat e_1+\varepsilon\hat e_2, x+\varepsilon\hat e_1)$$

$$\cdot\, W(x+\varepsilon\hat e_1, x)$$

$$= P\exp\left[-ig\varepsilon\,\hat e_2^\mu t^a A_\mu^a\left(x+\frac{\varepsilon}{2}\hat e_2\right)\right]\cdot P\exp\left[-ig\varepsilon\,\hat e_1^\mu t^a A_\mu^a\left(x+\varepsilon\hat e_2+\frac{\varepsilon}{2}\hat e_1\right)\right]$$

$$\cdot\, P\exp\left[ig\varepsilon\,\hat e_2^\mu t^a A_\mu^a\left(x+\varepsilon\hat e_1+\frac{\varepsilon}{2}\hat e_2\right)\right]\cdot P\exp\left[ig\varepsilon\,\hat e_1^\mu t^a A_\mu^a\left(x+\frac{\varepsilon}{2}\hat e_1\right)\right]$$

terms with $\partial^\nu A_\mu$

$$\simeq 1 - ig\varepsilon\,\hat e_2^\mu t^a \cancel{\partial^\nu A_\mu^a(x)\frac{\varepsilon}{2}\hat e_{2\nu}} - ig\varepsilon\,\hat e_1^\mu t^a\partial^\nu A_\mu^a(x)\left(\varepsilon\hat e_{2\nu} + \cancel{\frac{\varepsilon}{2}\hat e_{1\nu}}\right)$$

$$+ ig\varepsilon\,\hat e_2^\mu t^a\partial^\nu A_\mu(x)\left(\varepsilon\hat e_{1\nu} + \cancel{\frac{\varepsilon}{2}\hat e_{2\nu}}\right) + ig\varepsilon\,\hat e_1^\mu t^a \cancel{\partial^\nu A_\mu^a(x)\cdot\frac{\varepsilon}{2}\hat e_{1\nu}}$$

mixed terms from different exps

$$- g^2\varepsilon^2\Big(\hat e_2^\mu t^a A_\mu^a(x)\,\hat e_1^\nu t^b A_\nu^b(x) - \hat e_2^\mu t^a A_\mu^a\,\hat e_2^\nu t^b A_\nu^b$$

$$- \hat e_2^\mu t^a A_\mu^a\,\hat e_1^\nu t^b A_\nu^b - \hat e_1^\mu t^a A_\mu^a\,\hat e_2^\nu t^b A_\nu^b$$

$$- \hat e_1^\mu t^a A_\mu^a\cdot\hat e_1^\nu t^b A_\nu^b + \hat e_2^\mu t^a A_\mu^a\,\hat e_1^\nu t^b A_\nu^b$$

squares of individual exp's

$$+ 2\cdot\frac{1}{2}\hat e_2^\mu t^a A_\mu^a\,\hat e_2^\nu t^b A_\nu^b + 2\cdot\frac{1}{2}\hat e_1^\mu t^a A_\mu^a\,\hat e_1^\nu t^b A_\nu^b\Big)$$

$$= 1 + ig\varepsilon^2\left(\partial^1 A_2^a t^a - \partial^2 A_1^a t^a - ig\left[t^a A_1^a\, t^b A_2^b - t^a A_2^a\, t^b A_1^b\right]\right)$$

$\Rightarrow$ The quantity

$$\boxed{F_{\mu\nu} \equiv \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)t^a - ig\left[t^a A_\mu^a, t^b A_\nu^b\right]}$$

transforms as

$$F_{\mu\nu} \rightarrow V(x)\, F_{\mu\nu}\, V^{-1}(x)$$

and hence

$$\boxed{\mathcal{L}_{YM} \equiv -\frac{1}{2} \mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)}$$

Yang-Mills

is gauge-invariant

$\Rightarrow$ Triple and quartic gauge boson interactions

In QED:

$$\mathcal{L}_{gauge} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= \frac{1}{4e^2}\left(\left[D_\mu, D_\nu\right]\right)^2$$

$$= \frac{1}{4e^2}\left[(\partial_\mu + ieA_\mu)(\partial_\nu + ieA_\nu) - (\partial_\nu + ieA_\nu)(\partial_\mu + ieA_\mu)\right]^2$$

$$= \frac{1}{4e^2}\Big[ie(\partial_\mu A_\nu) + \cancel{ieA_\nu \partial_\mu} + \cancel{ieA_\mu \partial_\nu}$$

$$- ie(\partial_\nu A_\mu) - \cancel{ieA_\mu \partial_\nu} - \cancel{ieA_\nu \partial_\mu}\Big]^2$$

$$= -\frac{1}{4}\left[(\partial_\mu A_\nu) - (\partial_\nu A_\mu)\right]^2$$

□

$\Rightarrow$ Postulate:

$$\boxed{\mathcal{L}_{YM} \equiv \frac{1}{2g^2} \mathrm{tr}\left(\left[D_\mu, D_\nu\right]\right)^2 \equiv -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}}$$

Yang-Mills (non-Abelian pure gauge theory)

prefactor depends on $\mathrm{tr}\, t^a t^a$

with

$$\boxed{F^a_{\mu\nu} t^a \equiv \frac{i}{g}\left[D_\mu, D_\nu\right] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) t^a - ig\left[A^a_\mu t^a, A^b_\nu t^b\right]}$$

5.3 Lie algebras and Lie groups

Consider transformation

$$\underbrace{\mathcal{U} = \exp\left[i\alpha^a t^a\right]}_{\text{defining property of Lie group (for our purposes)}} \qquad (1)$$

Group properties imply

$$\mathcal{U}_1, \mathcal{U}_2 \in G \Rightarrow \mathcal{U}_1 \cdot \mathcal{U}_2 \in G$$

$$\Rightarrow t^a t^b = \sum c^a t^a$$

Define

$$\boxed{\left[t^a, t^b\right] \equiv i f^{abc} t^c} \qquad (2)$$

structure constants

Vector space spanned by the t^a (together with (1)) is called Lie algebra. The f^{abc} are a unique fingerprint of each Lie algebra.

Most relevant in particle physics:

- $\mathcal{U}(1)$: one generator $t = 1$

- $SU(N)$: special unitary group in N-dim
  - ↳ demand $\mathcal{U}\mathcal{U}^+ = 1$ and $\det \mathcal{U} = 1$

$$\Rightarrow 1 = \det e^{i\alpha^a t^a} = e^{i\,tr(\alpha^a t^a)}$$

$$\Rightarrow \boxed{tr\, t^a = 0}$$

end 15.01.2015

and

$$e^{i\alpha^a t^a} \cdot e^{-i\alpha^a t^{a+}} = 1$$

$$\Leftrightarrow \boxed{t^a = t^{a+}}$$

There are $\underbrace{2N^2}_{\substack{N \text{ real parts} \\ N \text{ imaginary parts}}} - \underbrace{2 \cdot \frac{N^2 - N}{2}}_{\substack{\text{constraints} \\ \text{on off-diagonal} \\ \text{elements}}} - \underbrace{N}_{\substack{\text{diagonal} \\ \text{real}}} - \underbrace{1}_{\substack{\text{require} \\ \operatorname{tr} t^a = 0}} = N^2 - 1$

such matrices.

Group representations

For each Lie algebra (defined by commutation relations (2)) there are many sets of matrices that satisfy (2)

Example: $su(2)$

- 2D representation $t^a_{2D} = \frac{\sigma^a}{2}$

$$\hookrightarrow \left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i \underbrace{\varepsilon^{abc}}_{= f^{abc}} \frac{\sigma^c}{2}$$

- 3D representation

$$t^1_{3D} = i\begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 0 \end{pmatrix} \; ; \; t^2_{3D} = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \; ; \; t^3_{3D} = i\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}$$

Corresponding transformations $U = e^{i\alpha^a t^a_r}$ act on 2D and 3D vectors, respectively

If basis can be chosen such that all t^a_r are in the same block-diagonal form, the rep. r is reducible.

In the following, we consider only irreducible reps. of $SU(N)$.

Normalization convention:

$\operatorname{tr} t^a_r t^b_r = D^{ab}$ is positive definite since t^a_r, t^b_r are Hermitian ($\Rightarrow t^a_r t^b_r$ is Hermitian and has only real eigenvalues. Thus, D^{ab} is symmetric. Go to basis where D^{ab} is diagonal. If it had a negative eigenvalue, we would have $\operatorname{tr} t^a_r t^a_r < 0$. ↯)

$\Rightarrow$ By convention choose basis of Lie algebra such that

$$\boxed{\operatorname{tr} t^a_r t^b_r = C(r)\, \delta^{ab}}$$

characteristic constant for each representation

It follows

$$\mathrm{tr}\left(\left[t_r^a, t_r^b\right] t_r^c\right) = i f^{abd}\, \mathrm{tr}\, t_r^d t_r^c$$

$$= i\, C(r)\, f^{abc}$$

$\Rightarrow$ f^{abc} is <u>totally antisymmetric</u> in a, b, c

[Proof: Use definition of $[\cdot,\cdot]$ and properties of tr.]

Given representation r, define <u>conjugate representation $\bar{r}$</u>:

$$\boxed{t_{\bar{r}}^a \equiv -\left(t_r^a\right)^* = -\left(t_r^a\right)^T}$$

[$\Rightarrow$ infinitesimal trafo $1 + i\alpha^a t_r^a \Longleftrightarrow 1 + i\alpha^a t_{\bar{r}}^a$

$$= 1 - i\alpha^a \left(t_r^a\right)^*$$

<u>Note</u>: r and $\bar{r}$ may equivalent if there is a unitary trafo U such that

$$t_{\bar{r}}^a = U t_r^a U^+ \qquad \forall a$$

Example: 2D-rep. of $SU(2)$: $t_2^a = \frac{\sigma^a}{2}$

$$t_{\bar{2}}^a = -\frac{\sigma^{a*}}{2} = \underbrace{\left(i\sigma^2\right)}_{=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} \frac{\sigma^a}{2} \left(i\sigma^2\right)^+$$

In $SU(N)$, the N-dim. (lowest-dimensional) rep. is called <u>fundamental representation</u>.

