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A TROTTER-TYPE APPROACH TO INFINITE RATE MUTUALLY CATALYTIC BRANCHING[1]

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Dawson and Perkins [*Ann. Probab.* **26** (1988) 1088–1138] constructed a stochastic model of an interacting two-type population indexed by a countable site space which locally undergoes a mutually catalytic branching mechanism. In Klenke and Mytnik [Preprint (2008), arXiv:0901.0623], it is shown that as the branching rate approaches infinity, the process converges to a process that is called the *infinite rate mutually catalytic branching process* (IMUB). It is most conveniently characterized as the solution of a certain martingale problem. While in the latter reference, a noise equation approach is used in order to construct a solution to this martingale problem, the aim of this paper is to provide a Trotter-type construction.

The construction presented here will be used in a forthcoming paper, Klenke and Mytnik [Preprint (2009)], to investigate the long-time behavior of IMUB (coexistence versus segregation of types).

This paper is partly based on the Ph.D. thesis of the second author (2008), where the Trotter approach was first introduced.

1. Introduction and main results.

1.1. *Background and motivation.* In [4], Dawson and Perkins studied a stochastic model of mutually catalytic (continuous-state) branching. Two populations live on a countable site space S and the amount of population of type $i = 1, 2$ at time t at site $k \in S$ is denoted by $Y_{i,t}(k) \in [0, \infty)$. The populations migrate according to a deterministic heatflow-like dynamics that is characterized by the (symmetric) q-matrix $\mathcal{A}$ of a Markov chain on S. Locally, the populations undergo critical continuous-state branching with a rate that is proportional to the size of the other type at the same place. Formally, this model can be described by a system of stochastic differential equations:

$$
\begin{aligned}
Y_{i,t}(k) &= Y_{i,0}(k) + \int_0^t \sum_{l \in S} \mathcal{A}(k,l) Y_{i,s}(l)\, ds \\
&\quad + \int_0^t (\gamma Y_{1,s}(k) Y_{s,2}(k))^{1/2}\, dW_{i,s}(k), \qquad t \geq 0, k \in S, i = 1, 2.
\end{aligned}
\tag{1.1}
$$

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Here, $(W_i(k), k \in S, i = 1, 2)$ is an independent family of one-dimensional Brownian motions and Y_0 is chosen from a suitable subspace of $([0,\infty)^2)^S$. The parameter $\gamma \geq 0$ can be thought of as being the branching rate for this model. Dawson and Perkins showed that there is a unique weak solution of (1.1) and studied the long-time behavior of this model. They also constructed the analogous model in the continuous setting on $\mathbb{R}$ instead of S.

For the model with $S = \mathbb{Z}$ and $\mathcal{A}$ the q-matrix of symmetric nearest neighbor random walk, the model tends to a state with spatially segregated types. In an approach to describing the cluster growth quantitatively, a space and time rescaling argument suggests that it is useful to first study the limit as $\gamma \to \infty$. Studying this limit requires a formal description of the limit process X, construction of the limit process and the establishing of convergence of Y as $\gamma \to \infty$.

This program is carried out for a process where S is a singleton in [9] and for a countable site space S in [10]. Furthermore, in [11], the long-time behavior is studied which shows a dichotomy between coexistence and segregation of types, depending on the potential properties of the matrix $\mathcal{A}$.

In [10], the process X is characterized both via a martingale problem and as the solution of a system of stochastic differential equations of jump type. While the construction of X was performed via the construction of approximate solutions of the stochastic differential equations, here, the aim is to present a different approach via a Trotter approximation scheme.

The main idea is described via the following heuristics. Denote by a_t the matrix of time t transition probabilities of the continuous-time Markov chain with q-matrix $\mathcal{A}$. Furthermore, let $Q_t(y, dy')$ denote the transition kernel for equation (1.1) with $\mathcal{A} = 0$. It is not hard to see that Q_t converges, as $t \to \infty$, to some kernel $\mathbf{Q}$. In fact, if $\mathcal{A} = 0$, then all colonies evolve independently and each colony is a time-transformed planar Brownian motion in $(0,\infty)^2$, stopped when it hits the boundary. Hence, $\mathbf{Q}$ is the product of the harmonic measures of planar Brownian motions in the upper-right quadrant. Now, let $\varepsilon > 0$, define $X_0^\varepsilon = Y_0$ and inductively let $X_{(k+1)\varepsilon}^\varepsilon$ be distributed, given $X_{k\varepsilon}^\varepsilon$, like $\mathbf{Q}(a_\varepsilon X_{k\varepsilon}^\varepsilon, dy')$. This amounts to an interlaced dynamics where deterministic heatflow and random infinite rate branching alternate. The main result of this paper is that the processes X^ε in fact converge, as $\varepsilon \to 0$, to the infinite rate mutually catalytic branching process X constructed in [10]. In Sections 1.2 and 1.3, we provide a formal description of this X.

The idea of using a Trotter-type approach for the construction of the infinite rate mutually catalytic branching process is taken from the Ph.D. thesis [13] and parts of the strategy of proof are based on that thesis.

While the noise equation approach of [10] relies on a duality of the processes in order to show convergence of a sequence of approximating processes, the Trotter approach works without this duality. This greater flexibility is exploited in [11] for the construction of a process X^K with state space $([0,K]^2 \setminus (0,K)^2)^S$

that approaches X and whose coordinate processes are driven by orthogonal L^2-martingales. For this process X^K that is used in order to study the long-time behavior of X, we do not have a duality and, thus, the noise equation approach does not seem to be feasible.

Furthermore, we hope that the Trotter-type approach could serve as a key tool for the construction of infinite rate symbiotic branching processes. Symbiotic branching processes with index $\varrho \in [-1, 1]$ are solutions of (1.1), but with $W_1(k)$ and $W_2(k)$ being *correlated* Brownian motions with correlation ϱ. These were introduced in [6]. Clearly, $\varrho = 0$ is the branching case considered here, $\varrho = -1$ is the case of interacting Wright–Fisher diffusions and $\varrho = 1$ is the parabolic Anderson model. The voter model can be considered as the infinite rate interacting Wright–Fisher diffusion model and can be obtained rather simply from this model via the Trotter approach. The other cases of ϱ are open. For symbiotic branching processes, there is a moment dual, but it is of limited use in many cases. Hence, the Trotter-type approach might also prove useful here to construct infinite rate versions of these processes.

1.2. *The infinite rate branching process.* We start with a definition of the state spaces of our processes. Define $E := [0, \infty)^2 \setminus (0, \infty)^2$. Let S be a countable set. For $u, v \in [0, \infty)^S$, define

$$\langle u, v\rangle = \sum_{k\in S} u(k)v(k) \in [0, \infty].$$

Similarly, for $x \in ([0, \infty)^2)^S$ and $\zeta \in [0, \infty)^S$, define

$$\langle x, \zeta\rangle = \sum_{k\in S} x(k)\zeta(k) \in [0, \infty]^2.$$

We can weaken the requirement that $\mathcal{A}$ be a q-matrix: let $\mathcal{A} = (\mathcal{A}(k, l))_{k,l\in S}$ be a matrix indexed by the countable set S satisfying

$$\mathcal{A}(k, l) \geq 0 \qquad \text{for } k \neq l \tag{1.2}$$

and

$$\|\mathcal{A}\| := \sup_{k\in S} \sum_{l\in S} |\mathcal{A}(k, l)| + |\mathcal{A}(l, k)| < \infty. \tag{1.3}$$

By Lemma IX.1.6 of [12], there exists a $\beta \in (0, \infty)^S$ and an $M \geq 1$ such that $\sum_{k\in S} \beta(k) < \infty$, and

$$\sum_{l\in S} \beta(l)\big(|\mathcal{A}(k, l)| + |\mathcal{A}(l, k)|\big) \leq M\beta(k) \qquad \text{for all } k \in S. \tag{1.4}$$

We fix this β for the rest of the paper.

