A Review on Spatial Catalytic Branching

Achim Klenke

Abstract. We review some results on spatial branching processes X^ϱ in random media ϱ. The local branching rate or law depends on the medium ϱ that may vary in space and time and can be random.

The main emphasis lies on catalytic super-Brownian motion where ϱ governs the local branching rate and is considered as a catalytic medium. We display the construction of X^ϱ and give results on absolute continuity of the states, longtime behaviour and so on.

Introduction

In the last 15 years there has been a lot of interest in spatial branching models – branching random walk, branching Brownian motion, super-Brownian motion and so on – where the branching mechanism may vary in space and/or time in a deterministic or random way. The protagonists in this field of research are Don Dawson and Klaus Fleischmann.

This survey focuses mainly on the best-studied subclass of models where the branching mechanism is critical with finite variance (super processes) or binary (particle systems). It is only the local *rate* at which the branching occurs that varies. The local branching rate is interpreted as the concentration of catalytic matter that enables the branching. This catalyst is a function (or distribution) in space and/or time and it may be random or deterministic. We even consider a case where two branching processes catalyse each other in a symmetric way.

The ambition of this article is to serve as a quick guide to the subject and to give a survey of the main results. It is by no means comprehensive and the author wishes to apologise to everyone whose work is not considered here. We do not aim at rigour in the exposition but appeal to the intuition of the reader. Some notions are explained loosely to give non-specialists a vague idea and enable them to go on reading.

1. Varying Branching Law

Galton–Watson processes in random environments have been studied for over 30 years. Since we shall focus on spatial models we refer only briefly to the books

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of Athreya and Ney (1972) and Jagers (1975), and for some more recent sources to d'Souza (1994), Fleischmann and Vatutin (1999).

We tried to keep the notation consistent throughout this article. The reader should be warned that this entails incongruencies with the notation in the original articles.

1.1. The Starting Point. Spatial catalytic branching processes in a random medium were considered first by Dawson and Fleischmann (1983) and (1985). They study discrete time branching random walk on $\mathbb{Z}^d$ where the branching law is critical but depends on the location of the particle. More precisely, let $a(i,j)$ be the transition kernel of a random walk on $\mathbb{Z}^d$, denote by $a^{(n)}$ the n-step transition probabilities, let $F = (f_i,\ i \in \mathbb{Z}^d)$ be a family of probability generating function with $f_i'(1-) = 1$, $i \in \mathbb{Z}^d$. Denote by $(p_i(k))_{k\in\mathbb{N}_0}$ the corresponding probability distribution on $\mathbb{N}_0$. The branching random walk (BRW) $(X_n)_{n\in\mathbb{N}_0}$ in the environment F is the particle system where in each time step at each site i the particles branch according to the (critical) distribution p_i. The resulting progeny moves according to the kernel a. The number of particles at site i at time n is denoted by $X_n(i)$.

We assume that the environment F is sampled from some probability law $\mathbb{P} = \mathcal{L}(F)$ but is fixed for all times. For fixed F denote by $\mathbf{P}^F$ the law of the branching random walk X (quenched law). $\mathbf{P}^F$ allows a random initial state X_0 whose distribution is assumed to be independent of F. Finally denote by $\mathcal{P} = \mathbb{E}\,\mathbf{P}^F = \int \mathbb{P}(dF)\mathbf{P}^F$ the annealed distribution.

The aspiration of Dawson and Fleischmann (1985) is to obtain criteria for the persistence of X. Recall that (for fixed F) X is called persistent if, roughly speaking, it maintains its spatial intensity of particles in the longtime limit. In Theorem 3.1 they obtain a Kallenberg criterion for persistence: Let $g_i(s) = f'(1-s)$, $s \in [0,1]$, and let (Z_n) be a random walk with kernel a. Then X is persistent iff for all $i, j \in \mathbb{Z}^d$:

$$\mathbf{P}^F\left[\sum_{n=1}^{\infty} g_{-Z_n}(a^{(n)}(-Z_n, j)) < \infty \middle| Z_0 = i\right] = 1. \tag{1.1}$$

The philosophy leading to (1.1) is simple: X is persistent if each $\{X_n(i), n \in \mathbb{N}_0\}$ is uniformly integrable. This is equivalent to the stochastic boundedness of the size biased random variables $\{\widehat{X_n(i)}, n \in \mathbb{N}_0\}$. (Recall that these are defined by $\mathbf{P}^F[\widehat{X_n(i)} = k] = k\mathbf{P}^F[X_n(i) = k]/\mathbf{E}^F[X_n(i)]$, $k \in \mathbb{N}_0$.) Now there is a nice representation of $\widehat{X_n(i)}$ going back to Olav Kallenberg. Trace back the ancestral line of a particle located at i (this is the random walk $(-Z_m)_{m=0,\dots,n}$). At each time m generate a random number Y_m of particles distributed according to the size biased offspring distribution $(kp_{-Z_n}(k))_{k\in\mathbb{N}_0}$ which has probability generating function f'_{-Z_n}. Now let $Y_m - 1$ offspring particles perform BRW and evaluate at time $n-m$. Adding over m yields the distribution of $\widehat{X_n(i)}$. This explains the quantities arising in (1.1). Some uniform integrability arguments are needed to establish the exact form of the criterion. In fact, this makes the proof a little involved.

If we sample F from a spatially homogeneous ergodic law $\mathbb{P}$ it is easily derived that (see Proposition 4.1)

$$\mathbb{P}[X \text{ is persistent}] \in \{0, 1\}. \tag{1.2}$$

The main result (Theorem 5.2) of Dawson and Fleischmann (1985) is the existence of a critical dimension for persistence. Assume that $\{f_i,\ i \in \mathbb{Z}^d\}$ is i.i.d. and that there exists a $\beta \in (0,1]$ such that $\lim_{s\to 0} s^{-\beta}\mathbb{E}[g_i(s)] \in (0,\infty)$ exists. Further assume that a is in the normal domain of attraction of a genuinely d-dimensional symmetric α–stable law for some $\alpha \in (0,2]$. Then

$$X \text{ is a.s. persistent} \iff d > \frac{\alpha}{\beta}. \tag{1.3}$$

Under somewhat weaker assumptions ($\mathbb{E}[g_i(s)]$ is only regularly varying, $\mathbb{P}$ is only ergodic, a is only in the domain of attraction) one gets (1.3) but excluding the case $d = \alpha/\beta$ (Theorem 5.1). Note that the result is qualitatively identical to the classical non-random homogeneous situation. Considering the fact that the medium is ergodic and the mean is constant a.s. this is understandable: the particles experience a mixture of the medium and this mixture converges to a deterministic limit. Since the branching is critical everywhere no sites are favoured and this limit coincides with mean with respect to $\mathbb{P}$.

1.2. Large Deviations for Non-critical Branching. The situation changes drastically if we allow the mean of the offspring distribution to vary also. This case has examined studied in Greven and den Hollander (1991), (1992) and Baillon, Clément, Greven and den Hollander (1993). We sketch the model and the result of the latter paper. The model is the same as the one introduced above but lives only on $\mathbb{Z}$. For the random walk kernel a we make the special choice $a(i,i) = 1-h$, $a(i,i+1) = h$, $i \in \mathbb{Z}$, for some parameter $h \in (0,1)$. We allow the mean $m_i = \sum_{k\in\mathbb{N}_0} kp_i(k)$ to be random. We assume that $\{p_i,\ i \in \mathbb{Z}\}$ is i.i.d. and that $M := \text{ess sup}_{\mathbf{P}}\ m_i < \infty$.