The generators f^{abc} form the <u>adjoint representation</u>

$$\boxed{\left(t_G^b\right)_{ac} \equiv i f^{abc}}$$

Srednicki uses opposite sign here

Proof: From the definition of SU(N) it follows that

$$\left[t^a, \left[t^b, t^c\right]\right] + \left[t^b, \left[t^c, t^a\right]\right] + \left[t^c, \left[t^a, t^b\right]\right] = 0$$

(Jacobi identity).

[In mathematics, this is actually part of the definition of a Lie algebra.

$$\Rightarrow \left[t^a, if^{bcd}t^d\right] + \left[t^b, if^{cad}t^d\right] + \left[t^c, if^{abd}t^d\right] = 0$$

$$if^{bcd}\underbrace{if^{ade}}_{=f^{dea}} + \underbrace{if^{cad}}_{=f^{dca}}\underbrace{if^{bde}}_{=-f^{bed}} + if^{abd}if^{cde} = 0$$

$$\left(t^c_G\, t^e_G\right)_{ba} - \left(t^e_G\, t^c_G\right)_{ba} = -if^{cde}\left(t^d_G\right)_{ba}$$

$$\left[t^c_G, t^e_G\right]_{ba} = if^{ced}\left(t^d_G\right)_{ba}$$

□

For SU(N), the adjoint rep. is N^2-1-dimensional (equal to number of generators).

The Casimir operator

Consider $\boxed{t^2 = \Sigma_a t^a t^a}$

Example: SU(2) acting on Pauli spinors $\rightarrow$ $t^2 = J^2$

Since

$$\left[t^b, t^a t^a\right] = \left[t^b, t^a\right]t^a + t^a\left[t^b, t^a\right]$$

$$= if^{bac}t^c t^a + \underbrace{if^{bac}t^a t^c}_{= f^{bca}t^c t^a \,=\, -f^{bac}t^c t^a}$$

$$= 0,$$

t^2_r can be represented in rep. r by $\boxed{t^2_r \equiv C_2(r)\mathbb{1}}$

For the adjoint rep.: $f^{acd}f^{dcb} = C_2(G)\,\delta^{ab}$

Since $\operatorname{tr} t^a_r t^b_r = C(r)\,\delta^{ab}$, we have

dim. of rep. G

$$\Sigma_a \operatorname{tr}(t^a_r t^a_r) = d(G) \cdot C(r)$$

$$= \operatorname{tr} t^2_r = C_2(r) \cdot \operatorname{tr} 1\!\!1_r$$

$$\Rightarrow \boxed{d(G)\, C(r) = d(r)\, C_2(r)}$$

In SU(N) with fundamental rep. N, adjoint rep. G:

$$\boxed{\begin{aligned} C(N) &= \frac{1}{2} \\ C_2(N) &= \frac{N^2-1}{2N} \\ C(G) = C_2(G) &= N \end{aligned}}$$

<u>Proof</u>: see Peskin sec. 15.4

Srednicki sec. 71
Peskin sec. 16.2.

5.4 Quantization of non-Abelian gauge fields

Consider $$\boxed{\underbrace{\mathcal{L}_{YM}}_{\text{Yang-Mills}} \equiv -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

Path integral

$$Z_0[J] \propto \int \mathcal{D}A \, \exp\Big[i \underbrace{\int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a}\right.}_{\equiv S_{YM} \text{ (Yang-Mills)}} \left. + J^{\mu,a} A^a_\mu \right) \Big]$$

Remember: For Abelian gauge field, we had to restrict integration to components of $\tilde{A}^\mu \perp k^\mu$ since gauge invariance implied that

$$\tilde{A}_\mu(k) \longrightarrow \tilde{A}_\mu(k) + \frac{i}{e} k_\mu \tilde{\alpha}(k)$$

does not change physics.

Here: gauge trafo non-linear

$$A_\mu \longrightarrow U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

$\Rightarrow$ subspace of gauge-equivalent field configurations more complicated

Solution (Fadeev - Popov): add gauge fixing δ-function

Let us illustrate this idea on a regular integral (not path integral) first: Consider

$$Z \propto \int dz \, dy \, e^{iS(z)}$$

In the Abelian case, we simply dropped $\int dy \Rightarrow Z = \int dz \, e^{iS(z)}$

We could equivalently have included a δ-function

$$Z = \int dz \, dy \, \delta(y - f(z)) \, e^{iS(z)}$$

The gauge-fixing condition $y = f(z)$ could also be given implicitely by requiring $G(z,y) = 0$, where here $G(z,y) = y - f(z)$.

$$\hookrightarrow \quad \delta(y - f(z)) = \delta\big(G(z,y)\big) \cdot \left|\frac{\partial G}{\partial y}\right|$$

$$Z = \int dz\, dy \left|\frac{\partial G}{\partial y}\right| \delta(G)\, e^{iS}$$

In multiple dimensions: n gauge-fixing functions $G_i(z,y)$

$$Z = \int d^n z\, d^n y \left|\det\left(\frac{\partial G_i(z,y)}{\partial y_j}\right)\right| \prod_i \delta(G_i(z,y))\, e^{iS}$$

Back to path integrals: $z, y \to A^a_\mu(x)$, where y corresponds to all gauge-equivalent configurations for given z.

We define

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x)$$

with arbitrary $\omega^a(x)$. (Generalization of Lorentz gauge $\partial^\mu A^a_\mu = 0$.

$$\hookrightarrow \quad Z_0[J] \propto \int \mathcal{D}A \, \det\left(\frac{\partial G^a(x)}{\partial \theta^b(y)}\right) \prod_{x,a} \delta(G^a(x))\, e^{iS_{YM}}$$

where $\theta^a(x)$ parameterizes the hypersurface of gauge-equivalent configurations.

Evaluation of $\det(\partial G/\partial\theta)$:

now infinitesimal

Infinitesimal gauge trafo $\mathcal{U}(x) = \mathbb{1} + ig\,\theta^a(x) t^a$

$$\Rightarrow G^a(x)t^a \to G^a(x)t^a + \partial^\mu\left[\mathcal{U} A^a_\mu t^a \mathcal{U}^\dagger + \frac{i}{g}\mathcal{U}(\partial_\mu \mathcal{U}^\dagger) - A^a_\mu t^a\right]$$

$$= G^a(x)t^a + \partial^\mu\Big[+ig\,\theta^a t^a A^b_\mu t^b - ig A^b_\mu t^b \theta^a t^a$$

$$+ \frac{i}{g}(-ig)\partial_\mu \theta^a t^a\Big]$$

Remember:

$D_\mu = \mathbb{1}\partial_\mu - ig\, t^a A^a_\mu$

$D^{ac}_\mu = \delta^{ac}\partial_\mu - ig (t^b)^{ac} A^b_\mu$

$$= G^a(x)t^a + \partial^\mu\Big[+ig\,\theta^a A^b_\mu\, i \underbrace{f^{abc}}_{-i(t^b_G)^{ac} \,=\, +i(t^b_G)^{ca}} t^c + \partial_\mu \theta^a t^a\Big]$$

antisymmetry of f^{abc}

$$= G^a(x)t^a + \partial^\mu \underbrace{D_{\mu,G}\,\theta}_{= D^{ac}_{\mu,G}\theta^c t^a}$$

$$\Rightarrow \frac{\partial G^a(x)}{\partial \theta^b(y)} = \partial^\mu D^{ab}_{\mu,G}\, \delta^{(4)}(x-y)$$

We write: (see sec. 3.1)

$$\boxed{\det \frac{\partial G^a(x)}{\partial \theta^b(y)} \propto \int \mathcal{D}c\, \mathcal{D}\bar{c}\; e^{iS_{gh}}}$$

auxiliary Grassmann fields

with $S_{gh} \equiv \int d^4x\, \mathcal{L}_{gh}$

$$\mathcal{L}_{gh} \equiv -\int d^4y\; \bar{c}^a(x)\, \partial^\mu D^{ab}_{\mu,G}(x)\, \delta^{(4)}(x-y)\, c^b(y)$$

$$= -\bar{c}^a(x)\; \partial^\mu D^{ab}_{\mu,G}(x)\; c^b(x)$$

$$= +(\partial^\mu \bar{c}^a)(\partial_\mu c^a) + (\partial^\mu \bar{c}^a)(-ig\underbrace{t^c_G}_{=if^{acb}})^{ab} A^c_\mu\, c^b$$

$$= (\partial^\mu \bar{c}^a)(\partial_\mu c^a) - g f^{abc} A^c_\mu (\partial^\mu \bar{c}^a)\, c^b$$

Dealing with $\delta(G^a(x))$

$G^a(x)$ contains $\omega^a(x)$, but $\omega^a(x)$ is arbitrary (does not affect physics)

↳ w/o loss of generality, can do a functional integral over many possible $\omega^a(x)$, with arbitrary weights (will only change normalization of $Z[J]$.)