Define the spaces

$$\mathbb{L}^{\beta} = \{u \in [0,\infty)^S : \langle u, \beta\rangle < \infty\},$$
$$\mathbb{L}^{\beta,2} = \{x \in ([0,\infty)^2)^S : \langle x, \beta\rangle \in [0,\infty)^2\},$$
$$\mathbb{L}^{f,2} = \{y \in ([0,\infty)^2)^S : y(k) \neq 0 \text{ for only finitely many } k \in S\},$$

as well as

$$\mathbb{L}^{\beta,E} = \mathbb{L}^{\beta,2} \cap E^S \quad \text{and} \quad \mathbb{L}^{f,E} = \mathbb{L}^{f,2} \cap E^S.$$

Finally, define the spaces

$$\begin{aligned} \mathbb{L}^{\beta}_{\infty} &= \{f \in [0,\infty)^S : \langle f, g\rangle < \infty \text{ for all } g \in \mathbb{L}^{\beta}\} \\ &= \Big\{f \in \mathbb{L}^{\beta} : \sup_{k \in S} f(k)/\beta(k) < \infty\Big\} \end{aligned} \tag{1.5}$$

and

$$\mathbb{L}^{\beta,E}_{\infty} = \{\eta = (\eta_1, \eta_2) \in E^S : \eta_1, \eta_2 \in \mathbb{L}^{\beta}_{\infty}\}.$$

Let $\mathcal{A} f(k) = \sum_{l \in S} \mathcal{A}(k,l) f(l)$ if the sum is well defined. Let $\mathcal{A}^n$ denote the nth matrix power of $\mathcal{A}$ [note that this is well defined and finite by (1.3)] and define

$$a_t(k,l) := e^{t\mathcal{A}}(k,l) := \sum_{n=0}^{\infty} \frac{t^n \mathcal{A}^n(k,l)}{n!}.$$

Let $\mathcal{S}$ denote the (not necessarily Markov) semigroup generated by $\mathcal{A}$, that is,

$$\mathcal{S}_t f(k) = \sum_{l \in S} a_t(k,l) f(l) \qquad \text{for } t \geq 0.$$

We will also use the notation $\mathcal{A}f$, $\mathcal{S}_t f$ and so on for $[0,\infty)^2$-valued functions f with the obvious coordinate-wise meaning.

For $u \in \mathbb{R}^S$, define

$$\|u\|_{\beta} = \sum_{k \in S} |u(k)| \beta(k). \tag{1.6}$$

Note that for $f \in \mathbb{L}^{\beta}$, the expressions $\mathcal{A}f$ and $\mathcal{S}_t f$ are well defined and that [recall M from (1.4)]

$$\|\mathcal{A} f\|_{\beta} \leq M \|f\|_{\beta} \quad \text{and} \quad \|\mathcal{S}_t f\|_{\beta} \leq e^{Mt} \|f\|_{\beta}. \tag{1.7}$$

That is, the spaces $\mathbb{L}^{\beta}$ and $\mathbb{L}^{\beta,2}$ are preserved under the dynamics of $(\mathcal{S}_t)$.

Let $D([0,\infty); \mathbb{L}^{\beta,E})$ be the Skorohod space of càdlàg $\mathbb{L}^{\beta,E}$-valued functions.

We will employ a martingale problem in order to characterize the infinite rate mutually catalytic branching process $X \in D([0,\infty); \mathbb{L}^{\beta,E})$. In order to formulate this martingale problem for X conveniently, for $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, we introduce the *lozenge product*

$$x \diamond y := -(x_1 + x_2)(y_1 + y_2) + i(x_1 - x_2)(y_1 - y_2) \tag{1.8}$$

(with $i = \sqrt{-1}$) and define

$$F(x, y) = \exp(x \diamond y). \tag{1.9}$$

Note that $x \diamond y = y \diamond x$, hence F is symmetric. For $x, y \in (\mathbb{R}^2)^S$, we write

$$\langle\langle x, y\rangle\rangle = \sum_{k\in S} x(k) \diamond y(k) \tag{1.10}$$

whenever the infinite sum is well defined and let

$$H(x, y) = \exp(\langle\langle x, y\rangle\rangle). \tag{1.11}$$

Note that the function $H(x, y)$ is well defined if either $x \in (\mathbb{R}^2)^S$ and $y \in \mathbb{L}^{f,E}$ or $x \in \mathbb{L}^{\beta,E}$ and $y \in \mathbb{L}^{\beta,E}_\infty$.

It is shown in [9], Corollary 2.4, that the vector space of finite linear combinations $\sum_{i=1}^n \alpha_i F(\cdot, y_i)$, $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $y_i \in E$, is dense in the space $C_l(E; \mathbb{C})$ of bounded continuous complex-valued functions on E with a limit at infinity. Hence, the family $H(\cdot, y)$, $y \in \mathbb{L}^{f,E}$, is measure-determining for probability measures on $\mathbb{L}^{\beta,E}$ (but not on $\mathbb{L}^{\beta,2}$).

In [10], the following theorem was established.

THEOREM 0. (a) *For all $x \in \mathbb{L}^{\beta,E}$, there exists a unique solution $X \in D([0,\infty); \mathbb{L}^{\beta,E})$ of the following martingale problem: for each $y \in \mathbb{L}^{f,E}$, the process $M^{x,y}$ defined by*

$$M_t^{x,y} := H(X_t, y) - H(x, y) - \int_0^t \langle\langle \mathcal{A}X_s, y\rangle\rangle H(X_s, y)\, ds \tag{MP}$$

is a martingale with $M_0^{x,y} = 0$.

(b) *For any $x \in \mathbb{L}^{\beta,E}$ and $y \in \mathbb{L}^{\beta,E}_\infty$, the process $M^{x,y}$ is well defined and is a martingale.*

(c) *Denote by P_x the distribution of X with $X_0 = x$. Then $(P_x)_{x\in\mathbb{L}^{\beta,E}}$ is a strong Markov family.*

Note that for the uniqueness, it is crucial that the single coordinates take values in E. If we required only values in $[0,\infty)^2$, then the finite rate mutually catalytic branching process Y is also a solution of the martingale problem for any $\gamma \geq 0$. In Proposition 1.1, we will see that our approximate process X^ε is also a solution to (MP) with the larger state space $\mathbb{L}^{\beta,2}$.

In [11], Theorem 1.3, it was shown that the processes Y defined in (1.1) converge to X as $\gamma \to \infty$ in the Meyer–Zheng topology. Hence, the name *infinite rate mutually catalytic branching process* for X is justified.