It is clear that the particles do not experience a $\mathbb{P}$–average of the medium. Rather they try to stay at those fertile sites i where $M - m_i$ is small. (Note that it is the distance $M - m_i$ to the optimal value M rather than the absolute value of m_i that determines the (relative) "fertility" of a site.) On the other hand, for a particle not to move costs a (entropy) price depending on h. We are the observers of a thrilling interplay of two opposed tendencies. It is not too hard to guess that we are peeking into the world of large deviations and that the best strategy for a particle can be characterised in terms of a variational problem.

However tempting it is to give a panorama, our focus lies on critical branching. We mention only briefly that the techniques developed in the three papers mentioned above have had a considerable spin-off to the theory of one-dimensional random polymers (see, e.g., Greven and den Hollander (1993), Baillon, Clément, Greven and den Hollander (1994), König (1996), van der Hofstad and den Hollander (1997), and van der Hofstad and Klenke (1998)). As an appetiser we present part of the result of Baillon et al. (1993) in a nutshell (cf. discussion on page 313). The interesting quantities are the Malthusian (global) growth rate $\rho(h)$ and the drift of the particles $\theta(h)$.

Assume that the law $\mathcal{L}(m_i)$ is non-trivial and that it has an atom at M. Here the flesh pots are abundant and travelling is too strenuous. On the other hand if $\mathcal{L}(m_i)$ has a "thin tail at M" the flesh pots are scarce and crossing the desert might be worthwhile. More precisely, we distinguish two cases.

Case 1:: $\mathbb{E}[(M-m_i)^{-1}]=\infty$. Here we have

$$\begin{array}{rcll}\rho(h) & = & \log(M(1-h)), & h\in(0,1),\\ \theta(h) & \equiv & 0. & \end{array}\tag{1.4}$$

Case 2:: $\mathbb{E}[(M-m_i)^{-1}]<\infty$. There exists an $h_c\in(0,1)$ at which a phase transition occurs: For $h\in(0,h_c)$ the values of $\rho(h)$ and $\theta(h)$ are as in (1.4). However for $h\in(h_c,1)$ these values are strictly exceeded. The functions ρ and θ are analytic on $(h_c,1)$ and ρ is continuous at h_c while θ is continuous at h_c iff $\mathbb{E}[(M-m_i)^{-2}]=\infty$.

1.3. Hydrodynamic Fluctuations.

As a last example where the branching law rather than the branching rate is affected by the random medium we mention a paper by Dawson, Fleischmann and Gorostiza (1989). They investigate a branching process in continuous time where particles move according to a symmetric α–stable process (with generator $\Delta_\alpha=-(-\Delta)^{\alpha/2})$) in $\mathbb{R}^d$. The particles branch at rate one according to a critical local offspring distribution p_x given by the probability generating function $f_x(s)=s+h(x)\cdot(1-s)^{1+\beta}$, $s\in[0,1]$, $x\in\mathbb{R}^d$. Here $\beta\in(0,1]$ is a fixed parameter and the stationary and ergodic random function $h:\mathbb{R}^d\to[0,(1+\beta)^{-1}]$ is the (time homogeneous) environment.

Now we perform the hydrodynamic limit procedure. Attach to each particle a mass ε^d, rescale time by $\varepsilon^{-\alpha}$ and space by ε. The corresponding process X_t^ε converges as $\varepsilon\to 0$ to the process Λ_t of deterministic mass flow governed by the α–stable semigroup.

Dawson, Fleischmann and Gorostiza (1989) scrutinise the fluctuations $Y_t^\varepsilon:=\varepsilon^k(X_t^\varepsilon-\Lambda_t)$, where $k:=(d\beta-\alpha)/(1+\beta)$. They show (Theorem 4.9) that (if the initial configurations converge) the process (Y_t^ε) converges as $\varepsilon\to 0$ to a generalised Ornstein-Uhlenbeck process Y_t which is the solution of a generalised Langevin equation

$$dY_t=\Delta_\alpha Y_t\,dt+dZ_t.\tag{1.5}$$

Here Z_t is a (distribution valued) process with independent increments. It is Gaussian if $\beta=1$ and asymmetric β–stable otherwise. The method of proof is a detailed study of the (random) cumulant equation of the Laplace functionals.

2. Catalytic Super-Brownian Motion

We come to the main object of the discussion. It is a continuous time branching model where the offspring distribution is fixed and only the local infinitesimal rate ϱ at which branching occurs varies. We could formulate the model in terms of (continuous time) branching random walk (BRW) on $\mathbb{Z}^d$ or any countable Abelian group. This has been done in some generality in Greven, Klenke and Wakolbinger (1999). However we follow a semi-chronological route and first present the setting where the underlying motion process (W_t) is Brownian motion in $\mathbb{R}^d$.

The Dawson-Watanabe process (or super-Brownian motion (SBM)) in $\mathbb{R}^d$ is the diffusion limit of (critical binary) branching Brownian motion (BBM). In order to model the varying branching rate ϱ assume that each particle has a clock $A(t)$ which is an additive functional of (W_t). If $\{\varrho_t(x),\ t\geq 0,\ x\in\mathbb{R}^d\}$ is a (nonnegative) function we can set $A(t)=\int_0^t\varrho_s(W_s)ds$. If ϱ is more generally a measure (on $[0,\infty)\times\mathbb{R}^d$) we have to be more careful with the definition. Under some regularity assumptions A is the collision local time $L_{[W,\varrho]}(0,t)$ of W with ϱ, that is, ϱ is the

Revuz measure of the time inhomogeneous additive functional A. In fact, it is a non-trivial piece of work to check in the examples that $L_{[W,\varrho]}$ can be well-defined. For example, this needs that the support of ϱ is not polar for Brownian motion and this restricts us in some cases to $d=1$ or $d\le 3$.

Denote by $(X_t^{\varrho,n})_{t\ge 0}$ the catalytic BBM (CBBM) where we assign to each particle a mass n^{-1} and where the clock is $nA(t)$ rather than $A(t)$. Let $\mu\in\mathcal{M}_F(\mathbb{R}^d)$ (=space of finite measures with the vague topology) and assume that $X_0^{\varrho,n}$ is a Poisson point process with intensity $n\mu$. We define catalytic SBM (CSBM) (X_t^ϱ) as the limit of $(X_t^{\varrho,n})$ as $n\to\infty$ and denote its law by $\mathbf{P}_\mu^\varrho$. Of course, it has to be justified that the limit exists and defines a Markov process with nice path properties. This intuitively appealing approach has been made by Delmas (1996) for stationary catalyst $\varrho_t\equiv\sigma$ with some additional energy assumption on σ and by Dynkin (1991) for more general additive functionals A but with a very restrictive exponential moment assumption. These assumptions have been relaxed in Dynkin (1994). Most of the recent papers rely on Dynkin's result and try to circumvent the moment assumptions by some approximation scheme (e.g., Dawson and Fleischmann (1997a), Fleischmann and Mueller (1995)). Due to the independence structure ("$\mathbf{P}_{\mu+\nu}^\varrho=\mathbf{P}_\mu^\varrho*\mathbf{P}_\nu^\varrho$") Laplace functionals are an important tool for the investigation of CSBM. For $\varphi\in C_c^+(\mathbb{R}^d)$ (=space of nonnegative continuous functions with compact support) we define the function $v_\varphi^\varrho(t;x)$ by

$$v_\varphi^\varrho(t;x)=-\log\mathbf{E}_{\delta_x}^\varrho[\exp(-\langle X_t^\varrho,\varphi\rangle)]. \tag{2.1}$$