Choose: $\int \mathcal{D}\omega\; \exp\left[-\frac{i}{2\xi} \int d^4x\, (\omega^a)^2\right] \delta(G(x))$

ξ ← arbitrary constant

$$= \exp\left[-\frac{i}{2\xi}\int d^4x\, (\partial^\mu A^a_\mu)^2\right]$$

$$\Rightarrow \boxed{Z_0[J] \propto \int \mathcal{D}A\, \mathcal{D}c\, \mathcal{D}\bar{c}\; e^{iS_{YM} + iS_{gh} + iS_{gf}}}$$

with $S_{YM} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

$$S_{gh} = \int d^4x \left((\partial^\mu \bar{c}^a)(\partial_\mu c^a) - g f^{abc} A^c_\mu (\partial^\mu \bar{c}^a) c^b\right)$$

$$S_{gf} = \int d^4x \left(-\frac{1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu)\right)$$

5.5 Feynman rules for non-Abelian gauge theories

Expand $\mathcal{L}$:

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a}$$

$$= -\frac{1}{4} \left[\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) + g f^{abc} A_\mu^b A_\nu^c \right]$$

$$\cdot \left[\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} \right) + g f^{ade} A^{\mu d} A^{\nu e} \right]$$

$$= -\frac{1}{2} \left(\partial_\mu A_\nu^a \right)\left(\partial^\mu A^{\nu a} \right) + \frac{1}{2} \left(\partial_\mu A_\nu^a \right)\left(\partial^\nu A^{\mu a} \right)$$

$$- g f^{abc} A_\mu^b A_\nu^c \, \partial^\mu A^{\nu a}$$

$$- \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$$

Together with $\mathcal{L}_{gf} \equiv -\frac{1}{2\xi} \left(\partial^\mu A_\mu^a \right)\left(\partial^\nu A_\nu^a \right)$, the terms quadratic in A become (after integration by parts)

$$\mathcal{L} \supset \frac{1}{2} A_\mu^a \left[g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu \left(1 - \frac{1}{\xi} \right) \right] A_\nu^a$$

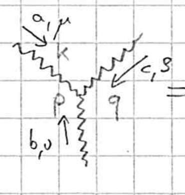
The gauge boson propagator is as usual the inverse of the operator in square brackets:

$$\boxed{D_{\mu\nu}^{ab}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i\delta^{ab}}{k^2 + i\varepsilon} \left[g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right] e^{-ik(x-y)}}$$

Proof: Check explicitly that

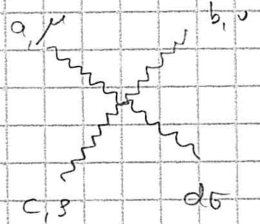
$$\frac{1}{k^2} \left(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right) \cdot k^2 \left(g_{\nu\rho} - \frac{k_\nu k_\rho}{k^2} \left(1 - \frac{1}{\xi} \right) \right) = g^\mu{}_\rho = \delta^\mu{}_\rho$$

The A^3 and A^4 terms lead to the vertices



$$= -ig f^{abc} \left[-ik^\nu g^{\mu\rho} + 5 \text{ permutations of } (a,\mu,k), (b,\nu,p), (c,\rho,q) \right]$$

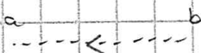
$$= g f^{abc} \left[g^{\mu\nu} (k-p)^\rho + g^{\nu\rho} (p-q)^\mu + g^{\rho\mu} (q-k)^\nu \right]$$



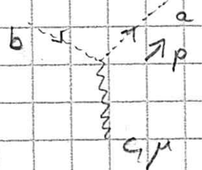
$$= -ig^2\Big[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$

Diagrams with ghosts:

$$\mathcal{L}_{gh} = (\partial^\mu \bar{c})(\partial_\mu c) - g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b$$

$\Rightarrow$ Ghost propagator: 

$= \dfrac{i\delta^{ab}}{p^2}$

Ghost – gauge boson vertex: 

$= +g f^{abc} p^\mu$

Note: Ghosts are merely a mathematical tool to cancel unwanted contributions in diagrams with gauge bosons.

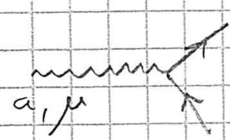
Ghosts cannot form an external line in a Feynman graph.

They only appear in loop diagrams.

Including fermions: (massless for simplicity)

$$\mathcal{L}_f \equiv \bar{\Psi}\, i\not{D}\, \Psi = \bar{\Psi}\, i\not{\partial}\, \Psi + g\,\bar{\Psi}\gamma^\mu t_r^a \Psi A_\mu^a$$

n-plets transforming in some rep. r of the gauge group.
E.g. QCD: $\Psi = (\Psi_r, \Psi_g, \Psi_b)$, $r = 3$

$\Rightarrow$ Vertex 

$= ig\gamma^\mu t_r^a$

5.6 The β function for non-Abelian gauge theories: Asymptotic freedom

Renormalized QCD Lagrangian, setting $\xi = \infty$ for simplicity.

$$
\begin{aligned}
\mathcal{L}_{QCD} = & -\tfrac{1}{4} Z_A \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)\left(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}\right) \\
& - g Z_{3A} f^{abc} A_\mu^b A_\nu^c \partial^\mu A^{\nu a} \\
& - \tfrac{1}{4} Z_{4A} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \\
& + Z_c (\partial^\mu \bar{c})(\partial_\mu c) - Z_{gc} g f^{abc} A_\mu^c (\partial^\mu \bar{c}^a) c^b \\
& + Z_\psi \bar{\Psi} i \not{\partial} \Psi - Z_m m \bar{\Psi}\Psi + Z_{g\psi} g \bar{\Psi} \gamma^\mu t^a \Psi A_\mu^a
\end{aligned}
$$

There are 8 counterterms $Z_i - 1$, but only 5 free parameters (normalization of A_μ^a, Ψ, c; coupling g; mass m)

↳ not all counterterms independent

Indeed, one can show that

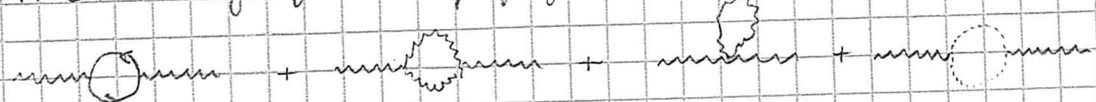
$$Z_{3A} = Z_A Z_{g\psi} Z_\psi^{-1} \qquad \text{(cf. Srednicki eq. 73.2)}$$

$$Z_{4A} = Z_A Z_{g\psi}^2 Z_\psi^{-2}$$

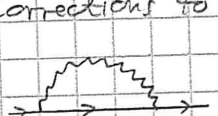
$$Z_{gc} = Z_{g\psi} Z_\psi^{-1} Z_c$$

⇒ Need to compute: (@ 1-loop)

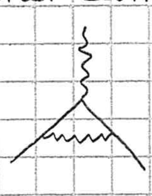
- Corrections to gauge boson propagator:



- Corrections to fermion propagator



- Vertex correction



- Corrections to ghost propagator cannot contribute in 1-loop calculation.

$\Rightarrow$ Determine counterterms, compute

$$\beta = \mu \frac{\partial g}{\partial \mu} = \mu \frac{\partial}{\partial \mu} \left[\frac{Z_\psi Z_A^{1/2}}{Z_{g_4}} g_0 \right]$$

<u>Result:</u>

$$\boxed{\beta(g) = -\frac{g^3}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right]}$$

n_f: number of fermion species (all in representation r)

For small n_f, $\beta(g)$ is <u>negative</u>

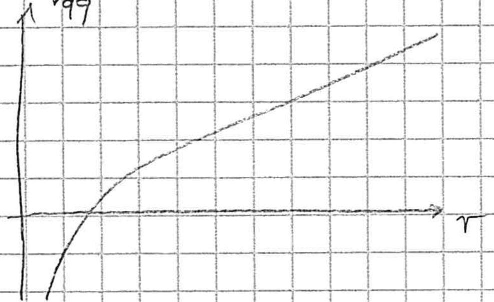
$$\hookrightarrow \quad g^2(\mu_2) = \frac{g^2(\mu_1)}{1 + \frac{2g^2(\mu_1)}{(4\pi)^2} \left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r) \right] \log \frac{\mu_2}{\mu_1}}$$

g becomes <u>smaller</u> at larger energy: <u>asymptotic freedom</u>

For example in QCD (gauge group $SU(3)$; $n_f = 5$ (below m_t); $r = 3$):

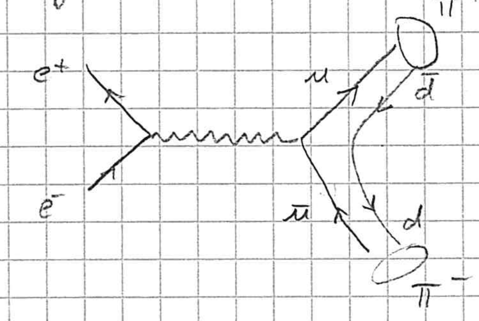
$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left[11 - \frac{10}{3} \right]$$

<u>Consequence:</u> Potential increases at large distances



↳ Separating colored particles (quarks, gluons) requires a lot of energy. Eventually, the potential is so strong that $q\bar{q}$ pairs can be produced from the vacuum. This happens until only colorless bound states ($q_c q_{\bar{c}}$, $q_r q_g q_b$) exist. These do not feel QCD any more and can separate

E.g.



end 29.01.2015

5.7 Quantum chromodynamics

Non-Abelian gauge theory with gauge group $SU(3)$
("unitary matrices in 3D with determinant 1)

8 generators (Gell-Mann-matrices): $t^a = \frac{\lambda_i}{2}$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} ; \quad \lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix} ; \quad \lambda_5 = \begin{pmatrix} 0 & & -i \\ & 0 & \\ i & & 0 \end{pmatrix} ;$$

$$\lambda_6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} ; \quad \lambda_7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} ; \quad \lambda_8 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$\boxed{\begin{aligned} \mathcal{L}_{QCD} = & -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu) \\ & + (\partial^\mu \bar{c})(\partial_\mu c) - g f^{abc} A^c_\mu (\partial^\mu \bar{c}^a) c^b \\ & + \underbrace{\bar{\Psi} i \not{D} \Psi} - m \bar{\Psi} \Psi \end{aligned}}$$

$$= \bar{\Psi} i \not{\partial} \Psi + g \bar{\Psi} \gamma^\mu t^a \Psi A^a_\mu$$

fermion ("quark") field
is understood to be a triplet
of $SU(3)$
↳ $\Psi = (\Psi_r, \Psi_g, \Psi_b)$ (red, green, blue)

Actually: 6 species of quarks
↳ $\sum_{q=u,d,s,c,t,b} \bar{\Psi}_q i \not{D} \Psi_q + \ldots$

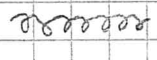
It is straightforward to also include electromagnetic interactions by adding gauge couplings

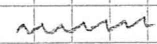
$$+ i e q_\Psi \bar{\Psi} \gamma^\mu \Psi A^{em}_\mu$$

charge of quark Ψ photon field

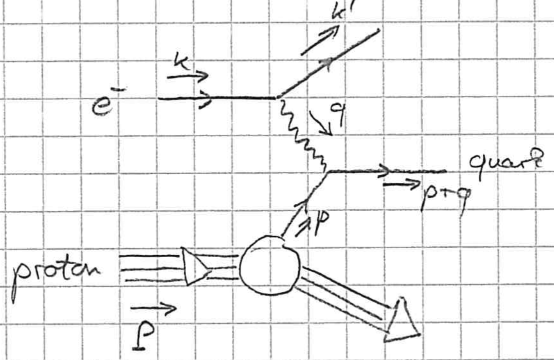
and gauge-kinetic term $-\frac{1}{4} F^{em}_{\mu\nu} F^{\mu\nu,em}$
(It is easy to check that the resulting $\mathcal{L}$ is invariant under $SU(3)$ and $U(1)$.)