1.3. *The main result.* We now define the approximating process X^ε in detail. In order to do so, we introduce the harmonic measure Q of planar Brownian motion B on $(0,\infty)^2$. That is, if $B=(B_1,B_2)$ is a Brownian motion in $\mathbb{R}^2$ started at $x\in[0,\infty)^2$ and $\tau=\inf\{t>0: B_t\notin(0,\infty)^2\}$, then we define

$$Q_x=\mathbf{P}_x[B_\tau\in\cdot]. \tag{1.12}$$

Now, for fixed $\varepsilon>0$, consider the stochastic process X^ε with values in $\mathbb{L}^{\beta,2}$ with the following dynamics:

(i) Within each time interval $[n\varepsilon,(n+1)\varepsilon)$, $n\in\mathbb{N}_0$, X^ε is the solution of (1.1) with $\gamma=0$; that is, for $k\in S$,

$$dX^\varepsilon_{i,t}(k)=(\mathcal{A}X^\varepsilon_{i,t})(k)\,dt \qquad \text{for } t\in\big[n\varepsilon,(n+1)\varepsilon\big).$$

Clearly, the explicit solution is

$$X^\varepsilon_{i,t}(k)=(\mathcal{S}_{t-n\varepsilon}X^\varepsilon_{i,n\varepsilon})(k) \qquad \text{for } t\in\big[n\varepsilon,(n+1)\varepsilon\big).$$

(ii) At time $n\varepsilon$, X^ε has a discontinuity. Independently, each coordinate $X^\varepsilon_{n\varepsilon-}(k)=\mathcal{S}_\varepsilon X^\varepsilon_{(n-1)\varepsilon}(k)$ is replaced by a random element of E drawn according to the distribution $Q_{X^\varepsilon_{n\varepsilon-}(k)}$.

If, for $x\in E^S$, we denote by $\mathbf{Q}(x,\cdot)=\bigotimes_{k\in S}Q_{x(k)}$ the Markov kernel of independent displacements, then $(X^\varepsilon_{n\varepsilon})_{n\in\mathbb{N}_0}$ is a Markov chain on $\mathbb{L}^{\beta,E}$ with transition kernel $\mathbf{Q}^\varepsilon(x,\cdot):=\mathbf{Q}(\mathcal{S}_\varepsilon x,\cdot)$. Note that X^ε is a càdlàg process with values in $\mathbb{L}^{\beta,2}$ (but not in $\mathbb{L}^{\beta,E}$!) and that, for any $y\in\mathbb{L}^{f,E}$,

$$H(X^\varepsilon_t)-\int_{n\varepsilon}^t\langle\langle\mathcal{A}X^\varepsilon_s,y\rangle\rangle H(X^\varepsilon_s,y)\,ds, \qquad t\in\big[n\varepsilon,(n+1)\varepsilon\big),$$

is a martingale. Furthermore, as we will show in Lemma 2.2, we have $\int H(x',y)\times\mathbf{Q}(x,dx')=H(x,y)$ for all $y\in\mathbb{L}^{f,E}$ and $x\in\mathbb{L}^{\beta,2}$. As an immediate consequence, we get the following proposition.

PROPOSITION 1.1. *For all $x\in\mathbb{L}^{\beta,E}$ and $y\in\mathbb{L}^{f,E}$, and for X^ε defined as above with $X_0=x$, we have that*

$$\begin{aligned} M_t^{\varepsilon,x,y}&:=H(X^\varepsilon_t,y)-H(X^\varepsilon_0,y)\\ &\qquad-\int_0^t\langle\langle\mathcal{A}X^\varepsilon_s,y\rangle\rangle H(X^\varepsilon_s,y)\,ds, \qquad t\ge 0,\ \textit{is a martingale.}\end{aligned} \tag{1.13}$$

We will show that X^ε converges to a process that takes values in $\mathbb{L}^{\beta,E}$ while preserving this martingale property.

The main theorem of this paper is the following.

THEOREM 1. *For any $x\in\mathbb{L}^{\beta,E}$, as $\varepsilon\to0$, the processes X^ε converge in distribution in the Skorohod spaces $D([0,\infty);\mathbb{L}^{\beta,2})$ to the unique solution X of the martingale problem* (MP).

With a small effort, this construction can be interpreted as a Trotter product approach. Recall that (under suitable assumptions on the spaces and cores of the operators involved), the Trotter product formula states the following (see, e.g., [7], Corollary 6.7): if $(S_t)_{t\geq 0}$, $(T_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ are strongly continuous contraction semigroups with generators A, B and $C = A + B$, respectively, then

$$\lim_{\varepsilon\downarrow 0}(T_\varepsilon S_\varepsilon)^{\lfloor t/\varepsilon\rfloor} = U_t \qquad \text{pointwise.}$$

In our setting, $T_t = \mathbf{Q}$ for all $t > 0$ and $T_0 = \mathrm{id}$, hence (T_t) is by no means strongly continuous. Nevertheless, Theorem 1 shows that the limit exists.

A nice by-product of this construction is the following statement concerning the distribution of X_t for fixed t.

THEOREM 2. *For all $t \geq 0$, $x \in \mathbb{L}^{\beta,E}$ and $y \in \mathbb{L}^{\beta}_{\infty}$, we have $\mathbf{E}_x[Q_{\langle X_t,y\rangle}] = Q_{\langle \mathcal{S}_t x,y\rangle}$. In particular, for all $k \in S$, we have*

$$\mathbf{P}_x[X_t(k) \in \cdot] = Q_{\mathcal{S}_t x(k)}.$$

As an application of Theorem 2, we consider the interface problem in dimension $d = 1$. Assume that $S = \mathbb{Z}$ and that $\mathcal{A}f(k) = \frac{1}{2}f(k+1) + \frac{1}{2}f(k-1) - f(k)$ is the q-matrix of symmetric simple random walk on $\mathbb{Z}$. Hence, a_t is the time t transition kernel of continuous-time rate 1 symmetric simple random walk. Let $u, v > 0$ and assume that $x(k) = (u, 0)$ for $k < 0$ and $x(k) = (0, v)$ for $k \geq 0$. Let X be the infinite rate mutually catalytic branching process on $\mathbb{Z}$ with $X_0 = x$. Define

$$b_{t,1} := \sup\{k \in \mathbb{Z} : X_{1,t}(k-1) > 0\}$$

and

$$b_{t,2} := \inf\{k \in \mathbb{Z} : X_{2,t}(k) > 0\}.$$

We conjecture that $b_{t,1} = b_{t,2}$ almost surely. In this case, the position $b_t := b_{t,1}$ could be considered as the interface between the type 1 population (left) and the type 2 population (right). It is a challenging task to determine the dynamics of $(b_t)_{t\geq 0}$. By work on the finite branching rate process of [2] and [3], we should have that $\limsup_{t\to\infty} b_t = \infty$ and $\liminf_{t\to\infty} b_t = -\infty$. That is, at any given site, the type changes over and over again at arbitrarily late times.

Theorem 2 provides an indication as to what the distribution of b_t is for fixed t.

COROLLARY 1.2. *If $b_{t,1} = b_{t,2}$ almost surely, then*

$$\mathbf{P}[b_t \leq k] = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{v_t(k)^2 - u_t(k)^2}{2u_t(k)v_t(k)}\right), \tag{1.14}$$

where

$$u_t(k) := u\sum_{l=k+1}^{\infty} a_t(0,l) \quad \textit{and} \quad v_t(k) := v\sum_{l=-k}^{\infty} a_t(0,l).$$

In particular, $\mathrm{median}(b_t) \sim \alpha\sqrt{t}$ *as* $t \to \infty$, *where* $\alpha = \Phi^{-1}(\frac{u}{u+v})$ *and* Φ *is the distribution function of the standard normal distribution*, *and* $\lim_{t\to\infty} \mathbf{P}[b_t \le 0] = \frac{1}{2} + \frac{1}{\pi}\arctan((v^2 - u^2)/2uv)$.

PROOF. By Theorem 2, we have $\mathbf{P}[b_t \le k] = \mathbf{P}[X_{2,t}(k) > 0] = Q_{\mathcal{S}_t x(k)}(\{0\} \times (0, \infty))$. By an explicit calculation using the density of Q (see Lemma 2.1), we get (1.14). The other two statements follow from the central limit theorem for a_t. $\square$

1.4. *Outline*. The rest of the paper is organized as follows. In Section 2, we collect some basic facts about the harmonic measure Q and prove Proposition 1.1. In Section 3, we derive a submartingale related to X^ε and show that the two types of X^ε are nonpositively correlated. In Section 4, we show relative compactness of the family $(X^\varepsilon, \varepsilon > 0)$. Finally, in Section 5, we complete the proofs of Theorems 1 and 2.