Note that $-\log\mathbf{E}_\mu^\varrho[\exp(-\langle X_t^\varrho,\varphi\rangle)]=\langle\mu,v_\varphi^\varrho(t;\bullet)\rangle$. The analytical means by which we scrutinise v_φ^ϱ is the *cumulant equation* (p_t is the heat kernel)

$$v_\varphi^\varrho(s,t;x)=(p_{t-s}\varphi)(x)-\int_s^t du\int_{\mathbb{R}}\varrho_u(dy)(v_\varphi^\varrho(u,t;y))^2p_{u-s}(x,y) \tag{2.2}$$

or formally

$$\begin{aligned}-\frac{d}{ds}v_\varphi^\varrho(s,t;x) &= \frac12\Delta v_\varphi^\varrho(s,t;x)-\frac{\varrho_s(dx)}{dx}(v_\varphi^\varrho(s,t;x))^2,\\ v_\varphi^\varrho(0,t) &= \varphi.\end{aligned} \tag{2.3}$$

As a rule we set $v_\varphi^\varrho(t;x)=v_\varphi^\varrho(0,t;x)$. (2.3) is the Kolmogorov backward equation of the Laplace functional. Since CSBM is in general time-inhomogeneous we work with this formulation rather than with the forward equation. From (2.2) it is not hard to derive a recursion scheme for the moments of $\langle X^\varrho,\varphi\rangle$. We only mention that the expectation and variance are given by

$$\begin{aligned}\mathbf{E}_\mu^\varrho[\langle X_t^\varrho,\varphi\rangle] &= \langle p_t\mu,\varphi\rangle,\\ \mathbf{V}\mathrm{ar}_\mu^\varrho[\langle X_t^\varrho,\varphi\rangle] &= \int_0^t ds\int_{\mathbb{R}}\varrho_s(dx)(p_s\mu)(x)(p_{t-s}\varphi)^2(x).\end{aligned} \tag{2.4}$$

Again it is not a priori clear that there exists a (unique) solution to (2.2) or (2.3). Establishing this by analytical methods for a certain catalyst ϱ was the starting point of Dawson and Fleischmann (1991). They use a smoothing procedure for the catalyst replacing ϱ_t by $p_\varepsilon\varrho_t$ (recall that p_ε heat kernel). Letting $\varepsilon\to 0$ they show that the cumulant equation could be uniquely solved. It is not too hard to deduce from this the existence of a unique Markov process (X_t^ϱ) connected to

v_φ^ϱ by (2.1). However, establishing path properties such as existence of a càdlàg or continuous version requires more work.

2.1. Single Point Catalyst in $\mathbf{d=1}$. The simplest catalyst which is not a function is a unit point mass $\varrho_t \equiv \delta_c$ at a point $c \in \mathbb{R}^d$. For $d \geq 2$ single points are polar for Brownian motion, so we have to assume $d = 1$. This model was studied first by Dawson and Fleischmann (1994). A remarkable insight via a nice representation in terms of the super process with respect to a $\frac{1}{2}$–stable sub-ordinator $(U_t)_{t\geq 0}$ is due to Fleischmann and Le Gall (1995) and we follow their exposition.

The unit mass δ_c is the Revuz measure of (Brownian) local time $L(\bullet, c)$ at c and this local time is a perfectly well understood object. On a heuristic level the "infinitesimal particles" of $(X_t^{\delta_c})$ branch at a high rate while they are at c and perform excursions from c otherwise. The length of the excursions can be described in terms of the jumps of the inverse local time $\widetilde{L}(t, c) = \inf\{s > 0 : \; L(s, c) \geq t\}$ which is a $\frac{1}{2}$–stable sub-ordinator $(R_t)_{t\geq 0}$. Associated with (R_t) is a super process $(U_t)_{t\geq 0}$ (derived as above from branching particles moving independently on $\mathbb{R}^+$ like (R_t)). Note that the former time-variable is now a space-variable. Define $V = \int_0^\infty U_t dt$. The reader might by now be willing to believe that the occupation density

$$\lambda_x^{\delta_c}(t) := \frac{1}{dx} \int_0^t X_s^{\delta_c}(dx) ds \tag{2.5}$$

exists at $x = c$ and that it should equal in distribution

$$\{\lambda_c^{\delta_c}(t), \; t \geq 0\} \overset{\mathcal{D}}{=} \{V([0, t]), \; t \geq 0\}. \tag{2.6}$$

This is in fact true (Theorem 1 of Fleischmann and Le Gall (1995)) if we define

$$U_0(dt) = -d\|Q_t^c \mu\|. \tag{2.7}$$

Here $(Q_t^c)_{t\geq 0}$ is the heat flow killed at c (note that $U_0(\{0\}) = \mu(\{c\})$) and $X_0^{\delta_c} = \mu$ a.s. Let $(q_t^c(x) = -d\|Q_t^c \delta_x\|/dt, \; t > 0, \; x \neq c)$ be the density of the first hitting time of Brownian motion at c:

$$q_t^c(x) = \frac{|x - c|}{(2\pi t^3)^{1/2}} \exp\left(-\frac{(x - c)^2}{2t}\right), \qquad t > 0.$$

Recalling the idea of infinitesimal particles performing excursions off c and considering the duality of excursions and Brownian motion killed at c we arrive at the following representation formula (ℓ is the Lebesgue measure):

$$X_t^{\delta_c} := \left(\int_0^t V(ds) q_{t-s}^c\right) \ell + Q_t^c \mu, \qquad t \geq 0, \tag{2.8}$$

is a version of CSBM in the medium $\varrho \equiv \delta_c$ (Theorem 1b). From this fancy formula one can derive a bunch of nice properties. E.g., for $x \neq c$ the density $\xi_t^{\delta_c}(x) = X_t^{\delta_c}(dx)/dx$ exists, is $\mathcal{C}^\infty$ and solves the heat equation (since q_t^c shares this property). Further $X_t^{\delta_c}$ contains all information about the past: $X_s^{\delta_c}$ can be reconstructed from $X_t^{\delta_c}$ if $0 \leq s \leq t$. Finally, if $\mu = \delta_c$ then $\lambda_c^{\delta_c}(\infty)$ is a $\frac{1}{2}$–stable random variable and $\lambda_x^{\delta_c}(\infty) = \lambda_c^{\delta_c}(\infty)$ a.s. for all $x \in \mathbb{R}^d$.

It is intriguing to know more about the behaviour of $X_t^{\delta_c}$ near c and at c. From infinite divisibility and a Palm formula Fleischmann and Le Gall (1995) derive that the support of V (and hence $\lambda_c^{\delta_c}$) has Hausdorff dimension 1. It is singular with respect to Lebesgue measure (Theorem 6) and diffusive: $V(\{t\}) = 0$ for all $t > 0$. The singularity of $\lambda_c^{\delta_c}$ had been shown earlier with a considerable technical effort by Dawson, Fleischmann, Li and Mueller (1995). They do not use (2.6) but construct historical CSBM $(\widetilde{X}_t^{\delta_c})$ and derive a Kallenberg type representation for the Palm canonical measure $Q_{s,w}^{r,\omega}$ of the historical occupation density $\widetilde{\lambda}_c^{\delta_c}$. From this they derive that $\varepsilon^{-1}\lambda_c^{\delta_c}((t-\varepsilon,t]) \to \infty$, $\varepsilon \to 0$, $Q_{s,w}^{r,\omega}$–a.s. (Theorem 4.2.2). Using standard arguments this gives the claim.

2.2. Extension to $\mathbf{d \geq 1}$. As mentioned above a single point catalyst makes sense only if $d = 1$. For $d \geq 1$ it is however possible to construct CSBM even for a singular catalyst ϱ if (roughly speaking) its carrying dimension is larger than $d-2$. More precisely, Delmas (1996) (Theorem 3.2 and 4.7) was able to construct (a continuous version of) (X_t^ϱ) for $\varrho_t \equiv \sigma$, where σ fulfils his "hypothesis (H)"

$$\exists \beta \in (0,1): \quad \sup_{x\in\mathbb{R}^d} \int_{\|y-x\|\leq 1} \frac{\sigma(dy)}{|x-y|^{d-2+2\beta}} < \infty. \tag{2.9}$$

For example, in $d = 1$ every finite measure $\sigma \in \mathcal{M}_f(\mathbb{R})$ fulfils (H) (take $\beta = 1/2$). In any dimension the Lebesgue measure ℓ fulfils (H). Note that σ does not charge polar sets and that the Hausdorff dimension of its support is at least $d-2+2\beta$.