To distinguish photons from gluons in Feynman diagrams one uses

 for gluons

 for photons

eskin sec. 17.3 **Examples 1. Deep inelastic scattering**



q is spacelike ($q^2 < 0$)
↳ define $Q^2 = -q^2$

Quark is kicked out of proton; subsequent soft processes create q-$\bar{q}$ pairs, and color-neutral hadrons ($q_c\, \bar{q}'_{\bar{c}}$ or $q_r\, q'_g\, q''_b$) bound states form. Reason: QCD potential grows with distance, so more and more energy is needed to separate colored particles further and further. It is more efficient to use some energy to form q-$\bar{q}$ pairs.

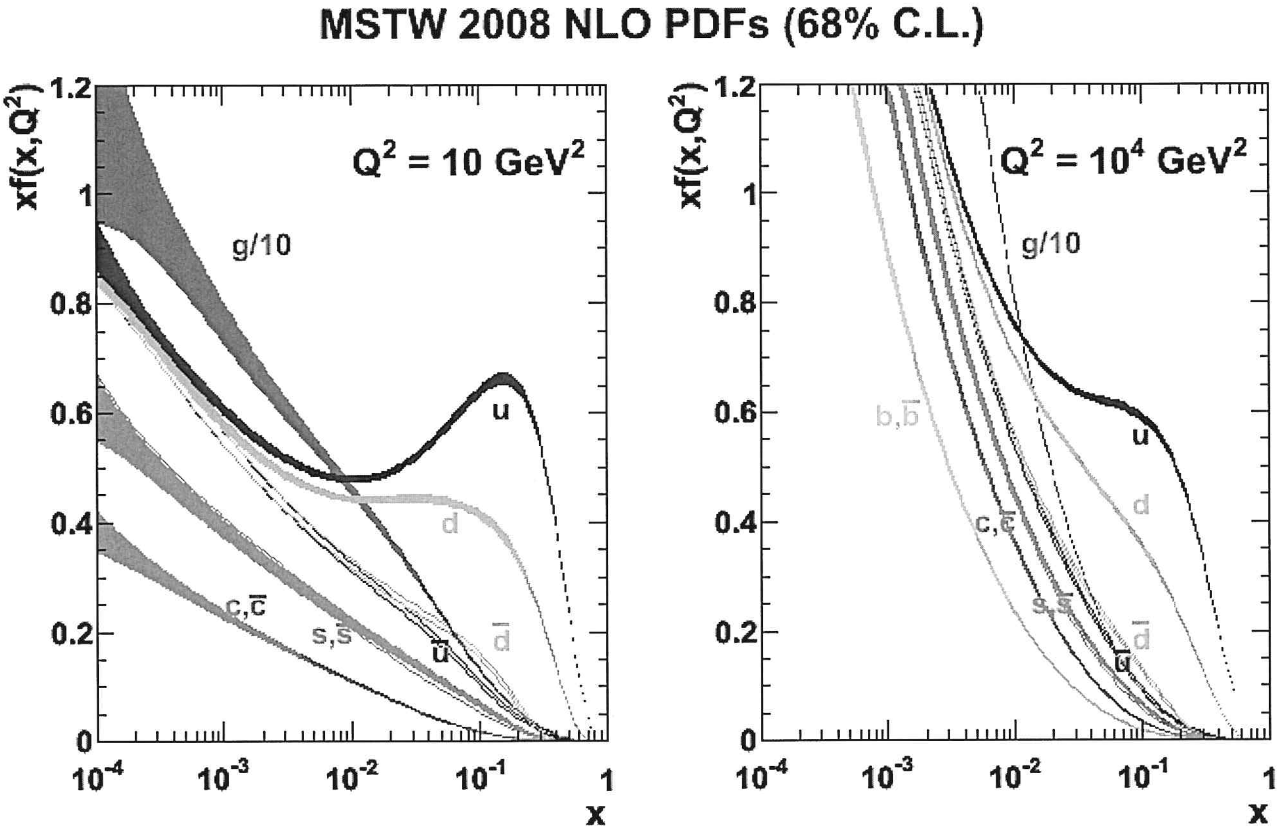
See sec. 5.6 for details

Consider frame in which p^+ and e^- move rapidly towards each other (e.g. e^--p^+ c.o.m. frame at energy $\gg$ GeV).

Assume m_p negligible.

Quark participating in the interaction carries fraction x of proton momentum P. Distribution of x given by <u>parton distribution function</u> $f_f(x)$

$$\boxed{f_f(x)\, dx = \begin{pmatrix} \text{probability of finding constituent} \\ \text{of type } f \in \{u,\bar{u},d,\bar{d},s,\bar{s},c,\bar{c},b,\bar{b},t,\bar{t},g\} \\ \text{with momentum fraction in } [x, x+dx] \end{pmatrix}}$$


MSTW 2008 NLO PDFs (68% C.L.)
$xf(x,Q^2)$
$Q^2 = 10$ GeV^2
g/10
u
d
c,$\bar{c}$
s,$\bar{s}$
$\bar{u}$
$\bar{d}$
$Q^2 = 10^4$ GeV^2
b,$\bar{b}$
x
0
0.2
0.4
0.6
0.8
1
1.2
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1

$$\Rightarrow \boxed{\begin{aligned}&\sigma\big(e^-(k) + p^+(P) \longrightarrow e^-(k') + \overset{\text{any hadronic final state}}{X}\big) \\ &= \int_0^1 dx \sum_f \underbrace{f_f(x)}_{\text{PDF}} \underbrace{\sigma\big(e^-(k) + q_f(\underbrace{xP}_{=p}) \rightarrow e^-(k') + q_f(\underbrace{p'}_{=p+q})\big)}_{\text{parton level x-sec.}}\end{aligned}}$$

Parton-level x-sec: (standard QED calculation)

$$\frac{d\sigma_f}{d\cos\theta_{CM}} = \frac{\pi\alpha^2 Q_f^2}{\hat{t}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{4}\right) \qquad \text{for parton flavor } f$$

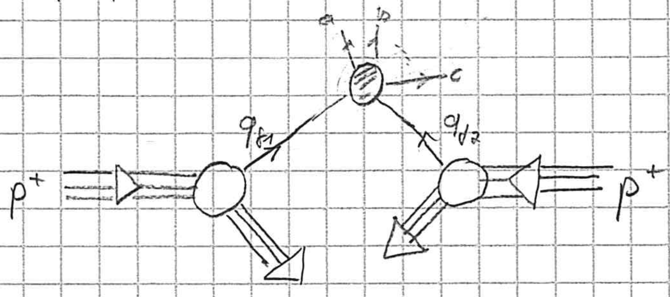
θ_{CM}: scattering angle in c.o.m. frame

where $\hat{s} = (p+k)^2 = 2pk = 2x\cdot P\cdot k = x\cdot s$

$\hat{t} = -Q^2$

$\hat{u} = (k'-p)^2$

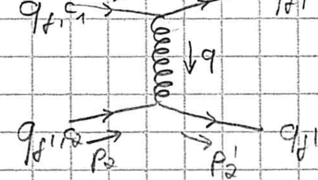
2. p-p interactions at the LHC



$$\boxed{\sigma(pp\rightarrow X) = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{f_1,f_2} \sigma\big(q_{f_1} q_{f_2} \longrightarrow \underbrace{a+b+\dots+c}_{\text{final state}}\big) f_{q_{f_1}}(x_1) f_{q_{f_2}}(x_2)}$$

Consider e.g. dijet production $pp\rightarrow q\bar{q}$:

- $q_f - q_{f'}$ scattering ($f\neq f'$):



$$= \bar{u}(p_1')(ig\gamma^\mu t^a)_{c_1'c_1} u(p_1)\cdot \frac{-ig_{\mu\nu}\delta^{ab}}{q^2+i\varepsilon} \cdot \bar{u}(p_2')(ig\gamma^\nu t^b)_{c_2'c_2} u(p_2)$$

we $\xi=1$ for simplicity

sum over colors, spins

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{1}{9}g^2 \; \mathrm{tr}(\not p_1' \gamma^\mu \not p_1 \gamma^\rho) \; \mathrm{tr}(\not p_2' \gamma_\mu \not p_2 \gamma_\rho) \left(\frac{1}{q^2}\right)^2 \cdot \Big[\underbrace{\mathrm{tr}(t^a t^c)}_{=C(r)\delta^{ac}}\Big]^2$$

average over initial colors, spins

$$= \frac{1}{36}\frac{g^2}{(q^2)^2} \cdot 16 \left(p_1'^\mu p_1^\rho - p_1' p_1 \, g^{\mu\rho} + p_1'^\rho p_1^\mu \right)$$

$$\cdot \left(p_2'^\mu p_2^\rho - p_2' p_2 \, g^{\mu\rho} + p_2'^\rho p_2^\mu \right) \cdot \frac{1}{4} \cdot \underbrace{\delta^{ac}\delta^{ac}}_{=8}$$

$$= \frac{8}{9}\frac{g^2}{(q^2)^2} \Big[(p_1' \cdot p_2')(p_1 \cdot p_2) - (p_1' \cdot p_1)(p_2' \cdot p_2) + (p_1' \cdot p_2)(p_1 \cdot p_2')$$

$$- (p_1' p_1)(p_2' p_2) + 4 (p_1' p_1)(p_2' p_2) - (p_1' p_1)(p_2' p_2)$$

$$+ (p_1' p_2)(p_1 p_2') - (p_1 \cdot p_1')(p_2 \cdot p_2') + (p_1 \cdot p_2)(p_1' \cdot p_2') \Big]$$

$$= \frac{8}{9}\frac{g^2}{(q^2)^2} \Big[2 (p_1' p_2')(p_1 p_2) + 2 (p_1' p_2)(p_1 p_2') \Big]$$