2. The harmonic measure Q.

2.1. *Harmonic measure and duality.* Recall that Q_x is the harmonic measure for planar Brownian motion in the upper-right quadrant started at $x \in [0, \infty)^2$ and stopped upon leaving $(0, \infty)^2$. If $x = (u, v) \in (0, \infty)^2$, then the harmonic measure Q_x has a one-dimensional Lebesgue density on E that can be computed explicitly:

$$Q_{(u,v)}(d(\bar{u}, \bar{v})) = \begin{cases} \dfrac{4}{\pi}\dfrac{uv\bar{u}}{4u^2v^2 + (\bar{u}^2 + v^2 - u^2)^2}\,d\bar{u}, & \text{if } \bar{v} = 0, \\ \dfrac{4}{\pi}\dfrac{uv\bar{v}}{4u^2v^2 + (\bar{v}^2 + u^2 - v^2)^2}\,d\bar{v}, & \text{if } \bar{u} = 0. \end{cases} \tag{2.1}$$

Furthermore, trivially, we have $Q_x = \delta_x$ if $x \in E$. Clearly,

$$x \mapsto Q_x \qquad \text{is continuous.} \tag{2.2}$$

LEMMA 2.1. *For all* $u, v > 0$ *and* $c \ge 0$, *we have*

$$Q_{(u,v)}\bigl(\{0\} \times [c, \infty)\bigr) = \frac{1}{2} + \frac{1}{\pi}\arctan\biggl(\frac{v^2 - u^2 - c^2}{2uv}\biggr).$$

PROOF. This follows from explicitly computing the integral $\int_c^\infty Q_{(u,v)}(d(0, \bar{v}))$ in (2.1). $\square$

Recall F from (1.9). Explicitly computing the Laplacian with respect to the first coordinate gives

$$\biggl(\frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2}\biggr)F(x, y) = 8y_1y_2F(x, y).$$

Hence, for $y \in E$, the function $F(\cdot, y)$ is harmonic for planar Brownian motion B and hence $(F(B_t, y))_{t\geq 0}$ is a bounded martingale. If τ denotes the first exit time of B from $(0,\infty)^2$, then we infer for $x \in [0,\infty)^2$ and $y \in E$ that

$$\int F(z,y) Q_x(dz) = \mathbf{E}_x[F(B_\tau, y)] = \mathbf{E}_x[F(B_0, y)] = F(x,y) \tag{2.3}$$

and, similarly (see [9], Corollary 2.3),

$$\int F(z,y) Q_x(dz) = \int F(x,z) Q_y(dz) \qquad \text{for } x, y \in [0,\infty)^2. \tag{2.4}$$

Similarly, since linear functions are harmonic for Brownian motion and using the fact that pth moments of $(B_t)_{t\leq\tau}$ are bounded for $p < 2$ (see Lemma 2.4), we can derive

$$\int z_i Q_x(dz) = x_i \qquad \text{for all } x \in [0,\infty)^2, i = 1, 2. \tag{2.5}$$

Note that (2.5) could also be computed explicitly using Lemma 2.3.

Recall that $\mathbf{Q}(x, \cdot) = \bigotimes_{k\in S} Q_{x(k)}$ for $x \in ([0,\infty)^2)^S$. From (2.3), we immediately get the following lemma.

LEMMA 2.2. *For all $x \in \mathbb{L}^{\beta,E}$ and $y \in \mathbb{L}^{f,E}$, we have*

$$\int H(z,y)\mathbf{Q}(x, dz) = H(x,y). \tag{2.6}$$

PROOF. Proof of Proposition 1.1 Note that, due to the definition of X^ε and the chain rule of calculus, we have

$$M_t^{\varepsilon,x,y} - M_s^{\varepsilon,x,y} = 0 \qquad \text{for } s, t \in \big[n\varepsilon, (n+1)\varepsilon\big), n \in \mathbb{N}_0.$$

Hence, the statement of Proposition 1.1 is an immediate consequence of Lemma 2.2. $\square$

2.2. *Moments of the harmonic measure.* Since the harmonic measure Q does not possess a second moment, our proofs will rely on pth moment estimates for $p \in (1,2)$. Here, we collect some of these estimates. Define $\arctan^\dagger$ as the inverse of the tangent function $\tan : [0,\pi] \to \bar{\mathbb{R}}$. That is,

$$\arctan^\dagger(x) = \arctan(x) + \pi \mathbb{1}_{\{x<0\}}.$$

Note that $\mathbb{R} \setminus \{0\} \to [0,\pi]$, $x \mapsto \arctan^\dagger(1/x)$ can be extended continuously to $x = 0$ with the convention that $\arctan^\dagger(1/0) = \arctan^\dagger(-1/0) = \pi/2$.

LEMMA 2.3. *For all $u, v > 0$, we have $\int_E x_1^2 Q_{(u,v)}(dx) = \infty$ and for $p \in (0,2)$,*

$$\int_E x_1^p Q_{(u,v)}(dx) = \frac{(u^2+v^2)^{p/2} \sin((p/2)\arctan^\dagger((2uv/v^2-u^2)))}{\sin((\pi/2)p)}.$$

PROOF. This follows from explicitly computing the integral using (2.1). □

LEMMA 2.4. *Let $B=(B_1,B_2)$ be a planar Brownian motion started in $B_0=(u,v)\in[0,\infty)^2$ and let*

$$\tau=\inf\{t>0\colon B_t\notin(0,\infty)^2\}.$$

Then, for any $p\in(0,2)$, we have

$$\mathbf{E}[\tau^{p/2}]\le\frac{8}{(2-p)(2\pi)^{p/2}}(uv)^{p/2}<\infty. \tag{2.7}$$

Furthermore, for any $p\in(1,2)$, we have

$$\mathbf{E}[\tau^{p/2}]\le\frac{4}{(2\pi)^{p/2}}\frac{p}{(p-1)(2-p)}\min(u^{p-1}v,uv^{p-1}). \tag{2.8}$$

PROOF. By the reflection principle and independence of B_1 and B_2, we get

$$\mathbf{P}[\tau>t]=4\mathcal{N}_{0,t}(0,u)\mathcal{N}_{0,t}(0,v),$$

where $\mathcal{N}_{0,t}(a,b)=(2\pi t)^{-1/2}\int_a^b e^{-r^2/2t}\,dr$ is the centred normal distribution with variance t. Hence, for any $p\in(0,2)$,

$$\begin{aligned}\mathbf{E}[\tau^{p/2}]&=\int_0^\infty\mathbf{P}[\tau>t^{2/p}]\,dt\\&\le4\int_0^\infty\bigl(1\wedge u(2\pi)^{-1/2}t^{-1/p}\bigr)\bigl(1\wedge v(2\pi)^{-1/2}t^{-1/p}\bigr)\,dt.\end{aligned}\tag{2.9}$$

We can continue this inequality as

$$\le\frac{4}{(2\pi)^{p/2}}(uv)^{p/2}+\frac{2uv}{\pi}\int_{(uv/2\pi)^{p/2}}^\infty t^{-2/p}\,dt=\frac{8}{(2-p)(2\pi)^{p/2}}(uv)^{p/2}.$$