Delmas can show that for $\varphi : \mathbb{R}^d \to \mathbb{R}$ bounded and measurable the evaluation process $(\langle X_t^\varrho, \varphi\rangle)_{t\geq 0}$ is a.s. continuous (Theorem 4.9).

He also shows an extension of the result that in one dimension the occupation measure densities exist (Proposition 5.1): For a measure η fulfilling a slightly stronger assumption than (2.9) (his "hypothesis (H')") there exists the weighted occupation measure $\Gamma_\eta(dt,dx)$. Formally Γ_η is defined by

$$\Gamma_\eta(dt,dx) = \xi_t^\varrho(x)\eta(dx)dt. \tag{2.10}$$

For $d = 1$ and $\nu = \delta_x$ we get back λ_x^ϱ. The main result (Theorem 7.1) is a representation formula analogous to (2.8). Let $D = \text{supp}(\sigma)$, and let ν be the Revuz measure of Brownian local time in D. For $x \in D$ let H^x be the excursion measure on paths ω starting at $\omega(0) = x$ (see Maisonneuve (1975)). Denote by $L(\omega)$ the length of the excursion and define $H_t^x \in \mathcal{M}_f(\mathbb{R}^d)$ by $H_t^x(A) = H^x(\{\omega : L(\omega) > t,\ \omega(t) \in A\})$. Finally let $(Q_t^D)_{t\geq 0}$ denote the heat flow killed at D. Then Delmas' formula is

$$X_t^\varrho \mathbf{1}_{D^c} = \int_{[0,t]\times D} \Gamma_\nu(ds,dx) H_{t-s}^x + Q_t^D \mu. \tag{2.11}$$

Using a refinement of the argument given above for the one-point catalyst Delmas deduces (Theorem 8.1) that on D^c the reactant X_t^ϱ has a density $\xi_t^\varrho(x)$ which is $\mathcal{C}^\infty$ and solves the heat equation

$$\Big(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\Big)\xi_t^\varrho = 0.$$

Finally we wish to mention that Delmas gives a characterisation (Proposition 9.1) of X^ϱ in terms of a martingale problem where the increasing process is $\Gamma_\sigma(dt,dx)$.

2.3. Multiple Point Catalyst. A natural extension of the single point catalyst is the multiple point catalyst. If the points are discrete, then it fits into the framework of Delmas (1996). If the points accumulate, the catalyst is locally not integrable. This causes a special behaviour and we delay the discussion of an example of a locally non-integrable catalyst.

The example we wish to examine here is that of a catalyst concentrated on points that are everywhere dense in $\mathbb{R}$. However these points c do not carry the unit mass δ_c but rather multiples that ensure local integrability. To be concrete, let σ be sampled from a random measure Γ on $\mathbb{R}$ with independent increments and with Lévy measure ν and define $\varrho_t \equiv \sigma$. We can define Γ in terms of Laplace transforms by

$$-\log \mathbb{E}[e^{-\langle \Gamma, \varphi \rangle}] = \int_0^\infty \nu(dx) \langle \ell, 1 - e^{-x\varphi} \rangle, \qquad \varphi \in C_c^+(\mathbb{R}). \tag{2.12}$$

We assume that $\int \nu(dx)(1 \wedge x) < \infty$ so that Γ is locally finite and is carried by a countable set. One example is the stable point process with index $\gamma \in (0,1)$, that is $\nu(dx) = c \cdot x^{-(1+\gamma)} dx$, $x > 0$, for some $c > 0$. If c is chosen appropriately this yields

$$-\log \mathbb{E}[e^{-\langle \Gamma, \varphi \rangle}] = \langle \ell, \varphi^\gamma \rangle, \qquad \varphi \in C_c^+(\mathbb{R}). \tag{2.13}$$

This model has been considered in Dawson, Li and Mueller (1995). They address the question if X^ϱ has a compact global support

$$\mathcal{G} = \text{closure}\left(\bigcup_{t \geq 0} supp X_t^\varrho \right) \tag{2.14}$$

$\mathbf{P}_{\delta_x}^\varrho$–a.s. Recall that $\mathcal{G}$ is in fact compact a.s. for classical SBM (see Iscoe (1988)). Dawson, Li and Mueller give complicated sufficient conditions for compactness of $\mathcal{G}$ (Theorem 1) and for non-compactness (Theorem 2) in terms of ν. One example for compactness (Corollary 1) is the γ–stable point process (recall that $\mathcal{P}_{\delta_x} = \mathbb{E}\,\mathbf{P}_{\delta_x}^\varrho$ is the annealed law)

$$\mathcal{P}_{\delta_x}[\mathcal{G} \text{ is compact}] = 1. \tag{2.15}$$

On the other hand (Corollary 2) if ν is finite or if, for instance, $\nu(dx) = x^{-1}\mathbf{1}_{(0,1]}(x)dx$ then

$$\mathcal{P}_{\delta_x}[\mathcal{G} \text{ is compact}] = 0. \tag{2.16}$$

The method employed is quite similar to the original approach of Iscoe. Consider a function $\psi \in C_c^+(\mathbb{R})$ and define

$$u_\psi(t;x) = -\log \mathbf{E}_{\delta_x}^\varrho[\exp(-\int_0^t \langle X_s^\varrho, \psi \rangle ds)]. \tag{2.17}$$

Hence u_ψ solves the integral equation (recall that p_t is the heat kernel)

$$u_\psi(t;x) = \int_0^t (p_{t-s}\psi)(x) ds - \int_0^t ds \int_{\mathbb{R}} \Gamma(dy) p_{t-s}(x,y) u_\psi^2(s,y). \tag{2.18}$$

Now let $t \to \infty$, and let $\psi \uparrow \mathbf{1}_{[\alpha_1,\alpha_2]^c}$ for some reals $\alpha_1 < \alpha_2$. Then (by Theorem 0) u_ψ approaches the solution $u(x) = u_{\alpha_1,\alpha_2}(x)$ of the formal boundary value problem

$$\begin{aligned} \frac{1}{2}\frac{d^2}{dx^2} u(x) &= u^2(x)\Gamma(dx), \quad x \in (\alpha_1, \alpha_2), \\ u(\alpha_1) &= u(\alpha_2) = \infty. \end{aligned} \tag{2.19}$$

Since $\mathbf{P}_{\delta_x}^\varrho[\mathcal{G} \subset [\alpha_1,\alpha_2]] = 1 - \exp(-u_{\alpha_1,\alpha_2}(x))$, it is clear that (2.15) holds if $u_{-\alpha,\alpha}(x) \to 0$, $\alpha \to \infty$. On the other hand, $u_{-\alpha,\alpha} \equiv \infty$ for all α implies (2.16). It is the content of Theorem 1 and 2 to give sufficient conditions on ν for either case to hold.

In a recent work Dawson, Fleischmann and Mueller (1998) study the question of finite time extinction for this model: Is it true that for $\mu \in \mathcal{M}_f(\mathbb{R})$

$$\mathbb{E}\,\mathbf{P}^{\varrho}_{\mu}[\|X^{\varrho}_t\| = 0 \text{ for } t \text{ large enough}] = 1? \tag{2.20}$$

Obviously, this is not the case if ϱ is supported by a non-dense set and $\mu((\text{supp }\varrho)^{\text{c}}) > 0$. In this case $X^{\varrho}_t \geq Q^{\text{supp }\varrho}_t \mu$ (recall that Q^D_t is the semigroup of the heat flow killed at D). Thus $\|X^{\varrho}_t\| > 0$ a.s. On the other hand, the γ–stable catalyst ϱ has a dense support. The dense support alone does not guarantee finite time extinction. However in this particular example Dawson, Fleischmann and Mueller can show that (2.20) holds.

