2nd problem sheet (20+4 points)
to be handed in on 24.11.2014 to the letterbox (foyer of Staudingerweg 7)

1. **Real $\phi^4$-theory**

   Consider a real scalar field with the Lagrangian

$$\mathcal{L} = -\frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 - \frac{1}{4!} Z_\lambda \lambda \phi^4 \,, \tag{1}$$

   that we can divide into a free term, a term containing the interaction and the counterterms:

$$\begin{aligned} \mathcal{L}_0 &= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 & (2)\\ \mathcal{L}_1 &= -\frac{1}{24}Z_\lambda\lambda\phi^4 & (3)\\ \mathcal{L}_{ct} &= -\frac{1}{2}(Z_\phi - 1)\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}(Z_m - 1)m^2\phi^2 \,. & (4) \end{aligned}$$

   The path integral is then given by

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{ct} + J\phi)} \,. \tag{5}$$

   Now ignore the counterterms.

   (a) **(3P.)** What expressions do you get for the propagator, the source and the vertex? How many line segments does a vertex join?

   (b) **(4P.)** Draw all connected diagrams with two sources and two vertices (there should be 3) and the diagram with four sources and one vertex. What is the symmetry factor of the latter?

   (c) **(1P.)** Why did we not have to include a counterterm linear in $\phi$ to cancel the tadpole diagrams?

2. **Complex $(\phi\phi^\dagger)^2$-theory**

   Consider a complex scalar field with the Lagrangian

$$\mathcal{L} = -Z_\phi \partial^\mu \phi^\dagger \partial_\mu \phi - Z_m m^2 \phi^\dagger \phi - \frac{1}{4} Z_\lambda \lambda (\phi^\dagger \phi)^2 \,, \tag{6}$$

   that we can again divide into a free term, a term containing the interaction and the counterterms:

$$\begin{aligned} \mathcal{L}_0 &= -\partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi & (7)\\ \mathcal{L}_1 &= -\frac{1}{4}Z_\lambda\lambda(\phi^\dagger\phi)^2 & (8)\\ \mathcal{L}_{ct} &= -(Z_\phi - 1)\partial^\mu\phi^\dagger\partial_\mu\phi - (Z_m - 1)m^2\phi^\dagger\phi \,. & (9) \end{aligned}$$

   This theory has two kinds of sources, $J$ and $J^\dagger$. The path integral is then given by

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{ct} + J^\dagger\phi + J\phi^\dagger)} \,. \tag{10}$$

To indicate the difference of the sources in the diagrams, we put an arrow on the attached propagator, pointing towards the source for $J^\dagger$ and away from it for J.

(a) **(6P.)** Ignoring the counterterms, what expressions do you get here for the propagator, the source and the vertex? Mind the arrows for the vertex!

(b) **(6P.)** Draw all connected diagrams with two sources and two vertices (there should be again 3) and the diagram with four sources and one vertex. What is the symmetry factor of the latter?

3. **(Bonus 4P.) Regularisation of the Feynman propagator**

In the lecture you derived the counterterm to cancel the contribution of so called tadpole diagrams. But since the contribution $D_F(0)$ of the tadpole is infinite, it needs to be regularised first, before the counterterm Y can be chosen such that they cancel.

To regularise the propagator, choose an ultraviolett (UV) cutoff $\Lambda$, introduced such that $D_F(x-y)$ remains Lorentz-invariant:

$$D_F(x-y) \quad \to \quad \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2-m^2+i\epsilon} \left(\frac{\Lambda^2}{k^2-\Lambda^2+i\epsilon}\right)^2 \tag{11}$$

$$D_F(0) \quad \to \quad \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon} \left(\frac{\Lambda^2}{k^2-\Lambda^2+i\epsilon}\right)^2 . \tag{12}$$

Use Wick's theorem to rotate the contour to the imaginary axis in $k^0$ and define the Euclidean vector $\kappa = (ik^0, \mathbf{k})$. Then change the integration variable $\kappa$ to spherical coordinates:

$$d^4\kappa \to d\Omega_3 \kappa^3 d\kappa \, . \tag{13}$$

For the final integral you can neglect $m$, since the cutoff $\Lambda$ is assumed to go to infinity and therefore we only want the leading order contribution. You can also drop the $i\epsilon$. The result should be proportional to $\Lambda^2$.