This gives (2.7). For $p\in(1,2)$, we can continue (2.9) as

$$\begin{aligned}&\le4u(2\pi)^{-1/2}\int_0^{v^p/(2\pi)^{p/2}}t^{-1/p}\,dt+\frac{2uv}{\pi}\int_{v^p/(2\pi)^{p/2}}^\infty t^{-2/p}\,dt\\&=\frac{4}{(2\pi)^{p/2}}\frac{p}{(p-1)(2-p)}uv^{p-1}.\end{aligned}\tag{2.10}$$

Interchanging the roles of u and v in (2.10) gives (2.8). □

LEMMA 2.5. *For $p\in(1,2)$, there exists a constant $C_p<\infty$ such that for every $x\in[0,\infty)^2$ and $i=1,2$, we have*

$$\int_E y_i^p\,Q_x(dy)\ge x_i^p \tag{2.11}$$

and

$$\int_E|y_i-x_i|^p\,Q_x(dy)\le C_p\min(x_1^{p-1}x_2,x_1x_2^{p-1})\le C_p(x_1x_2)^{p/2}. \tag{2.12}$$

PROOF. By the Burkholder–Davis–Gundy inequality (see, e.g., [5], Theorem VII.92) and Lemma 2.4, $(B_{i,t})_{t\le\tau}$ is a uniformly integrable martingale. Hence, by Jensen's inequality,

$$x_i^p = \mathbf{E}_x[B_{i,t}]^p \le \mathbf{E}_x[B_{i,t}^p] = \int_E y_i^p Q_x(dy).$$

The claim (2.12) could be checked either by a direct computation using Lemma 2.3 or by proceeding as follows. Let B and τ be as in Lemma 2.4. Using the Burkholder–Davis–Gundy inequality and then Lemma 2.4, we get

$$\begin{aligned}\int_E |y_i - x_i|^p Q_x(dy) &= \mathbf{E}_x[|B_{i,\tau} - x_i|^p] \le (4p)^p \mathbf{E}_x[\tau^{p/2}] \\ &\le \frac{(4p)^{p+1}}{(p-1)(2-p)(2\pi)^{p/2}} \min(x_1^{p-1}x_2, x_1x_2^{p-1}). \qquad \square\end{aligned}$$

3. The approximating process X^ε.

3.1. *Martingale property of X^ε.*

PROPOSITION 3.1. *Let $x \in \mathbb{L}^{\beta,E}$ and $k \in S$. Define the process $N^{\varepsilon,x}$ for $i = 1, 2$, $k \in S$ and $t \ge 0$ by*

$$N_{i,t}^{\varepsilon,x}(k) := X_{i,t}^\varepsilon(k) - X_{i,0}^\varepsilon(k) - \int_0^t (\mathcal{A}X_{i,s}^\varepsilon)(k)\, ds.$$

(i) *For each $i = 1, 2$ and $k \in S$, the process $(N_{i,t}^{\varepsilon,x}(k))_{t\ge 0}$ is a martingale with respect to the natural filtration. In particular,*

$$\mathbf{E}_x[X_{i,t}^\varepsilon(k)] = (\mathcal{S}_t x_i)(k) \qquad \text{for all } t \ge 0, k \in S, i = 1, 2. \tag{3.1}$$

(ii) *Define $\lambda := \sup_{k\in S}(-\mathcal{A}(k,k))$ and note that $|\lambda| < \infty$ by assumption (1.3). Define*

$$Z_{i,t}^\varepsilon(k) := e^{-\mathcal{A}(k,k)t} X_{i,t}^\varepsilon(k)$$

and

$$\bar{Z}_{i,t}^\varepsilon := e^{\lambda t} \|X_{i,t}^\varepsilon\|_\beta.$$

$Z_i^\varepsilon(k)$ and $\bar{Z}_i^\varepsilon$ are then nonnegative submartingales.

PROOF. (i) This is an immediate consequence of the definition of X^ε and (2.5).
(ii) Since $\mathcal{A}(k,l) \ge 0$ for all $k \ne l$, we have

$$\frac{d}{dt} Z_{i,t}^\varepsilon(k) = \sum_{l\ne k} \mathcal{A}(k,l) Z_{i,t}^\varepsilon(l) \ge 0 \qquad \text{for } t \in \big(n\varepsilon, (n+1)\varepsilon\big).$$

Together with (2.5), this shows that Z_i^ε is a submartingale. As a sum of submartingales, $\bar{Z}_i^\varepsilon$ is also a submartingale. $\square$

COROLLARY 3.2. *For every $K, T > 0$ and any set $G \subset S$, we have*

$$\mathbf{P}_x\Big[\sup_{t\in[0,T]} \|(X^\varepsilon_{1,t} + X^\varepsilon_{2,t})\mathbb{1}_G\|_\beta \ge K\Big] \le K^{-1}e^{\lambda T}\big\|\big(\mathcal{S}_T(x_1+x_2)\big)\mathbb{1}_G\big\|_\beta. \tag{3.2}$$

In particular,

$$\mathbf{P}_x\Big[\sup_{t\in[0,T]} \|(X^\varepsilon_{1,t} + X^\varepsilon_{2,t})\|_\beta \ge K\Big] \le K^{-1}e^{(\lambda+M)T}\|x_1+x_2\|_\beta. \tag{3.3}$$

PROOF. This is an immediate consequence of Proposition 3.1 and Doob's inequality. □

3.2. *One-dimensional distributions*.

LEMMA 3.3. *Let $a(1), a(2), \dots$ be nonnegative numbers and let $x(1), x(2), \dots \in [0,\infty)^2$ be such that*

$$\bar{x} := \langle a, x\rangle = \sum_{k=1}^\infty a(k)x(k) \in [0,\infty)^2.$$

Let $\xi(1), \xi(2), \dots$ be independent random variables with $\mathbf{P}[\xi(k)\in\cdot] = Q_{x(k)}$. Define $\bar{\xi} := \langle a, \xi\rangle = \sum_{k=1}^\infty a(k)\xi(k)$ and assume that X is an E-valued random variable such that $\mathbf{P}[X\in\cdot\,|\bar{\xi}] = Q_{\bar{\xi}}$. Then $\mathbf{P}[X\in\cdot] = Q_{\bar{x}}$. In other words, $\mathbf{E}[Q_{\bar{\xi}}] = Q_{\bar{x}}$.

PROOF. First, note that $\mathbf{E}[\xi_i(k)] = x_i(k)$ and, hence, $\bar{\xi} \in [0,\infty)^2$ almost surely. Recall F from (1.9). By (2.3), for all $y \in E$, we have

$$\begin{aligned}\mathbf{E}[F(X,y)] = \mathbf{E}[F(\bar{\xi},y)] &= \prod_{k=1}^\infty \mathbf{E}[F(\xi(k), a(k)y)] = \prod_{k=1}^\infty F(x(k), a(k)y)\\ &= F(\bar{x}, y) = \int_E F(z,y)\,Q_{\bar{x}}(dz).\end{aligned}$$

Since $F(\cdot, y)$, $y\in E$, is measure-determining (see [9], Corollary 2.4), this yields the claim. □

COROLLARY 3.4. *For any $\varepsilon > 0$, $n \in \mathbb{N}_0$ and $k \in S$, we have*

$$\mathbf{P}[X^\varepsilon_{n\varepsilon}(k)\in\cdot] = Q_{\mathcal{S}_{n\varepsilon}x(k)}.$$

PROOF. Fix $n\in\mathbb{N}$. We show by induction on m that

$$\mathbf{P}[X^\varepsilon_{n\varepsilon}(k)\in\cdot\,|X^\varepsilon_{m\varepsilon}] = Q_{(\mathcal{S}_{(n-m)\varepsilon}X^\varepsilon_{m\varepsilon})(k)} \qquad \text{for all } m = 0,\dots,n.$$