2.4. Moving Multiple Point Catalyst. We come to our first example where the catalyst is not time homogeneous. Assume that Γ is the γ–stable point process, $\gamma \in (0,1)$, introduced in (2.13). Consider the representation

$$\Gamma = \sum_{i=1}^{\infty} g^i \delta_{x^i}, \tag{2.21}$$

where $\{g^i\}$ are the "action weights" of the points $\{x^i\}$. Now allow the points to perform independent Brownian motion and carry their weights with them. More precisely, let $\{(x^i_t)_{t\geq 0}\}$ be independent Brownian motions, $x^i_0 = x^i$, and define

$$\varrho_t = \sum_{i=1}^{\infty} g^i \delta_{x^i_t}. \tag{2.22}$$

This model has been studied by Dawson, Fleischmann and Roelly (1991) and Dawson and Fleischmann (1991). In fact, these papers cover a moderately more general situation, namely with the Brownian motion replaced by the fashionable symmetric α–stable process. The finite variance branching is replaced by a certain offspring law in the normal domain of attraction of a β-stable law, $\beta \in (0,1]$, (in the cumulant equation replace u^2 by $u^{1+\beta}$). Here we no not draw a bead on these details, the reader may think of $\alpha = 2$, $\beta = 1$.

We need the following definitions. Let $\mathcal{M}(\mathbb{R}^d) = \{\text{Radon measures on } \mathbb{R}^d\}$ be equipped with the vague topology. This is a Polish space (see Kallenberg (1983)). For $p > d$ define the space $\mathcal{M}_p(\mathbb{R}^d)$ of p–tempered measures by

$$\mathcal{M}_p(\mathbb{R}^d) := \left\{\mu \in \mathcal{M}(\mathbb{R}^d) :\ \langle \mu, \phi_p\rangle < \infty\right\}, \tag{2.23}$$

where $\phi_p(x) = (1 + \|x\|^2)^{-p/2}$. Note that $\ell \in \mathcal{M}_p(\mathbb{R}^d)$.

Dawson and Fleischmann construct (X^{ϱ}_t) as a Markov process with values in $\mathcal{M}_p(\mathbb{R}^d)$ (Theorem 1.8.2). However they do not make a statement on whether a càdlàg version exists. The reason for this flaw originates in the construction. Rather than following the currently appreciated approach via "branching functionals" they show that the cumulant equation (2.2) could be uniquely solved. They replace ϱ_t by the function $p_\varepsilon \varrho_t$, $\varepsilon > 0$, and then let $\varepsilon \to 0$. This yields a solution, uniqueness follows by standard arguments, and also existence of (X^{ϱ}_t). Dawson, Fleischmann and Roelly (Theorem 1.19) show absolute continuity of the states of X^{ϱ}_t with respect to Lebesgue measure if $\alpha = 2$ or if $(\beta\gamma)(1+\alpha) > 1$. The "main result" of Dawson and Fleischmann (1991) is a scaling limit (Theorem 1.9.4). Fix $\eta > 0$ and define for $K > 0$

$${}^K X^{\varrho}_t = K^{-\eta} X^{\varrho}_{tK}(K^{\eta}\bullet). \tag{2.24}$$

If $\eta = \eta_c := ((\gamma(\alpha-1)+1)/\alpha\beta\gamma$ then for $t > 0$ $\mathcal{L}_\ell^\varrho({}^K X_t^\varrho) \Longrightarrow \mathcal{L}_\ell({}^\infty X_t^\varrho)$, $K \to \infty$ in $\mathbb{P}$–probability (note that $\Longrightarrow$ denotes weak convergence). Here for $t > 0$, ${}^\infty X_t$ is a homogeneous independent point process on $\mathbb{R}^d$ with Lévy measure ν_t^∞ characterised as follows. For fixed ϱ let v_t^ϱ be the Lévy measure of $\mathbf{P}_{\delta_0}^\varrho[\|X_t^\varrho\| \in \bullet]$. Then $\nu_t^\infty = \mathbb{E}[\nu_t^\varrho]$. In terms of the Laplace functionals this reads

$$-\log \mathbf{E}[\exp(-\langle X_t^\infty, f\rangle)] = \int dx\ \mathbb{E}[v^\varrho_{f(z)\mathbf{1}}(t;0)]. \tag{2.25}$$

If $\eta > \eta_c$ then there holds a law of large numbers: ${}^\infty X_t = \delta_\ell$ a.s. (Theorem 1.10.1).

2.5. Hyperplanes. Dawson and Fleischmann (1995) also have an example for a time homogeneous catalyst ϱ in $\mathbb{R}^d$, $d > 1$, whose support may be everywhere dense. However they practically assume that ϱ factors into a function of $d-1$ coordinates and a measure in one dimension. So you can keep in mind the example where ϱ consists of "hyperplanes": $\varrho = \sigma \otimes \ell_{d-1}$, where $\sigma \in \mathcal{M}(\mathbb{R})$ and ℓ_{d-1} is the $(d-1)$–dimensional Lebesgue measure. Under some conditions on ϱ they show the existence of CSBM (Lemma 2.3.4). Further under some very restrictive assumptions on the branching rate functional A (e.g., all moments exist and fulfil a growth condition, see Definition 2.4.7 and 2.6.1) they show absolute continuity of X_t^ϱ with respect to ℓ_d (Theorem 2.6.2). This is proved by showing that there exists a proper solution of the cumulant equation with terminal condition δ_x, $x \in \mathbb{R}^d$. Compare this result with SBM in $d \geq 2$ which is singular with respect to the Lebesgue measure.

2.6. Locally Infinite Catalyst. As far as the author knows there is only one paper studying a model where the catalyst is locally not integrable. Fleischmann and Mueller (1997) construct one-dimensional CSBM (Theorem 1) with time homogeneous catalyst ϱ whose density is given by $\varrho_t(dx)/dx = \theta|x-c|^{-\sigma}$, where $c \in \mathbb{R}$, $\theta \in (0,\infty)$ and $\sigma \in [1,2]$. In particular, $\langle \varrho_t, \mathbf{1}_{[c-1,c+1]}\rangle = \infty$. They show that for $\langle X_0^\varrho, \mathbf{1}\rangle < \infty$ the total mass process $\langle X_t^\varrho, \mathbf{1}\rangle$ is a super martingale but not a martingale. It has finite variance iff $\sigma < 2$ (Theorem 3).

For Brownian motion W let $\tau = \{t > 0 :\ W_t = c\}$ and recall that $A(t) = \theta\int_0^t |W_s - c|^{-\sigma} ds$. Note that for $x \neq c$

$$\mathbf{P}_x[A(\tau) = \infty] = \begin{cases} 1, & \sigma = 2, \\ 0, & \sigma < 2. \end{cases}$$

Since the critical Galton–Watson process dies out eventually this implies that if $\sigma = 2$ then "no infinitesimal particle of X_t^ϱ ever reaches c". More precisely, there exists an increasing sequence of stopping times (τ_n) of Brownian motion W such that any τ_n is strictly smaller than τ and such that $\lim_{n\to\infty} \mathbf{P}_{\delta_x}^\varrho[X_{\tau_n}^\varrho = 0] = 1$ (Theorem 4). Here $X_{\tau_n}^\varrho$ is understood in the sense of Dynkin's stopped measure (see Dynkin (1991)).

