1st problem sheet (20+4 points)
to be handed in on 10.11.2014 to the letterbox (foyer of Staudingerweg 7)

1. **Cross section of electron – muon scattering**

   (a) **(3P.)** Use the Feynman rules to write down the amplitude for the process $e^- \, \mu^- \to e^- \, \mu^-$

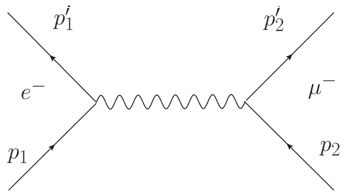

Figure 1: Feynman diagram for electron – muon scattering.

   (b) **(5P.)** Show that the corresponding squared amplitude, averaged and summed over the spins, is equal to

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\right] , \qquad (1)$$

   when neglecting the electron mass $m_e = 0$.

   (c) **(4P.)** Assume to be in the centre of mass (com) system and calculate the diffenrential cross section $\frac{d\sigma}{d\Omega}$ (again for vanishing electron mass, $m_e = 0$). Start from the general equation for a scattering process of $2 \to N$ particles

$$d\sigma = \frac{|\mathcal{M}(p_1, p_2 \to \{p_f\})|^2}{2E_1 2E_2|v_1 - v_2|}\cdot\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}(p_1 + p_2 - \sum p_f)\,, \quad (2)$$

   where p_1 and p_2 are the momenta of the incoming particles and p_f the ones of the outgoing particles and show it gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{com},\ m_e = 0} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{com}^2} \qquad (3)$$

   for $2 \to 2$ scattering with one massless particle. Then calculate $|\mathcal{M}|^2$ with the parametrisation given in Figure 2.

   What is the high energy limit of the cross section (where also the muon mass vanishes)?

   (d) **(Bonus 4P.)** Take the nonrelativistic limit of the scattering amplitude to deduce the form of the Coulomb potential $V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r}$.

   Show that in the nonrelativistic limit of

$$\bar{u}(p')\gamma^\mu u(p) \qquad (4)$$

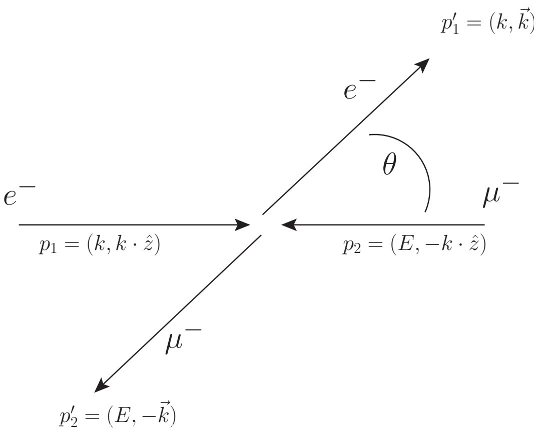

Figure 2: Parametrisation of the momenta of the in- and outgoing electron and muon in the com system.

only the γ^0 part in nonzero, when setting $p = p' = 0$, and that it is equal to

$$\bar{u}(p')\gamma^0 u(p) = 2m\xi'^{\dagger}\xi\,, \tag{5}$$

where ξ and ξ' are normalised two-component vectors. Note that the factors of $2m$ come from the relativistic normalisation convention and have to be dropped when comparing to states with nonrelativistic normalisation.
Compare the solution of the nonrelativistic matrix element to the Born approximation of the scattering amplitude in nonrelativistic quantum mechanics:

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{q}) \cdot 2\pi \cdot \delta(E_{p'} - E_p). \tag{6}$$

To get the potential in position space you have to Fourier transform $\tilde{V}(\mathbf{q})$:

$$V(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\tilde{V}(\mathbf{q})e^{i\mathbf{q}\mathbf{x}}\,. \tag{7}$$

2. **Path integrals in Quantum Mechanics**

In the lecture you derived the amplitude for the propagation of a state from q' to q'' to be

$$\langle q''(t)|e^{-iHt}|q'(0)\rangle = \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} e^{ip_j(q_{k+1}-q_k)-ip_j^2/2m-iV(\bar{q}_k)} \tag{8}$$

$$= \int \mathcal{D}q\mathcal{D}p \exp\left(i\int_0^t dt L(p,q)\right)\,, \tag{9}$$

with

$$L(p,q) = p\dot{q} - H(p,q) = p\dot{q} - \frac{p^2}{2m} - V(q) \quad \text{and} \quad V(\bar{q}_k) = 1/2(q_k + q_{k+1})\,. \tag{10}$$

(a) **(4P.)** Note that the integrals over p_j are gaussian and thus give a constant. Derive this constant. (When solving the integrals treat them as a real integrals.)

(b) **(4P.)** Now set the potential to zero $V(q) = 0$ (free particle) and solve also the integrals over dq_k.
Hint: integrate over q_1, then q_2, etc. and try to look for a pattern.