Define Mandelstam variables $\hat{s} = (p_1 + p_2)^2 = 2 p_1 p_2 = 2 p_1' p_2'$

$$\hat{t} = (p_1' - p_1)^2 = -2 p_1 p_1' = -2 p_2 p_2'$$

$$\hat{u} = (p_1' - p_2)^2 = -2 p_1' p_2 = -2 p_1 p_2'$$

$$\hookrightarrow \overline{|\mathcal{M}|^2} = \frac{4}{9}\frac{g^2}{\hat{t}^2}\left(\hat{s}^2 + \hat{u}^2\right)$$

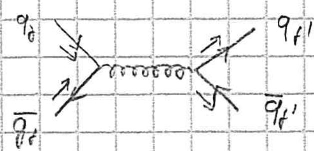
$$\frac{d\sigma}{d\hat{t}} = \frac{1}{64\pi \hat{s}} \frac{1}{\hat{s}/4} \cdot \overline{|\mathcal{M}|^2}$$

particle data group Kinematics review

$$= \frac{1}{4\pi \hat{s}^2} \cdot \frac{1}{9}\frac{g^2}{\hat{t}^2}\left(\hat{s}^2 + \hat{u}^2\right)$$

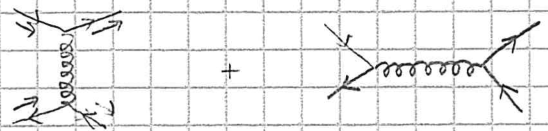
$$\boxed{\frac{d\sigma}{d\hat{t}}(q_f q_{f'} \to q_f q_{f'}) = \frac{4\pi\alpha_s}{9\hat{s}^2}\left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right)}$$ with $\alpha_s = \frac{g^2}{4\pi}$

• $q_f + \overline{q}_f \rightarrow q_{f'} + \overline{q}_{f'} \quad (f \neq f')$



$$\frac{d\sigma}{d\hat{t}}\left(q_f\overline{q}_f \rightarrow q_{f'}\overline{q}_{f'}\right) = \frac{4\pi\alpha_s}{\hat{s}^2}\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right)$$

• $q_f + \overline{q}_f \longrightarrow q_f + \overline{q}_f$



$$\frac{d\sigma}{d\hat{t}}\left(q_f\overline{q}_f \rightarrow q_f\overline{q}_f\right) = \frac{4\pi\alpha_s^2}{9\hat{s}^2}\left[\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2} + \frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2} - \frac{2}{3}\frac{\hat{u}^2}{\hat{s}\hat{t}}\right]$$

<u>Note:</u> Computation involves $\mathrm{tr}(t^a t^b t^a t^b)$

↳ use $t^b t^a t^b = t^b t^b t^a + t^b [t^a, t^b]$

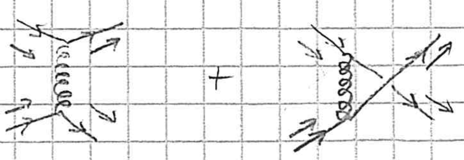
$$= C_2(r)\, t^a + i t^b f^{abc} t^c$$

$$= C_2(r)\, t^a + \frac{1}{2} i f^{abc} [t^b, t^c]$$

$$= C_2(r)\, t^a - \frac{1}{2} \underbrace{f^{abc} f^{bcd}}_{\substack{= -f^{bac} f^{cdb} \\ = -C_2(G)\,\delta^{ad}}} t^d$$

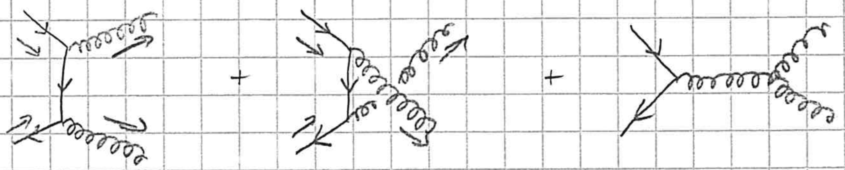
$$= \left(C_2(r) - \frac{1}{2} C_2(G)\right) t^a$$

• $q_f q_f \rightarrow q_f q_f$



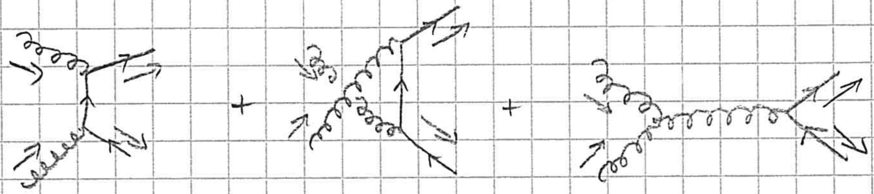
$$\frac{d\sigma}{d\hat{t}}\left(q_f q_f \rightarrow q_f q_f\right) = \frac{4\pi\alpha_s}{9\hat{s}^2}\left[\frac{\hat{u}^2+\hat{s}^2}{\hat{t}^2} + \frac{\hat{t}^2+\hat{s}^2}{\hat{u}^2} - \frac{2}{3}\frac{\hat{s}^2}{\hat{u}\hat{t}}\right]$$

- $q_f \overline{q_f} \to gg$



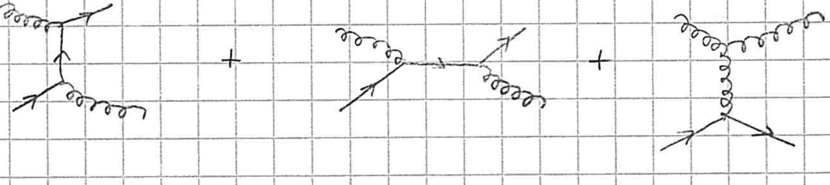
$$\frac{d\sigma}{d\hat{t}}(q_f\overline{q}_f \to gg) = \frac{32\pi\alpha_s^2}{27\hat{s}^2}\left[\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{9}{4}\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right)\right]$$

- $gg \to q_f\overline{q}_f$



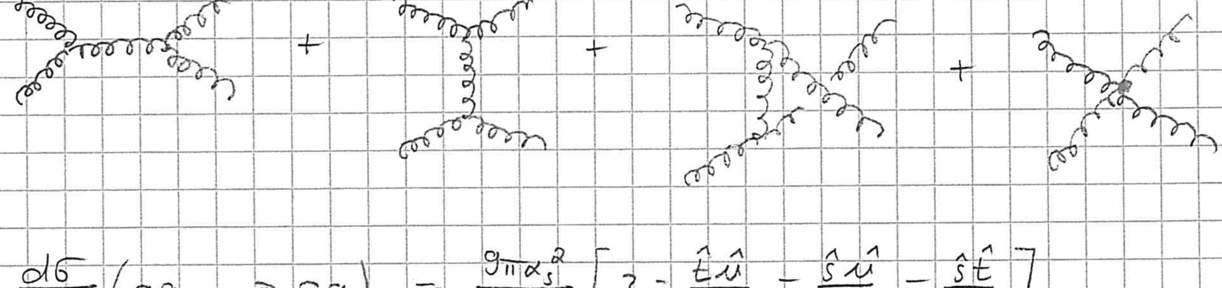
$$\frac{d\sigma}{d\hat{t}}(gg \to q_f\overline{q}_f) = \frac{\pi\alpha_s^2}{6\hat{s}^2}\left[\frac{\hat{u}}{\hat{t}} + \frac{\hat{t}}{\hat{u}} - \frac{9}{4}\left(\frac{\hat{t}^2+\hat{u}^2}{\hat{s}^2}\right)\right]$$

- $\overset{(-)}{q}g \to \overset{(-)}{q}g$



$$\frac{d\sigma}{d\hat{t}}(qg \to qg) = \frac{4\pi\alpha_s^2}{9\hat{s}^2}\left[-\frac{\hat{u}}{\hat{s}} - \frac{\hat{s}}{\hat{u}} + \frac{9}{4}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right)\right]$$

- $gg \to gg$



$$\frac{d\sigma}{d\hat{t}}(gg \to gg) = \frac{9\pi\alpha_s^2}{2\hat{s}^2}\left[3 - \frac{\hat{t}\hat{u}}{\hat{s}^2} - \frac{\hat{s}\hat{u}}{\hat{t}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2}\right]$$

6. Spontaneous symmetry breaking

6.1 The Abelian Higgs mechanism

Peskin sec. 20.1

Consider complex scalar field ϕ coupled to $U(1)$ gauge theory (e.g. electrodynamics)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - V(\phi)$$

Let $V(\phi) = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2} (\phi^\dagger \phi)^2$