3. CSBM in a SBM Medium

In an intriguing model we consider a random time-inhomogeneous catalyst ϱ: The catalyst is itself a sample path of SBM on $\mathbb{R}^d$. Let $\mathbb{P}_\mu$ denote its law with initial condition $\mu \in \mathcal{M}_p(\mathbb{R}^d)$ (recall (2.23)). Recall that SBM is a $(d \wedge 2)$–dimensional object. Hence the support is polar for Brownian motion if $d \geq 4$. We thus have to restrict ourselves to $d \leq 3$.

This CSBM can serve as a model for a biological one-way interaction between two species (green and red, say). The green species (the catalyst) does not even notice the red species and simply performs its migration and resampling scheme, however influences thereby the reproduction rate of the red species. This model has been studied by Dawson and Fleischmann (1997a), (1997b), Etheridge and Fleischmann (1998), and Fleischmann and Klenke (1999). On a BRW level it is treated in Greven, Klenke and Wakolbinger (1999).

A model with a symmetric interaction between two species is due to Dawson and Perkins (1998). We discuss it briefly in Section 4.

3.1. The Construction. Dawson and Fleischmann (1997a) construct X^ϱ in a somewhat more general framework. Instead of aiming at the concrete model directly they first construct Hölder continuous versions of CSBM for a certain class of Hölder continuous branching functionals A.

Define $\mathbf{K}$ to be the set of continuous additive functionals of W such that for all $t_0 \geq 0$ (recall (2.23))

$$\sup_{x\in\mathbb{R}^d} \mathbf{E}_{s,x}\left[\int_s^t A(dr)\phi_p(W_r)\right] \to 0, \qquad s,t\to t_0. \tag{3.1}$$

Further define for $\xi \in (0,1]$ the subclass $\mathbf{K}^\xi$ by imposing the additional requirement that for $T>0$ there exists $c_T>0$ such that

$$\mathbf{E}_{s,x}\left[\int_s^t A(dr)(\phi_p(W_r))^2\right] \leq c_T|t-s|^\xi \phi_p(x), \qquad 0\leq s\leq t\leq T. \tag{3.2}$$

They show (Proposition 1) that for $A\in\mathbf{K}$ the cumulant equation

$$v_\varphi(s,t;x) = (p_{t-s}\varphi)(x) - \mathbf{E}_{s,x}\left[\int_s^t A(dr)v_\varphi(r,t;W_r)^2\right], \qquad \varphi\in C_c^+(\mathbb{R}^d), \tag{3.3}$$

has a unique solution. The proof relies on an approximation of A by functionals that fit into Dynkin's (1994) framework. As an application (Proposition 2 (sic!), page 230) one gets the existence of a time-inhomogeneous multiplicative $\mathcal{M}_p(\mathbb{R}^d)$–valued Markov process $(X^A, \mathbf{P}^A_{t,\mu}, t\geq 0, \mu\in\mathcal{M}_p(\mathbb{R}^d))$ with log-Laplace function

$$v_\varphi(s,t;x) = -\log \mathbf{E}^A_{s,\delta_x}[\exp(-\langle X^A_t,\varphi\rangle)]. \tag{3.4}$$

The moments of $\langle X^A_t,\varphi\rangle$ can be expressed in terms of the derivatives of $v_{\theta\varphi}$ at $\theta=0$. In particular the first and second moments are

$$\mathbf{E}^A_{s,\mu}[X_t] = p_{t-s}\mu,$$

$$\mathbf{Cov}^A_{s,\mu}[\langle X^A_{t_1},\varphi_1\rangle, \langle X^A_{t_2},\varphi_2\rangle] = \int \mu(dx)\mathbf{E}^A_{s,x}\Big[\int_s^{t_1\wedge t_2} A(dr)(p_{t_1-s}\varphi_1)(W_r)(p_{t_2-s}\varphi_2)(W_r)\Big]. \tag{3.5}$$

Dawson and Fleischmann develop a recursion formula for the n-th derivatives of $v_{\theta\varphi}$ at $\theta=0$ and for the n-th centred moments $\mathbf{E}^A_{s,\mu}[|\langle Z^A_t,\varphi\rangle|^n]$, where $Z^A_t = X^A_t - \mathbf{E}^A_{s,\mu}[X^A_t]$. These moments are finite if, for instance, $A\in\mathbf{K}^\xi$ for some

$\xi \in (0,1]$ (Lemma 5). In this case the following estimate holds (Lemma 6)

$$\begin{aligned}&\mathbf{E}^A_{s,\mu}[|\langle Z^A_{t+h} - Z_t, \varphi\rangle|^{2n}] \\ &\qquad \le \text{const}\left(\left\|\frac{p_h\varphi - \varphi}{\phi_p}\right\|^{2n}_\infty + h^{\xi n}\left\|\frac{\varphi}{\phi_p}\right\|^{2n}_\infty\right)\big(\langle \mu, \phi_p\rangle + 1\big)^{2n}.\end{aligned} \tag{3.6}$$

Using a measure-valued version of Kolmogorov's method of moments one obtains for every $\varepsilon \in (0, \xi/2)$ a Hölder-ε-continuous version of the centred process (Z^A_t) (Theorem 1). Here we assumed a certain underlying metric that generates the vague topology on $\mathcal{M}_p(\mathbb{R}^d)$ but which we do not specify here. In particular, (X^A_t) has a continuous version since $t \mapsto p_{t-s}\mu$ is continuous.

There is a simple criterion for absolute continuity of X^A_t with respect to Lebesgue measure ℓ. Formally we could define the density $\xi^A_t(x) = \langle X^A_t, \delta_x\rangle$, $x \in \mathbb{R}^d$. This expression makes sense as the limit of $\langle X^A_t, p_\varepsilon\delta_x\rangle$ as $\varepsilon \to 0$. If $t > s$ then $\mathbf{E}^A_{s,\mu}[X_t] = p_{t-s}\mu$ is absolutely continuous and it suffices to check that the variances converge along some sequence $\varepsilon_n \downarrow 0$

$$\begin{aligned}&\lim_{n\to\infty}\int \mu(dx)\mathbf{E}^A_{s,x}\left[\int_s^t A(dr)p_{t-r+\varepsilon_n}(W_r, z)^2\right] \\ &\qquad = \int \mu(dx)\mathbf{E}^A_{s,x}\left[\int_s^t A(dr)p_{t-r}(W_r, z)^2\right] < \infty.\end{aligned} \tag{3.7}$$

If (3.7) holds then $\xi^A_t(z)$ exists as the L^2–limit of $\langle X^A_t, p_{\varepsilon_n}\delta_z\rangle)$, $n \to \infty$, and is the density of X^A_t (Proposition 4). It has first and second moment

$$\begin{aligned}&\mathbf{E}^A_{s,\mu}[\xi^A_t] = p_{t-s}\mu, \\ &\mathbf{Cov}^A_{s,\mu}[\xi^A_{t_1}(z_1), \xi^A_{t_2}(z_2)] = \\ &\qquad \int \mu(dx)\mathbf{E}^A_{s,x}\left[\int_s^{t_1\wedge t_2} A(dr)p_{t_1-s}(W_r, z_1)p_{t_2-s}(W_r, z_2)\right].\end{aligned} \tag{3.8}$$

A criterion similar to the one above is derived for the absolute continuity of the occupation time measure $\int_s^t X^A_r dr$ (Proposition 5).