For the induction base $m = n$, this is true by the definition of X^ε. Now, assume that we have shown the statement for some $m \geq 1$. Using the induction hypothesis in the first line and Lemma 3.3 in the second line, we get

$$\begin{aligned}\mathbf{P}\big[X^\varepsilon_{n\varepsilon}(k) \in \cdot | X^\varepsilon_{(m-1)\varepsilon}\big] &= \mathbf{E}\big[Q_{(\mathcal{S}_{(n-m)\varepsilon} X^\varepsilon_{m\varepsilon})(k)} | X^\varepsilon_{(m-1)\varepsilon}\big] \\ &= \mathbf{E}\big[Q_{(\mathcal{S}_{(n-m)\varepsilon} X^\varepsilon_{m\varepsilon-})(k)} | X^\varepsilon_{(m-1)\varepsilon}\big] \\ &= Q_{(\mathcal{S}_{(n-(m-1))\varepsilon} X^\varepsilon_{(m-1)\varepsilon})(k)}.\end{aligned}$$

Note that we have used the fact that $X^\varepsilon_{m\varepsilon-} = \mathcal{S}_\varepsilon X^\varepsilon_{(m-1)\varepsilon}$ in the last line. □

COROLLARY 3.5. *Let $y \in \mathbb{L}^\beta_\infty$. Then $\mathbf{E}_x[Q_{\langle X^\varepsilon_t, y\rangle}] = Q_{\langle \mathcal{S}_t x(k), y\rangle}$.*

PROOF. The proof is similar to the proof of Corollary 3.4. (Note that $\langle X^\varepsilon_t, y\rangle \in [0, \infty)^2$ almost surely since $X^\varepsilon_t \in \mathbb{L}^{\beta,2}$ almost surely.) □

3.3. *Correlations.*

LEMMA 3.6. *Let Y and Z be nonpositively correlated nonnegative random variables and assume that $h : [0, \infty) \to [0, \infty)$ is concave and monotone increasing. Then $\mathbf{E}[Yh(Z)] \leq \mathbf{E}[Y]h(\mathbf{E}[Z])$.*

PROOF. If $\mathbf{E}[Z] = 0$, then we even have equality. Now, assume that $\mathbf{E}[Z] > 0$. By concavity of h, there exists a $b \in \mathbb{R}$ such that for all $z \geq 0$,

$$h(z) \leq h(\mathbf{E}[Z]) + (z - \mathbf{E}[Z])b.$$

Since h is nondecreasing, we have $b \geq 0$ and thus

$$\mathbf{E}[Yh(Z)] \leq \mathbf{E}\big[Y\big(h(\mathbf{E}[Z]) + (Z - \mathbf{E}[Z])b\big)\big] \leq \mathbf{E}[Y]h(\mathbf{E}[Z]). \qquad \square$$

LEMMA 3.7. *For any $\varepsilon > 0$, $n \in \mathbb{N}_0$ and $k \in S$, the random variables $X^\varepsilon_{1,n\varepsilon}(k)$ and $X^\varepsilon_{2,n\varepsilon}(k)$ are nonpositively correlated, in the sense that*

$$\begin{aligned}\mathbf{E}_x[X^\varepsilon_{1,n\varepsilon}(k)X^\varepsilon_{2,n\varepsilon}(k)] &\leq \mathbf{E}_x[X^\varepsilon_{1,n\varepsilon}(k)]\mathbf{E}_x[X^\varepsilon_{2,n\varepsilon}(k)] \\ &= (\mathcal{S}_{n\varepsilon}x_1(k))(\mathcal{S}_{n\varepsilon}x_2(k)).\end{aligned} \tag{3.4}$$

PROOF. Let $t \geq 0$. Recall that $\mathcal{F}$ is the natural filtration of X^ε. Then

$$\begin{aligned}&\mathbf{E}_x\big[\mathcal{S}_t X^\varepsilon_{1,n\varepsilon}(k)\mathcal{S}_t X^\varepsilon_{2,n\varepsilon}(k)|\mathcal{F}_{(n-1)\varepsilon}\big] \\ &\quad = \sum_{l_1 \neq l_2} a_t(k, l_1)a_t(k, l_2)\mathbf{E}_x\big[X^\varepsilon_{1,n\varepsilon}(l_1)X^\varepsilon_{2,n\varepsilon}(l_2)|\mathcal{F}_{(n-1)\varepsilon}\big] \\ &\quad = \sum_{l_1 \neq l_2} a_t(k, l_1)a_t(k, l_2)\big(\mathcal{S}_\varepsilon X^\varepsilon_{1,(n-1)\varepsilon}\big)(l_1)\big(\mathcal{S}_\varepsilon X^\varepsilon_{2,(n-1)\varepsilon}\big)(l_2)\end{aligned}$$

$$\begin{aligned}&\leq \sum_{l_1,l_2} a_t(k,l_1)\big(\mathcal{S}_\varepsilon X^\varepsilon_{1,(n-1)\varepsilon}\big)(l_1)a_t(k,l_2)\big(\mathcal{S}_\varepsilon X^\varepsilon_{2,(n-1)\varepsilon}\big)(l_2)\\&= \mathcal{S}_{t+\varepsilon}X^\varepsilon_{1,(n-1)\varepsilon}(k)\mathcal{S}_{t+\varepsilon}X^\varepsilon_{2,(n-1)\varepsilon}(k).\end{aligned}$$

Inductively, we get

$$\mathbf{E}_x[\mathcal{S}_t X^\varepsilon_{1,n\varepsilon}(k)\mathcal{S}_t X^\varepsilon_{2,n\varepsilon}(k)] \leq \mathcal{S}_{t+n\varepsilon}x_1(k)\mathcal{S}_{t+n\varepsilon}x_2(k).$$

Applying this with $t = 0$ yields the claim. □

4. Tightness.

The goal of this section is to show the following proposition.

PROPOSITION 4.1. *The family of processes* $(X^\varepsilon)_{\varepsilon>0}$ *is relatively compact in the Skorohod spaces of càdlàg functions* $D([0,\infty);\mathbb{L}^{\beta,2})$.

By Prohorov's theorem, in order to show relative compactness of (X^ε), it is enough to show tightness of (X^ε).

The strategy of proof is to check the compact containment condition for X^ε (Lemma 4.4) and then use Aldous's tightness criterion for functions $h(X^\varepsilon_t)$, where $h:\mathbb{L}^{\beta,2}\to\mathbb{R}$ is Lipschitz continuous and depends on only finitely many coordinates.

We start by collecting some basic facts about compact sets and separating function spaces. The proofs of the following statements are standard and are therefore omitted here.

LEMMA 4.2. *A set* $C \subset \mathbb{L}^{\beta,2}$ *is relatively compact if and only if the following hold*:

(i) $B_C := \sup_{x\in C}\|x_1+x_2\|_\beta < \infty$;

(ii) *for any* $\eta > 0$, *there exists a finite subset* $S_\eta \subset S$ *such that* $\sup_{x\in C}\|(x_1 + x_2)\mathbb{1}_{S\setminus S_\eta}\|_\beta < \eta$.

LEMMA 4.3. *Let* $C_b(\mathbb{L}^{\beta,2};\mathbb{R})$ *be the space of real-valued bounded continuous functions* $\mathbb{L}^{\beta,2}\to\mathbb{R}$ *with the topology of uniform convergence on compact sets*. *Denote by* $\mathrm{Lip}_f(\mathbb{L}^{\beta,2};\mathbb{R})$ *the space of Lipschitz continuous bounded functions* $\mathbb{L}^{\beta,2}\to\mathbb{R}$ *that depend on only finitely many coordinates*. *Then* $\mathrm{Lip}_f(\mathbb{L}^{\beta,2};\mathbb{R}) \subset C_b(\mathbb{L}^{\beta,2};\mathbb{R})$ *is dense*.