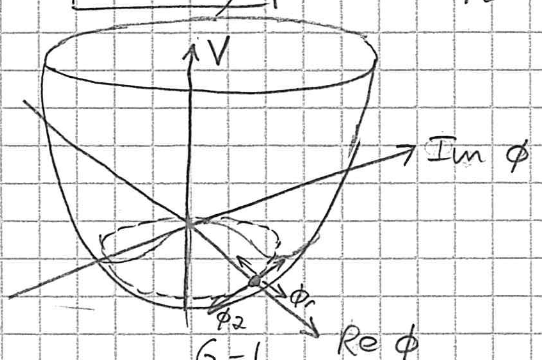
$V(\phi)$ has a minimum at $\boxed{\langle \phi \rangle \equiv v \equiv \sqrt{\frac{\mu^2}{\lambda}}.}$

Write $\phi(x) = v + \frac{1}{\sqrt{2}} \left(\phi_1(x) + i \phi_2(x) \right)$

real scalar fields

$$\hookrightarrow V(\phi) = -\mu^2 \left[v + \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \right]\left[v + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \right]$$
$$+ \frac{\lambda}{2} \left\{ \left[v + \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) \right]\left[v + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \right] \right\}^2$$
$$= -\frac{\mu^4}{\lambda} - \cancel{\mu^2 \sqrt{\frac{\mu^2}{\lambda}} \frac{1}{\sqrt{2}} \phi_1 \cdot 2} - \cancel{\mu^2 \cdot \frac{1}{2} (\phi_1^2 + \phi_2^2)}$$
$$+ \frac{\mu^4}{2\lambda} + \cancel{\frac{\lambda}{2} \cdot \frac{\mu^2}{\lambda} \sqrt{\frac{\mu^2}{\lambda}} \cdot \frac{1}{\sqrt{2}} \phi_1 \cdot 4} + \cancel{\frac{\lambda}{2} \frac{\mu^2}{\lambda} \frac{1}{2} (\phi_1^2 + \phi_2^2) \cdot 2}$$
$$+ \frac{\lambda}{2} \frac{\mu^2}{\lambda} \cdot \frac{1}{2} \phi_1^2 \cdot 4 + (\text{cubic and quartic terms})$$
$$= -\frac{\mu^4}{2\lambda} + \frac{1}{2} \cdot 2\mu^2 \phi_1^2 + (\text{cubic and quartic terms})$$

$\Rightarrow$ ϕ_1 has mass $\boxed{m = \sqrt{2}\mu}$ and ϕ_2 is <u>massless</u>.



The symmetry is no longer apparent once $\mathcal{L}$ is rewritten in terms of ϕ_1 and ϕ_2.

$\mathcal{L}$ still possesses the $U(1)$ symmetry, though. This implies that all configurations with

$$v = \sqrt{\frac{\mu^2}{\lambda}}\, e^{i\alpha}$$

α: $U(1)$ gauge parameter

are equivalent. Only when we arbitrarily gauge-fix the model by choosing $\alpha = 0$, the symmetry is hidden.

Now consider the kinetic term of ϕ:

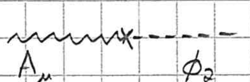
$$|D_\mu \phi|^2 = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \sqrt{2}\, e \cdot v A_\mu \partial^\mu \phi_2$$

$D_\mu = \partial_\mu + ieA_\mu$

$$+ \underbrace{e^2 v^2}_{\frac{1}{2} m_A^2} A_\mu A^\mu \quad + \text{(cubic and quartic terms)}$$

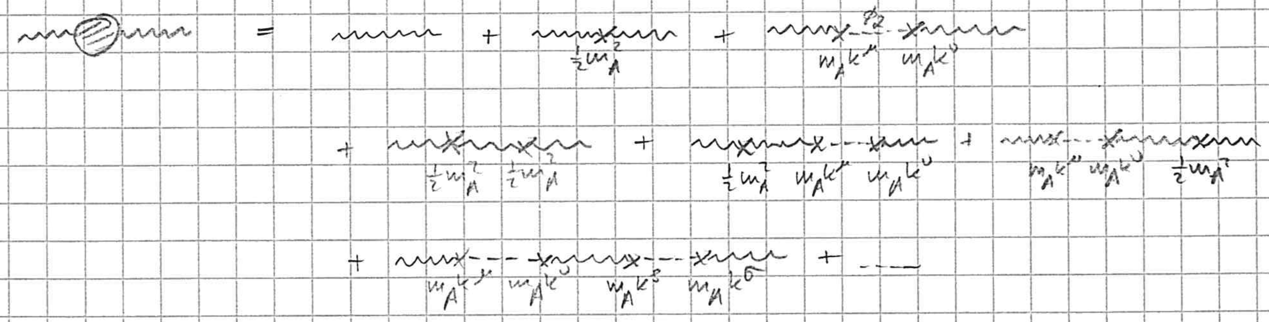
$\Rightarrow$ we have generated a <u>gauge boson mass term</u>!

The appearance of this term is intricately connected to the massless field ϕ_2:



$$= i\sqrt{2}\, e v(-ik^\mu) = m_A k^\mu$$

$\Rightarrow$



$$= \frac{-ig^{\mu\nu}}{k^2+i\varepsilon} + \frac{-ig^{\mu\rho}}{k^2+i\varepsilon} \cdot \frac{i}{2} m_A^2 \frac{-ig_\rho^{\ \nu}}{k^2+i\varepsilon} + \frac{-ig^{\mu\rho}}{k^2+i\varepsilon} m_A k_\rho \frac{i}{k^2+i\varepsilon}(-m_A k_\sigma) \frac{-ig^{\sigma\nu}}{k^2+i\varepsilon}$$

$$+ \frac{-ig^{\mu\rho}}{k^2+i\varepsilon} \frac{i}{2} m_A^2 \frac{-ig_{\rho\sigma}}{k^2+i\varepsilon} \frac{i}{2} m_A^2 \frac{-ig^{\sigma\nu}}{k^2+i\varepsilon}$$

$$+ \frac{-ig^{\mu\rho}}{k^2+i\varepsilon} \frac{i}{2} m_A^2 \frac{-ig_{\rho\sigma}}{k^2+i\varepsilon} m_A k^\sigma \frac{i}{k^2+i\varepsilon}(-m_A k^\alpha) \frac{-ig_\alpha^{\ \nu}}{k^2+i\varepsilon} \cdot 2$$

$$+ \frac{-ig^{\mu\rho}}{k^2+i\varepsilon} m_A k^\rho \frac{i}{k^2+i\varepsilon}(-m_A k^\sigma) \frac{-ig_{\sigma\alpha}}{k^2+i\varepsilon} m_A k^\alpha \frac{i}{k^2+i\varepsilon}(-m_A k^\beta) \frac{-ig_\beta^{\ \nu}}{k^2+i\varepsilon} + \dots$$

$$= \frac{-ig^{\mu\rho}}{k^2+i\varepsilon}\Big[g_\rho{}^\nu + \frac{m_A^2}{k^2}\Big(g_\rho{}^\nu - \frac{k_\rho k^\nu}{k^2}\Big) + \Big(\frac{m_A^2}{k^2}\Big)^2 \underbrace{\Big(g_\rho{}^\nu - 2 g_{\rho\sigma}\frac{k^\sigma k^\nu}{k^2} + \frac{k^\rho k^\sigma g_{\sigma\alpha} k^\alpha k^\nu}{k^2\cdot k^2}\Big)}_{= g_\rho{}^\nu - \frac{k_\rho k^\nu}{k^2}} + \dots \Big]$$

$$= \frac{-ig^{\mu\rho}}{k^2+i\varepsilon}\Big(1 + \frac{m_A^2}{k^2} + \Big(\frac{m_A^2}{k^2}\Big)^2 + \dots\Big)\Big(g_\rho{}^\nu - \frac{k_\rho k^\nu}{k^2}\Big) - \frac{-ig^{\mu\rho}}{k^2+i\varepsilon}\Big(-\frac{k_\rho k^\nu}{k^2}\Big)$$

can be neglected since $k^\nu j_\nu = 0$.

$$= \frac{-i\big(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\big)}{k^2 - m_A^2 + i\varepsilon}$$

<u>Note:</u> One can always make a gauge transformation $\phi(x) \to e^{i\alpha(x)}\phi'(x)$ such that $\phi_2(x) \equiv 0 \quad \forall x$.
↳ "The field ϕ_2 gets eaten by the gauge boson."
It disappears from the theory, but a new degree of freedom (the longitudinal polarization state of the gauge boson) appears.

$k^\mu \phi_2$ and A_μ have identical quantum numbers and are interchangeable thanks to gauge invariance.

Massive gauge boson propagator (Abelian)

$$= \frac{-i}{k^2 - m_A^2}\Big[g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi m_A^2}(1-\xi)\Big]$$

gauge fixing parameter

ϕ_1 propagator

$$\phi_1 \qquad = \frac{i}{k^2 - m^2}$$

ϕ_2 propagator

$$\phi_2 \qquad = \frac{i}{k^2 - \xi m_A^2}$$

Ghost propagator (now, ghosts appear even in Abelian theory, coupled to ϕ_1)

$$= \frac{i}{k^2 - \xi m_A^2}$$

6.2 Goldstone's theorem

"Theories with spontaneously broken symmetries contain massless bosons"

Intuitively: There must be ∞ many equivalent minima of the scalar potential, related by the symmetry. Field excitations along these directions correspond to massless bosons

Formal proof: Consider theory with scalar fields $\phi = (\phi^1, \dots \phi^n)$

$$\mathcal{L} = (\text{terms with derivatives}) - V(\phi)$$

Assume $V(\phi)$ has a minimum at

$$\frac{\partial}{\partial \phi^a} V(\phi) \Big|_{\phi^a(x) = \phi_0^a} = 0 \qquad \forall a = 1 \dots n$$

Expand V around minimum

$$V(\phi) = V(\phi_0) + \frac{1}{2} (\phi - \phi_0)^a (\phi - \phi_0)^b \underbrace{\left[\frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \right]_{\phi_0}}_{\equiv m^{ab}} + \dots$$

(mass matrix of ϕ^a fields)

Now assume $\mathcal{L}$ is invariant under symmetry

can depend on x, but doesn't have to

$$\phi^a(x) \longrightarrow \phi^a(x) + \alpha(x) \Delta^a(\phi)$$

infinitesima

This must be true in particular for x-independent field configurations and x-independent α.