We come back to the situation where ϱ is a sample path of SBM and $A = L_{[W,\varrho]}$ is the *collision local time* of W with ϱ. As above with the density of X^A_t we would like to define $L_{[W,\varrho]}(s,t) = \int_s^t dr\ (\varrho_r(dx)/dx)\big|_{x=W_r}$. If $d = 1$ this makes perfect sense since ϱ_r is absolutely continuous. However for $d \ge 2$ it is not. Hence we define for $\varepsilon > 0$

$$L^\varepsilon_{[W,\varrho]}(s,t) = \int_s^t dr(p_\varepsilon\varrho_r)(W_r). \tag{3.9}$$

We hope that it makes sense to define

$$L_{[W,\varrho]}(s,t) = \lim_{\varepsilon\to 0} L^\varepsilon_{[W,\varrho]}(s,t). \tag{3.10}$$

The reader might guess that this is non-trivial. However, Evans and Perkins (1994, Theorem 4.1) show that if $d \le 3$ and if $\nu \in \mathcal{M}_f(\mathbb{R}^d)$ is absolutely continuous, then for $\mathbb{P}_\nu$–a.a. ϱ the limit on the r.h.s. of (3.10) makes sense in L^2 and defines a continuous additive functional. A simple approximation argument extends this to absolutely continuous $\nu \in \mathcal{M}_p(\mathbb{R}^d)$. However we typically want singular initial conditions. For instance, if $d = 3$ and ν is a sample from the equilibrium of SBM, then ν is singular with respect to ℓ. We outline the steps that yield in fact that (3.10) makes sense in L^2 for the examples we have in mind and that we even have $A = L_{[W,\varrho]} \in \mathbf{K}^\xi$ for some $\xi \in (0, 1/4)$.

Dawson and Fleischmann improve a result of Sugitani (1989) which states that the occupation measure Y on $[0,\infty) \times \mathbb{R}^d$ defined by

$$Y([s,t] \times B) = \int_s^t \varrho_r(B)dr$$

is absolutely continuous and has a (jointly) continuous density $(t, x) \mapsto y(t, x)$. In fact, using moment estimates and Kolmogorov's method they get that for the centred density $\overline{y} = y = \mathbb{E}_\nu[y]$ on any set $[0,T]\times\mathbb{R}^d$, $T > 0$, the function $\overline{y}(t, x)\phi_p(x)$ is Hölder-ξ-continuous, $\xi \in (0, 1/4)$ (Theorem 2). Hence $y(t, x)\phi_p(x)$ is Hölder-ξ-continuous iff

$$(3.11) \qquad (t,x) \mapsto \int_0^t (p_r\nu)(x)\phi_p(x)dr \text{ is Hölder-}\xi\text{-continuous.}$$

Denote by $\mathcal{M}_p^\xi(\mathbb{R}^d)$ the space of measures $\nu \in \mathcal{M}_p(\mathbb{R}^d)$ for which (3.11) holds. We do not give a characterisation of $\mathcal{M}_p^\xi(\mathbb{R}^d)$ but only mention some examples established by Fleischmann and Klenke (1999): In $d = 1$ we have $\mathcal{M}_p^\xi(\mathbb{R}) = \mathcal{M}_p(\mathbb{R})$. In any dimension $\mathcal{M}_p^\xi(\mathbb{R}^d)$ contains any $\nu \ll \ell_d$ with bounded density. If $\nu \in \mathcal{M}_p(\mathbb{R}^d)$ and $\delta > 0$, then $\mathbb{P}_\nu[\varrho_0 \in \mathcal{M}_p^\xi(\mathbb{R}^d)] = 1$. Note that $\delta_x \notin \mathcal{M}_p^\xi(\mathbb{R}^d)$ if $d \geq 2$.

Dawson and Fleischmann use the Hölder continuity to imitate the existence proof for the collision local time of Evans and Perkins. They can show that for $\nu \in \mathcal{M}_p^\xi(\mathbb{R}^d)$ the limit (3.10) makes sense in L^2 and that $A = L_{[W,\varrho]} \in \mathbf{K}^\xi$. In partiaular, for $\mathbb{P}_\nu$–a.a. ϱ there exists a continuous version of (X_t^ϱ).

3.2. Longtime Behaviour. In this subsection we study the longtime behaviour of CSBM in a SBM medium.

It is well known (see Dawson (1977)) that SBM is persistent iff $d > 2$. More precisely, $\mathbb{P}_\ell[\varrho_t(B) > 0] \to 0$, $t \to \infty$, for any compact set B if $d = 1, 2$. However, if $d \geq 3$ then there exist equilibria $\nu_i \in \mathcal{M}_1(\mathcal{M}_p(\mathbb{R}^d))$ with intensity $\int m\nu(dm) = i\ell$, $i \in [0,\infty)$ and such that $\mathbb{P}_{i\ell}[\varrho_t \in \bullet] \Rightarrow \nu_i$, $t \to \infty$. The situation is quite different for our CSBM X^ϱ.

First we consider $d = 3$. This is the maximal dimension for which CSBM exists. The catalyst is persistent and we assume that $(\varrho_t)_{t\in\mathbb{R}}$ is the stationary process with intensity $i_c > 0$. We start X^ϱ in $i_r\ell$ for some $i_r > 0$. Instead of starting at time 0 and evaluating at time t it is more convenient to start at time $-t$ and evaluate at time 0. The advantage is that we can fix ϱ and exploit monotonicity of the cumulant equation (3.3). Recall that this is a backward equation and it is immediate that $\langle i_r\ell, v_\varphi^\varrho(-t, 0)\rangle$ is monotone decreasing in t. Hence its limit as $t \to \infty$ exists and so does the limit of $\mathbf{P}^\varrho_{-t,i_r\ell}[X_0^\varrho \in \bullet]$. Note that the variances are monotone in t and converge to a finite limit since Brownian motion in $\mathbb{R}^3$ is transient. Hence for $\varphi \in C_c^+(\mathbb{R}^3)$ the random variable $\langle X_0^\varrho, \varphi\rangle$ is uniformly integrable under the sequence $\mathbf{P}^\varrho_{-t,i_r\ell}$ and thus $\mathbf{E}^\varrho_{-\infty,i_r\ell}[X_0^\varrho] = i_r\ell$. In other words, three-dimensional CSBM is persistent (see Theorem 1 of Dawson and Fleischmann (1997b)).

In dimension $d = 1$ the catalyst does not only die out locally in distribution but even a.s. for any compact set B,

$$\mathbb{P}_{i_c\ell}[\varrho_t(B) = 0 \text{ for } t \text{ large enough}] = 1.$$

Dawson and Fleischmann (1997a) show that (see Proposition 7) there exists a random time τ such that

$$\mathbb{E}_{i_c\ell}[\mathbf{P}_0[L_{[W,\varrho]}(\tau,\infty) = 0]] = 1.$$

Furthermore (Proposition 8) for all $x \in \mathbb{R}$,

$$\mathbb{P}_{i_c\ell}[\mathbf{E}_x[L_{[W,\varrho]}(0,\infty)^2] < \infty] = 1. \tag{3.12}$$

Note that for finite initial mass $\mu \in \mathcal{M}_f(\mathbb{R})$ and fixed ϱ the total mass process $\langle X_t^\varrho, \mathbf{1}\rangle$ is a nonnegative L^2–martingale with variance

$$\mathbf{Var}_\mu^\varrho[\langle X^\varrho, \mathbf{1}\rangle] = 2\int \mu(dx)\mathbf{E}_x[L_{[W,\varrho]}(0,t)]. \tag{3.13}$$

By (3.12) this martingale is bounded in L^2 and by the martingale convergence theorem it converges to a limit with finite variance and expectation $\|\mu\|$ (Theorem 5). In other words, we have persistence even of a finite initial mass. Having in mind a law of large numbers it is clear that for $\mathbb{P}_{i_c\ell}$–a.a. ϱ we have (see Theorem 6)

$$\mathbf{P}_{i_r\ell}^\varrho[X_t^\varrho \in \bullet] \Longrightarrow i_r\ell, \quad t \to \infty. \tag{3.14}$$

For $d = 2$ the situation is a little more involved. Neither do we have a non-trivial equilibrium nor a.s. extinction of the catalyst. In fact, any non-trivial open set is visited at arbitrarily large times. The key to the long-time behaviour lies in the self similarity (see Dawson and Fleischmann (1997b, Proposition 13))

$$\mathbb{P}_{i_c\ell}[\mathbf{P}_{i_r\ell}^\varrho[K^{-1}X_{Kt}^\varrho(K^{1/2}\bullet) \in \bullet] \in \bullet] = \mathbb{P}_{i_c\ell}[\mathbf{P}_{i_r\ell}^\varrho[X_t^\varrho(\bullet) \in \bullet] \in \bullet], \ K, t > 0, \tag{3.15}$$

and a study of the detailed behaviour for fixed time.

