3rd problem sheet (20 points)
to be handed in on 08.12.2014 to the letterbox (foyer of Staudingerweg 7)

1. **Grassmann Numbers**

   In the lecture you showed that for the Grassmann numbers $\theta_1, \ldots, \theta_n$

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j} = \det(B)\,. \tag{1}$$

   (a) **(2P.)** Show the following relation, proceeding in a similar way as in the lecture:

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) \theta_k \theta_l^* e^{-\theta_i^* B_{ij} \theta_j} = \det(B) \cdot (B^{-1})_{kl}\,. \tag{2}$$

   (b) **(2P.)** Then verify the equality

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* \theta_j - \eta_i^* \theta_i + \theta_i^* \eta_i} = e^{-\eta_i^* \eta_i} \tag{3}$$

   by induction.

   (c) **(2P.)** Use the result from the lecture, equation (1), and equation (3) to show that

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j - \eta_i^* \theta_i + \theta_i^* \eta_i} = \det(B) \cdot e^{-\eta_i^* (B^{-1})_{ij} \eta_j}\,. \tag{4}$$

2. **(6P.) Fermionic Path Integral**

   For a free Dirac field with

$$\mathcal{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta \tag{5}$$

   show that the path integral

$$Z_0[\eta, \bar{\eta}] = \int D\psi D\bar{\psi} e^{i \int d^4x \mathcal{L}_0} \tag{6}$$

   gives

$$Z_0[\eta, \bar{\eta}] = \exp[-\int d^4x\, d^4x'\, \bar{\eta}(x)\, S_F(x - x')\, \eta(x')] \tag{7}$$

   with the Feynman propagator for Dirac fermions

$$S_F(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{(k\!\!\!/ + m) i e^{-ik(x-x')}}{k^2 - m^2 + i\epsilon}\,. \tag{8}$$

3. **Propagator corrections to ϕ^3-Theory**

$$\mathcal{L} = \frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_m m^2 \phi^2 + \frac{1}{3!} Z_g g \phi^3 + Y \phi^3 + \phi J \tag{9}$$

   (a) **(2P.) Order g^2**
   Sketch the Feynman diagrams that give corrections to the propagator at order g^2. There should be one loop diagram and one counterterm diagram. Use the Feynman rules to write down the amplitude of the process.

   (b) **(2P.) Order g^4**
   At order g^4 sketch the three one- and two-loop diagrams and the two diagrams containing counterterms that give corrections to the propagator. Again write down the amplitude.

4. **Yukawa Theory**

$$\mathcal{L} = \bar{\psi}(i Z_\psi \not{\partial} - Z_m m)\psi + \frac{1}{2} Z_\phi \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} Z_M M^2 \phi^2 + Z_g g \phi \bar{\psi} \psi + \phi J + \bar{\eta} \psi + \bar{\psi} \eta \tag{10}$$

   (a) **(2P.) $\bar{\psi}\psi \to \phi\phi$**
   Sketch the Feynman diagrams for the process of two fermions annihilating to two bosons on tree level and again give the amplitude.

   (b) **(2P.) Scalar propagator**
   At order g^2 sketch the loop diagram and the counterterm that give corrections to the propagator. Again write down the amplitude.