LEMMA 4.4 (Compact containment condition). *Fix* $x \in \mathbb{L}^{\beta,2}$. *For any* $\eta > 0$ *and* $T > 0$, *there exists a compact set* $\Gamma \subset \mathbb{L}^{\beta,2}$ *such that*

$$\mathbf{P}_x\big[X^\varepsilon_t \in \Gamma \textit{ for all } t \in [0,T]\big] \geq 1-\eta \qquad \textit{for all } \varepsilon > 0. \tag{4.1}$$

PROOF. Let $T > 0$ and $\eta > 0$. Recall M from (1.4) and λ from Proposition 3.1(ii). Choose a $K > \frac{2}{\eta} e^{(\lambda+M)T} \|x_1 + x_2\|_\beta$ and let $A_K := \{y \in \mathbb{L}^{\beta,2} : \|y_1 + y_2\|_\beta < K\}$. According to Corollary 3.2, we have

$$\mathbf{P}_x\big[X_t^\varepsilon \in A_K \text{ for all } t \in [0, T]\big] \geq 1 - \frac{\eta}{2}.$$

Now, for any $n \in \mathbb{N}$, choose a finite $S_n \subset S$ such that

$$n e^{\lambda T} \|\mathcal{S}_T(x_1 + x_2)\mathbb{1}_{S\setminus S_n}\|_\beta < 2^{-n-1}\eta$$

and define

$$B_n := \{y \in \mathbb{L}^{\beta,2} : \|(y_1 + y_2)\mathbb{1}_{S\setminus S_n}\|_\beta < 1/n\}.$$

According to Corollary 3.2, we have

$$\mathbf{P}_x\big[X_t^\varepsilon \in B_n \text{ for all } t \in [0, T]\big] \geq 1 - 2^{-n-1}\eta.$$

Now, let Γ by the closure of $A_K \cap \bigcap_{n=1}^\infty B_n$. Then

$$\mathbf{P}_x\big[X_t^\varepsilon \in \Gamma \text{ for all } t \in [0, T]\big] \geq 1 - \eta$$

and, by Lemma 4.2, Γ is compact. $\square$

LEMMA 4.5. *Fix $h \in \mathrm{Lip}_f(\mathbb{L}^{\beta,2}; \mathbb{R})$. For $\varepsilon > 0$, define the process Y^ε by*

$$Y_t^\varepsilon := h(X_t^\varepsilon), \qquad t \geq 0.$$

$(Y^\varepsilon)_{\varepsilon>0}$ is then tight in the Skorohod space $D([0,\infty); \mathbb{R})$ of càdlàg functions $[0,\infty) \to \mathbb{R}$.

PROOF. The idea is to use Aldous's criterion for tightness in $D([0,\infty); \mathbb{R})$. As h is bounded, $(Y_t^\varepsilon)_{\varepsilon>0}$ is tight for each $t \geq 0$. Hence, by Aldous's criterion (see, e.g., [1], equation (13), or [8], Section VI.4a), we need to show the following: for any $\eta > 0$ and $T > 0$, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for any stopping time $\tau \leq T$, we have

$$\sup_{\delta' \in [0,\delta]} \sup_{\varepsilon \in (0,\varepsilon_0]} \mathbf{P}_x[|Y_{\tau+\delta'}^\varepsilon - Y_\tau^\varepsilon| > \eta] \leq \eta. \tag{4.2}$$

Since h is Lipschitz continuous and depends on only finitely many coordinates, it is enough to consider the case where $h(x) = x_i(k)$ for some $k \in S$ and $i = 1, 2$. Using Markov's inequality, it is enough to show that for any $\eta > 0$ and $T > 0$, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that for any stopping time $\tau \leq T$, we have

$$\sup_{\delta' \in [0,\delta]} \sup_{\varepsilon \in (0,\varepsilon_0]} \mathbf{E}_x[|X_{i,\tau+\delta'}^\varepsilon(k) - X_{i,\tau}^\varepsilon(k)|] \leq \eta. \tag{4.3}$$

Define

$$N := \lfloor \tau/\varepsilon \rfloor \quad \text{and} \quad N' := \lfloor (\tau + \delta')/\varepsilon \rfloor.$$

Then

$$\mathbf{E}_x[|X^{\varepsilon}_{i,\tau+\delta'}(k) - X^{\varepsilon}_{i,\tau}(k)|] \le E_1 + E_2 + E_3 + E_4,$$

where

$$\begin{aligned}
E_1 &:= \mathbf{E}_x[|X^{\varepsilon}_{i,\tau}(k) - X^{\varepsilon}_{i,N\varepsilon}(k)|],\\
E_2 &:= \mathbf{E}_x[|X^{\varepsilon}_{i,\tau+\delta'}(k) - X^{\varepsilon}_{i,N'\varepsilon}(k)|],\\
E_3 &:= \mathbf{E}_x[|X^{\varepsilon}_{i,N'\varepsilon}(k) - X^{\varepsilon}_{i,N\varepsilon}(k)|].
\end{aligned}$$

Now, by (1.7), we get

$$\begin{aligned}
E_1 &= \mathbf{E}_x[|\mathcal{S}_{\tau-N\varepsilon}X^{\varepsilon}_{i,N\varepsilon}(k) - X^{\varepsilon}_{i,N\varepsilon}(k)|]\\
&\le \mathbf{E}_x\biggl[\int_0^{\tau-N\varepsilon} |\mathcal{A}\mathcal{S}_s X_{i,N\varepsilon}(k)|\,ds\biggr]\\
&\le \frac{Me^{\delta M}\delta}{\beta(k)}\mathbf{E}_x[\|X^{\varepsilon}_{i,N\varepsilon}\|_\beta] \le \frac{Me^{(T+2\delta)M}}{\beta(k)}\|x_i\|_\beta\delta.
\end{aligned}$$

Similarly, we get

$$E_2 \le \frac{Me^{(T+2\delta)M}}{\beta(k)}\|x_i\|_\beta\delta.$$

Note that $N' - N$ takes only the values $\lfloor\delta'/\varepsilon\rfloor$ and $\lceil\delta'/\varepsilon\rceil$. Hence, $E_3 \le E_3' + E_3''$, where

$$E_3' := \mathbf{E}_x\bigl[\bigl|X^{\varepsilon}_{i,(N+\lfloor\delta'/\varepsilon\rfloor)\varepsilon}(k) - X^{\varepsilon}_{i,N\varepsilon}(k)\bigr|\bigr]$$

and

$$E_3'' := \mathbf{E}_x\bigl[\bigl|X^{\varepsilon}_{i,(N+\lceil\delta'/\varepsilon\rceil)\varepsilon}(k) - X^{\varepsilon}_{i,N\varepsilon}(k)\bigr|\bigr].$$

Define

$$\bar{E}_3'' := \mathbf{E}_x\bigl[\bigl|X^{\varepsilon}_{i,(N+\lceil\delta'/\varepsilon\rceil)\varepsilon}(k) - \mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^{\varepsilon}_{i,N\varepsilon}(k)\bigr|\bigr].$$

Using the triangle inequality and proceeding as for E_1, we get

$$E_3'' \le \bar{E}_3'' + \mathbf{E}_x\bigl[\bigl|X^{\varepsilon}_{i,N\varepsilon}(k) - \mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^{\varepsilon}_{i,N\varepsilon}(k)\bigr|\bigr] \le \bar{E}_3 + \frac{Me^{(T+2\delta)M}}{\beta(k)}\|x_i\|_\beta\delta.$$