$\hookrightarrow V(\phi) = V(\phi + \alpha \Delta(\phi))$

[Derivative terms vanish for constant ϕ, α]

$$\Rightarrow \quad V(\phi) = V(\phi) + \alpha \Delta^a(\phi) \frac{\partial V}{\partial \phi^a}$$

$$\Rightarrow \quad 0 = \frac{\partial \Delta^a(\phi)}{\partial \phi^b} \frac{\partial V}{\partial \phi^a} + \Delta^a(\phi) \frac{\partial^2 V}{\partial \phi^b \partial \phi^a}$$

Now set $\phi = \phi_0$ $\Rightarrow$ first term on RHS vanishes since ϕ_0 is minimum
$\Rightarrow$ 2nd term must vanish

If transformation leaves ϕ_0 invariant (unbroken symmetry, $\Delta^a = 0$), this is trivial. Otherwise (broken symmetry), m^{ab} has zero eigenvalue with eigenvector $\Delta^a(\phi_0)$.

$\square$

6.3 The Glashow - Salam - Weinberg theory of electroweak interactions

Peskin sec. 20.2

We begin with the gauge group $SU(2) \times U(1)$ and a scalar field ϕ (the Higgs field) in rep. 2 of $SU(2)$, with charge $\frac{1}{2}$ under the $U(1)$:

$$\phi(x) \xrightarrow[\text{trafo}]{\text{gauge}} e^{i\alpha^a(x)\frac{\sigma^a}{2}} \cdot e^{i\beta(x)\cdot\frac{1}{2}} \phi(x)$$

generators of $SU(2)$ in 2-rep.

$$\boxed{\mathcal{L}_{Higgs} \supset (D_\mu\phi)^+(D_\mu\phi) \underbrace{+\mu^2\phi^+\phi - \lambda(\phi^+\phi)^2}_{\equiv -V(\phi)}}$$

Potential has minimum at

$$\boxed{2v^2 \equiv \langle \phi^+\phi \rangle = \frac{\mu^2}{\lambda}}$$

We write $\boxed{\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix}}$ (but any $SU(2)$ transform of this would be equivalent)

<u>Note</u>: $\langle\phi\rangle$ breaks $SU(2)\times U(1)$ except for transformations with $\alpha^1 = \alpha^2 = 0$, $\alpha^3 = \beta$ $\Rightarrow$ 3 of 4 generators broken, expect 3 Goldstone bosons $\rightarrow$ 3 massive gauge bosons and 1 massive boson (Higgs boson), 1 massless gauge boson (photon)

$$\Rightarrow \text{ with } D_\mu\phi = \partial_\mu\phi - ig\underset{\substack{\uparrow\\ SU(2)\text{ gauge fields}}}{A^a_\mu}\frac{\sigma^a}{2}\phi - i\frac{1}{2}g'\underset{\substack{\uparrow\\ U(1)\text{ gauge field}}}{B_\mu}\phi :$$

$$\mathcal{L} \supset \frac{1}{2}(0,\ v)\left(gA^a_\mu\frac{\sigma^a}{2} + \frac{1}{2}g'B_\mu\right)\left(gA^{\mu b}\frac{\sigma^b}{2} + \frac{1}{2}g'B^\mu\right)\begin{pmatrix}0\\v\end{pmatrix}$$

$$= \frac{1}{2}\frac{v^2}{4}\left(g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + g^2(A^3_\mu)^2 - 2gg'A^3_\mu B^\mu + g'^2(B_\mu)^2\right)$$

$\{\sigma^a,\sigma^b\} = 2\delta^{ab}\mathbb{1}$

$$= \frac{1}{2}\frac{v^2}{4}\left[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (A^3_\mu\ \ B_\mu)\underbrace{\begin{pmatrix}g^2 & -gg'\\ -gg' & g'^2\end{pmatrix}}\begin{pmatrix}A^{\mu 3}\\ B^\mu\end{pmatrix}\right]$$

eigenvalues $g^2+g'^2$ and 0

eigenvectors $\begin{pmatrix}g\\-g'\end{pmatrix}$ and $\begin{pmatrix}g'\\g\end{pmatrix}$

$$\boxed{\begin{aligned} W^{\pm}_\mu &\equiv \frac{1}{\sqrt{2}}\left(A^1_\mu \mp i A^2_\mu\right) && \text{with } m_W = g\frac{v}{2} \\ Z^0_\mu &\equiv \frac{1}{\sqrt{g^2+g'^2}}\left(g A^3_\mu - g' B_\mu\right) && \text{with } m_Z = \sqrt{g^2+g'^2}\,\frac{v}{2} \\ A_\mu &\equiv \frac{1}{\sqrt{g^2+g'^2}}\left(g' A^3_\mu + g B_\mu\right) && \text{with } m_A = 0 \end{aligned}}$$

Define:

$$\Rightarrow A^1_\mu = \frac{1}{\sqrt{2}}\left(W^+_\mu + W^-_\mu\right)$$
$$A^2_\mu = \frac{-i}{\sqrt{2}}\left(-W^+_\mu + W^-_\mu\right)$$
$$A^3_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(g Z^0_\mu + g' A_\mu\right)$$
$$B_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(-g' Z^0_\mu + g A_\mu\right)$$

(The meaning of the indices $\pm$ on the W fields will become clear shortly, as will the identification of A_μ with the photon.)

$$\Rightarrow D_\mu = \partial_\mu - \frac{1}{\sqrt{2}} i g\, t^1 \left(W^+_\mu + W^-_\mu\right) + \frac{i}{\sqrt{2}} i g\, t^2 \left(-W^+_\mu + W^-_\mu\right)$$
$$- i g t^3 \frac{1}{\sqrt{g^2+g'^2}}\left(g Z^0_\mu + g' A_\mu\right)$$
$$- i \underbrace{Y}_{u(1)\text{ charge} = \text{"hypercharge"}} g' \frac{1}{\sqrt{g^2+g'^2}}\left(-g' Z^0_\mu + g A_\mu\right)$$

$$= \partial_\mu - \frac{ig}{\sqrt{2}}\Big(W^+_\mu \underbrace{t^+}_{\equiv t^1 + i t^2 = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} + W^-_\mu \underbrace{t^-}_{\equiv t^1 - i t^2 = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)}\Big)$$
$$- i \frac{1}{\sqrt{g^2+g'^2}}\left[Z_\mu \cdot \left(g^2 t^3 - g'^2 Y\right) + A_\mu\, g g' \left(t^3 + Y\right)\right]$$

From the last term (coupling to massless gauge boson = photon) we identify:

$$\boxed{\underbrace{e}_{\text{e.m. coupling}} \cdot \underbrace{Q}_{\text{charge of particle}} = \frac{g g'}{\sqrt{g^2+g'^2}} \cdot \left(t^3 + Y\right)}$$

Examples: • left-handed quarks live in $SU(2)$ doublet with $Y = \frac{1}{6}$

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \Rightarrow \quad t^3 q_L = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

$$\Rightarrow Q_{u_L} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$Q_{d_L} = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$$

- right-handed quarks live in SU(2) singlets ($\to t^3 = 0$) with $Y = \frac{2}{3}$ for u_R, $Y = -\frac{1}{3}$ for d_R
- left-handed leptons: SU(2) doublets with $Y = -\frac{1}{2}$

Another common definition: <u>Weinberg angle Θ_W</u>

$$\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\Theta_W & -\sin\Theta_W \\ +\sin\Theta_W & \cos\Theta_W \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}$$

$$\hookrightarrow \boxed{\sin\Theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \quad ; \quad \cos\Theta_W = \frac{g}{\sqrt{g^2+g'^2}}}$$

$$\Rightarrow \boxed{\begin{aligned} D_\mu = \partial_\mu &- \frac{ig}{\sqrt{2}}\left(W^+_\mu t^+ + W^-_\mu t^-\right) \\ &- \underbrace{\frac{ig}{\cos\Theta_W}}_{= i\sqrt{g^2+g'^2}} Z_\mu \big(t^3 - \underbrace{\sin^2\Theta_W\, Q}_{= \frac{g'^2}{g^2+g'^2}(t^3+Y)}\big) - ieA_\mu Q \end{aligned}}$$

$U(1)_{em}$ transformation of $W_\mu^\pm$

Consider a pure $W_\mu^\pm$ configuration, i.e. $A_\mu^1 = \mp i A_\mu^2$ and $A_\mu^3 = B_\mu = 0$

$U(1)_{em}$ gauge trafo: $A_\mu^1 (t^1 \pm i t^2) \longrightarrow U(x)\left(A_\mu^1(x)(t^1 \pm i t^2) + \frac{i}{g}\partial_\mu\right)U^\dagger(x)$

with $U(x) = \exp[i\alpha(x)\, t^3]$. ($U(1)_Y$ part of the trafo does not affect A_μ^a fields)

$$(t^1 \pm i t^2)\, e^{-i\alpha(x) t^3} = (t^1 \pm i t^2) \sum_{n=0}^{\infty} \frac{1}{n!}(-i\alpha t^3)^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} (-i\alpha)^n \left(-it^2 + t^3 t^1 \mp t^1 \pm i t^3 t^2\right)(t^3)^{n-1}$$

$[t^a, t^b] = i\varepsilon^{abc} t^c$

$$+ \frac{1}{0!}(-i\alpha t^3)^0 (t^1 \pm i t^2)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}(-i\alpha)^n (t^3 \mp 1)(t^1 \pm i t^2)(t^3)^{n-1}$$

$$+ \frac{1}{0!}(-i\alpha)^0 (t^3 \mp 1)^0 (t^1 \pm i t^2)$$

$$= \exp[-i\alpha(t^3 \mp 1)]\,(t^1 \pm i t^2)$$

$$= \exp[-i\alpha t^3]\,\exp[\pm i\alpha]$$

$$\Rightarrow A_\mu^1 (t^1 \pm i t^2) \rightarrow \underbrace{e^{\pm i\alpha} A_\mu^1 (t^1 \pm i t^2)}_{\text{gauge trafo of } W_\mu^\pm} + \underbrace{\frac{i}{g}\partial_\mu \alpha\, t^3}_{\text{contributes to gauge trafo of } A_\mu^3 \text{ and thus } Z_\mu^0 \text{ and } A_\mu}$$

Coupling to fermions

Left-handed fields $\psi_L \equiv P_L \psi \equiv \frac{1-\gamma^5}{2}\psi$ and right-handed fields $\psi_R \equiv P_R\psi = \frac{1+\gamma^5}{2}\psi$ can be in different representations of the gauge group (they belong to different reps. of the Lorentz group already).

	$SU(3)$	$SU(2)$	$U(1)_Y$
LH leptons $E_L = \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	1	2	$-\frac{1}{2}$
RH charged leptons e_R	1	1	-1
LH quarks $Q_L = \begin{pmatrix}u_L \\ d_L\end{pmatrix}$	3	2	$\frac{1}{6}$
RH up quarks u_R	3	1	$\frac{2}{3}$
RH down quarks d_R	3	1	$-\frac{1}{3}$

$$\mathcal{L}_{\text{fermions,kin}} = \bar{E}_L i\not{D} E_L + \bar{e}_R i\not{D} e_R + \bar{Q}_L i\not{D} Q_L + \bar{u}_R i\not{D} u_R + \bar{d}_R i\not{D} d_R$$

$$= \bar{E}_L i\not{\partial} E_L + \bar{e}_R i\not{\partial} e_R + \bar{Q}_L i\not{\partial} Q_L + \bar{u}_R i\not{\partial} u_R + \bar{d}_R i\not{\partial} d_R + g\left(W_\mu^+ J_W^{\mu+} + W_\mu^- J^{\mu-} + Z_\mu^0 J_Z^\mu\right) + e A_\mu J_{em}^\mu$$

with

$$J^{\mu+} \equiv \frac{1}{\sqrt{2}}\left(\bar{\nu}_L \gamma^\mu e_L + \bar{u}_L \gamma^\mu d_L\right)$$

$$J^{\mu-} \equiv \frac{1}{\sqrt{2}}\left(\bar{e}_L \gamma^\mu \nu_L + \bar{d}_L \gamma^\mu u_L\right)$$

Example: $\bar{E}_L i\not{D} E_L = \bar{E}_L \frac{g\gamma^\mu}{\sqrt{2}}\left(W_\mu^+ t^+ + W_\mu t^-\right) E_L + \ldots$

$$= \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \frac{g\gamma^\mu}{\sqrt{2}}\left[W_\mu^+ \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix} + W_\mu^- \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + \ldots$$

$$= \bar{\nu}_L \frac{g\gamma^\mu}{\sqrt{2}} W_\mu^+ e_L + \bar{e}_L \frac{g\gamma^\mu}{\sqrt{2}} W_\mu^- \nu_L + \ldots$$

$$J_Z^\mu = \frac{1}{\cos\theta_W}\Big[\bar{\nu}_L \gamma^\mu \tfrac{1}{2}\nu_L + \bar{e}_L\gamma^\mu\left(-\tfrac{1}{2}+\sin^2\theta_W\right)e_L$$

$$+ \bar{e}_R\gamma^\mu \sin^2\theta_W\, e_R + \bar{u}_L\gamma^\mu\left(\tfrac{1}{2}-\tfrac{2}{3}\sin^2\theta_W\right)u_L$$

$$+ \bar{u}_R\gamma^\mu\left(-\tfrac{2}{3}\sin^2\theta_W\right)u_R + \bar{d}_L\gamma^\mu\left(-\tfrac{1}{2}+\tfrac{1}{3}\sin^2\theta_W\right)d_L$$

$$+ \bar{d}_R\gamma^\mu\left(\tfrac{1}{3}\sin^2\theta_W\right)d_R$$

$$= \frac{1}{\cos\theta_W}\Big[\bar{\nu}\gamma^\mu(1-\gamma^5)\cdot\tfrac{1}{4}\nu + \bar{e}\gamma^\mu\Big[\left(-\tfrac{1}{2}+\sin^2\theta_W\right)\frac{1-\gamma^5}{2} + \sin^2\theta_W\frac{1+\gamma^5}{2}\Big]e$$

$$+ \bar{u}\gamma^\mu\Big[\left(\tfrac{1}{2}-\tfrac{2}{3}\sin^2\theta_W\right)\frac{1-\gamma^5}{2} + \left(-\tfrac{2}{3}\sin^2\theta_W\right)\frac{1+\gamma^5}{2}\Big]u$$

$$+ \bar{d}\gamma^\mu\Big[\left(-\tfrac{1}{2}+\tfrac{1}{3}\sin^2\theta_W\right)\frac{1-\gamma^5}{2} + \tfrac{1}{3}\sin^2\theta_W\frac{1+\gamma^5}{2}\Big]d$$

$$= \frac{1}{4\cos\theta_W}\Big[\bar{\nu}\gamma^\mu(1-\gamma^5)\nu + \bar{e}\gamma^\mu\left[-1+4\sin^2\theta_W+\gamma^5\right]e$$

$$+ \bar{u}\gamma^\mu\left[1-\tfrac{8}{3}\sin^2\theta_W-\gamma^5\right]u$$

$$+ \bar{d}\gamma^\mu\left[-1+\tfrac{4}{3}\sin^2\theta_W+\gamma^5\right]d$$

Check vs. Cheng/Li!?

$$J^\mu_{em} \equiv \bar{e}\gamma^\mu(-1)e + \bar{u}\gamma^\mu\left(\tfrac{2}{3}\right)u + \bar{d}\gamma^\mu\left(-\tfrac{1}{3}\right)d$$

Fermion masses

Explicit mass terms like $-m\,\bar{e}_L e_R$ are forbidden by gauge symmetries. But the following terms are allowed

$$\mathcal{L}_Y \supset -y_e \underset{(2,-\frac{1}{2})}{\bar{E}_L}\;\underset{(2,\frac{1}{2})}{\phi}\;\underset{(1,-1)}{e_R} + h.c.$$

$$= -y_e\,\bar{e}_L\langle\phi\rangle e_R + \ldots$$

$$= -m_e\,\bar{e}_L e_R + \ldots \qquad \text{with} \quad \boxed{m_e \equiv \frac{y_e v}{\sqrt{2}}}$$

Analogously for down quarks. For up quarks, use

$$\mathcal{L}_Y \supset -y_u \overline{Q}_L \tilde{\phi} u_R + h.c.$$

with $\tilde{\phi} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^*$ and thus $\langle \tilde{\phi} \rangle = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \end{pmatrix}$

Note that under $SU(2)$ trafo, $\tilde{\phi} \longrightarrow e^{i\alpha^a \frac{\sigma^a}{2}} \tilde{\phi}$ (just like ϕ)

<u>Proof:</u> $\phi \to e^{i\alpha^a \frac{\sigma^a}{2}} \phi$

$$\Rightarrow \phi^* \to e^{-i\alpha^a \frac{(\sigma^a)^*}{2}} \phi^*$$

Infinitesimally: $\phi^* \to \left(1 - i\alpha^a \frac{(\sigma^a)^*}{2}\right) \phi^*$

$$\tilde{\phi} \to \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= i\sigma^2} \left(1 - i\alpha^a \frac{(\sigma^a)^*}{2}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\phi}$$

$$= \left(1 - i\alpha^a \sigma^2 \frac{\sigma^{a*}}{2} \sigma^2\right) \tilde{\phi}$$

$$\underset{\substack{\sigma^1 = \sigma^{1*} \\ \sigma^2 = -\sigma^{2*} \\ \sigma^3 = \sigma^{3*} \\ \text{and } \{\sigma^a, \sigma^b\} = 2}}{=} \left(1 + i\alpha^a \frac{\sigma^a}{2}\right) \tilde{\phi}$$

Finite (non-infinitesimal) trafos can be composed of many infinitesimal ones. $\square$

<u>The Higgs boson</u>

By suitable $SU(2) \times U(1)$ gauge trafos, $\phi(x)$ can be brought to the form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

(<u>unitary gauge</u>).

This leads to the following couplings

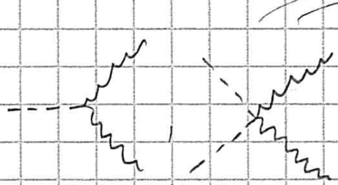
$$\mathcal{L}_{Higgs} = (D_\mu \phi^+)(D_\mu \phi) + \mu^2 \phi^+ \phi - \lambda (\phi^+ \phi)^2$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{g^2}{2} W_\mu^+ W_\mu^- \frac{1}{2}(v+h)^2$$

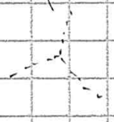
$$+ \frac{g^2}{\cos^2\theta_W} Z_\mu Z^\mu \frac{1}{4}\frac{1}{2}(v+h)^2$$

$$+ \mu^2 \frac{1}{2}(v+h)^2 - \frac{\lambda}{4}(v+h)^4$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \underbrace{m_W^2}_{=\frac{g^2 v^2}{4}} W_\mu^+ W^- \left(1+\frac{h}{v}\right)^2 + \frac{1}{2}\underbrace{m_Z^2}_{=\frac{g^2 v^2}{4\cos^2\theta_W}} \left(1+\frac{h}{v}\right)^2 Z_\mu Z^\mu$$



$$+ h^2 \underbrace{\left(\frac{1}{2}\mu^2 - \frac{\lambda}{4} 6 \overbrace{v^2}^{=\frac{\mu^2}{\lambda}}\right)}_{-\mu^2} - \lambda v h^3 - \frac{\lambda}{4} h^4$$



Higgs boson mass $m_h = \mu$