Fix a $p \in (1,2)$. Using the Markov property of X^ε and conditioning on $X^{\varepsilon}_{N\varepsilon}$, by Corollary 3.4 and Jensen's inequality, we get

$$\begin{aligned}
\bar{E}_3'' &= \mathbf{E}_x\biggl[\int_E |y_i - X^{\varepsilon}_{i,N\varepsilon}(k)|\,Q_{\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^{\varepsilon}_{N\varepsilon}(k)}(dy)\biggr]\\
&\le \biggl(\mathbf{E}_x\biggl[\int_E |y_i - X^{\varepsilon}_{i,N\varepsilon}(k)|^p\,Q_{\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^{\varepsilon}_{N\varepsilon}(k)}(dy)\biggr]\biggr)^{1/p}.
\end{aligned}$$

Applying Lemma 2.5, there exists a constant $C = C_p < \infty$ such that

$$(\bar{E}_3'')^p \le C\mathbf{E}_x\big[\big(\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{1,N\varepsilon}(k)\big)^{p-1}\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{2,N\varepsilon}(k)\mathbb{1}_{\{X^\varepsilon_{2,N\varepsilon}(k)=0\}}\big]$$
$$+ C\mathbf{E}_x\big[\big(\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{2,N\varepsilon}(k)\big)^{p-1}\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{1,N\varepsilon}(k)\mathbb{1}_{\{X^\varepsilon_{1,N\varepsilon}(k)=0\}}\big].$$

By symmetry, it is enough to consider the first summand. Since the first and the second type are nonpositively correlated (Lemma 3.7), by Lemma 3.6 [with $h(z) = z^{p-1}$], the first summand can be estimated by

$$\begin{aligned}
&\mathbf{E}_x\big[\big(\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{1,N\varepsilon}(k)\big)^{p-1}M\delta e^{M\delta}\beta(k)^{-1}\|X^\varepsilon_{N,\varepsilon,2}\|_\beta\big]\\
&\quad\le \mathbf{E}_x\big[\big(\mathcal{S}_{\lceil\delta'/\varepsilon\rceil\varepsilon}X^\varepsilon_{1,N\varepsilon}(k)\big)^{p-1}\big]Me^{M\delta}\delta\beta(k)^{-1}\mathbf{E}_x[\|X^\varepsilon_{N,\varepsilon,2}\|_\beta]\\
&\quad\le \big(e^{M(T+\delta+\varepsilon_0)}\|x_1\|_\beta\big)^{p-1}M\delta\beta(k)^{-1}e^{M(T+2\delta)}\|x_2\|_\beta.
\end{aligned}$$

The estimate for E_3' is analogous. Summing up, by choosing δ sufficiently small (independently of $\varepsilon \le \varepsilon_0$), we can get $E_j < \eta/3$, $j = 1, 2, 3$ and hence (4.3). $\square$

PROOF OF PROPOSITION 4.1. The space $\mathbb{L}^{\beta,2}$ is Polish and hence so is the Skorohod space $D([0,\infty);\mathbb{L}^{\beta,2})$ of càdlàg paths $[0,\infty) \to \mathbb{L}^{\beta,2}$ (see [7], Chapter III.5). Hence, by Prohorov's theorem, it is enough to show tightness of $(X^\varepsilon)_{\varepsilon>0}$ in $D([0,\infty);\mathbb{L}^{\beta,2})$. By [7], Theorem III.9.1, it is enough to check two conditions:

(i) the compact containment condition—this is done in Lemma 4.4;

(ii) there is a dense (in the topology of uniform convergence on compacts) space $H \subset C_b(\mathbb{L}^{\beta,2};\mathbb{R})$ such that for every $h \in H$, the family $h(X^\varepsilon)$, $\varepsilon > 0$, is tight in $D([0,\infty);\mathbb{R})$—we have checked this for $H = \mathrm{Lip}_f(\mathbb{L}^{\beta,2};\mathbb{R})$ in Lemmas 4.3 and 4.5. $\square$

5. The martingale problem.

In this section, we complete the proofs of Theorems 1 and 2.

5.1. *Proof of Theorem* 1. From Proposition 4.1 we know that X^ε, $\varepsilon > 0$, is weakly relatively compact. From Theorem 0, we know that the martingale problem (MP) has a unique solution. Hence, it remains to show that any weak limit point of X^ε, $\varepsilon > 0$, is a solution of (MP).

Let $x \in \mathbb{L}^{\beta,E}$. Fix a sequence $\varepsilon_n \downarrow 0$ such that X^{ε_n} converges and denote the limit by X. Without loss of generality, we may assume that the processes are defined on one probability space such that $X^{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} X$ almost surely. Let $y \in \mathbb{L}^{f,E}$ and define $M^{x,y}$ as in (MP) and $M^{\varepsilon,x,y}$ as in (1.13). We know from Proposition 1.1 that $M^{\varepsilon_n,x,y}$ is a martingale. Hence, it is enough to show that

$$M_t^{\varepsilon_n,x,y} \overset{n\to\infty}{\longrightarrow} M_t^{x,y} \qquad \text{in } L^1 \text{ for all } t \ge 0. \tag{5.1}$$

Note that the integrand in (1.13) converges pointwise to the integrand in (MP). Since H is bounded, in order to show (5.1), it is enough to show that $\langle\langle \mathcal{A}X_s^{\varepsilon_n}, y\rangle\rangle$

is uniformly integrable (with respect to Lebesgue measure on $[0,t]$ and $\mathbf{P}_x$). Let $p\in(1,2)$. Since $y(k)\neq 0$ for only finitely many $k\in S$, it is enough to show that for $i=1,2$ and $t>0$, we have

$$\sup_{\varepsilon>0}\sup_{s\in[0,t]}\mathbf{E}[|\mathcal{A}X^{\varepsilon}_{i,s}(k)|^p]<\infty. \tag{5.2}$$

Recall that $|\mathcal{A}X^{\varepsilon}_{i,s}(k)|\le M\|X^{\varepsilon}_{i,s}\|_\beta/\beta(k)$. Let Z be an E-valued random variable such that $\mathbf{P}[Z\in\cdot|X^\varepsilon]=Q_{\|X^\varepsilon_s\|_\beta}$. Then $\mathbf{E}[Z_i^p]\ge\mathbf{E}_x[\|X^{\varepsilon}_{i,s}\|_\beta^p]$, by Lemma 2.5. However, by Corollary 3.5, we have $\mathbf{P}_x[Z\in\cdot]=Q_{\|\mathcal{S}_s x\|_\beta}$. Hence, again by Lemma 2.5 and using (1.7), we get

$$\begin{aligned}\mathbf{E}[\|X^{\varepsilon}_{i,s}\|_\beta^p]&\le\mathbf{E}[Z_i^p]\le 2^{p-1}(\mathbf{E}[|Z_i-\|\mathcal{S}_s x_i\|_\beta|^p]+\|\mathcal{S}_s x_i\|_\beta^p)\\&\le C_p\big((\|\mathcal{S}_s x_1\|_\beta\|\mathcal{S}_s x_2\|_\beta)^{p/2}+\|\mathcal{S}_s x_i\|_\beta^p\big)\\&\le C_p e^{pMs}\big((\|x_1\|_\beta\|x_2\|_\beta)^{p/2}+\|x_i\|_\beta^p\big).\end{aligned}$$

This shows (5.2) and completes the proof of Theorem 1.

5.2. *Proof of Theorem* 2. Theorem 2 is a direct consequence of Theorem 1, Corollary 3.5 and (2.2).

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