Fleischmann and Klenke (1999) show in their Theorem 1 that CSBM in $d = 1, 2, 3$ is absolutely continuous on the complement $Z(\varrho)$ of the space-time support of the catalyst and that the density $\xi_t^\varrho(z)$ is $\mathcal{C}^\infty$ on $Z(\varrho)$ and solves the heat equation. This is established by similar means as in Delmas (1996). In $d = 2, 3$ the catalyst is singular with respect to Lebesgue measure ℓ_d and hence X_t^ϱ is absolutely continuous everywhere. Plugging this result into (3.15) they derive (Corollary 2) that for $d = 2$, CSBM is in fact persistent and that

$$\mathbb{P}_{i_c\ell}[\mathbf{P}_{i_r\ell}^\varrho[X_t^\varrho \in \bullet] \in \bullet] \Longrightarrow \mathbb{P}_{i_c\ell}[\mathbf{P}_{i_r\ell}^\varrho[\xi_1^\varrho(0)\ell \in \bullet] \in \bullet]. \tag{3.16}$$

Hence the limit of X_t^ϱ is a random multiple of the Lebesgue measure with full expectation. The randomness reflects the catalyst as experienced by a "reactant particle".

Earlier Dawson and Fleischmann (1997b, Theorem 18) established that if $d = 2$ then the normalised occupation measures $t^{-1}\int_0^t X_s^\varrho ds$ converge to $(\int_0^1 \xi_s^\varrho(0)ds) \cdot \ell$ (in the sense of (3.16)). Note however that Dawson and Fleischmann do not show absolute continuity of X_t^ϱ but only of $\int_0^t X_s^\varrho ds$.

3.3. Catalytic Branching Random Walk. Recall that we defined CSBM as the diffusion limit of CBBM. Here we consider a model where neither the particle nor the spatial diffusion limit has been taken: catalytic branching random walk (CBRW).

Greven, Klenke and Wakolbinger (1999) study CBRW in some detail. They construct by elementary means the process in the following setting. The process lives on a countable Abelian group G as site space. For the moment the catalyst can be any measurable function $\varrho : [0,\infty) \times G \to [0,\infty)$, $(t,g) \mapsto \varrho_t(g)$. The reactant X^ϱ performs a continuous rate 1 random walk on G with q–matrix $\mathcal{B}$. Further each particle branches at the local rate $\varrho_t(g)$ according to the (global) offspring law $(q_k)_{k\in\mathbb{N}_0}$ with probability generating function Q. Formally the (time-inhomogeneous) process X^ϱ can be defined by its Laplace functionals

$$v^\varrho_\varphi(r,t;g) = \mathbf{E}^\varrho_{r,\delta_g}[\exp(-\langle X^\varrho_t, \varphi\rangle)] \tag{3.17}$$

that solve the backward equation, $v^\varrho_\varphi(t,t) = \varphi$,

$$-\frac{d}{dr} v^\varphi_\varrho(r,t;g) = \varrho_r(g)\big[Q(v^\varrho_\varphi(r,t;g)) - v^\varrho_\varphi(r,t;g)\big] + (\mathcal{B}v^\varrho_\varphi(r,t))(g). \tag{3.18}$$

Henceforth, let us restrict to critical binary branching $q_0 = q_2 = \frac{1}{2}$. We are interested in the case where ϱ is itself a sample of critical binary branching random walk (BRW) on G with q–matrix $\mathcal{A}$. Compared with the CSBM model, there is an enormous freedom in the choice of the system parameters. For example, one can choose $\mathcal{A}$ transient while $\mathcal{B}$ is recurrent, or one can add a drift to one of the kernels.

Greven, Klenke and Wakolbinger investigate the longtime behaviour of (ϱ, X^ϱ) in the case where $\mathcal{L}(\varrho_0, X^\varrho_0) \in \mathcal{E}_{i_c,i_r}$ is ergodic with intensities $i_c,\ i_r \in (0,\infty)$. Let $\delta_{\underline{0}}$ be the Dirac measure on the empty configuration. Denote by $\mathcal{H}_i$ the Poisson point process on G with uniform intensity $i \in (0,\infty)$. Finally denote by $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ the symmetrization of $\mathcal{A}$ and $\mathcal{B}$ respectively, $\widehat{\mathcal{A}}(g,h) = \frac{1}{2}(\mathcal{A}(g,h) + \mathcal{A}(h,g))$.

Consider first the case where $\widehat{\mathcal{A}}$ is transient so that ϱ is persistent. As discussed earlier the particles of X^ϱ experience an average of the medium ϱ. Thus X^ϱ is persistent iff $\widehat{\mathcal{B}}$ is transient (Theorem 1).

The situation is far more delicate if $\widehat{\mathcal{A}}$ is recurrent. The catalyst goes to extinction but there is a subtle difference between a.s. extinction (as for $\mathcal{A}$ Bernoulli on $\mathbb{Z}$) and extinction only in probability (as for $\mathcal{A}$ Bernoulli on $\mathbb{Z}^2$). For this reason only the special cases $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$ are considered.

Assume first $G = \mathbb{Z}$ and additionally $\mathcal{A}$ and $\mathcal{B}$ have the properties $\sum_{x\in\mathbb{Z}} \mathcal{A}(0,x)x = \sum_{x\in\mathbb{Z}} \mathcal{B}(0,x)x = 0$ and

$$\sum_{x\in\mathbb{Z}} \mathcal{A}(0,x)|x|^\alpha < \infty, \qquad \sum_{x\in\mathbb{Z}} \mathcal{B}(0,x)|x|^\beta < \infty, \qquad \text{for some } \alpha > 2 \text{ and } \beta > 1.$$

As with CSBM in $d = 1$ we have (Theorem 2a)

$$\mathcal{L}_{\mathcal{H}_{i_c},\mathcal{H}_{i_r}}(\varrho, X^\varrho_t) \Longrightarrow \delta_{\underline{0}} \otimes \mathcal{H}_{i_r}, \quad t \to \infty. \tag{3.19}$$

If $G = \mathbb{Z}^2$ and $\mathcal{A}$ and $\mathcal{B}$ are Bernoulli, then the situation is similar to CSBM in $d = 2$. The law of X^ϱ_t converges to a limit with random homogeneous intensity (Theorem 3):

$$\mathbb{P}_{\mathcal{H}_{i_c}}[\mathbf{P}^\varrho_{\mathcal{H}_{i_r}}[X^\varrho_t \in \bullet] \in \bullet] \Longrightarrow \mathbb{P}_{i_c\ell}[\mathbf{E}^{\widetilde{\varrho}}_{i_r\ell}[\mathcal{H}_{\widetilde{\xi}_1(0)}] \in \bullet], \tag{3.20}$$

where $\widetilde{\xi}_t(x)$ is the density of the 2-dimensional CSBM $\widetilde{X}^{\widetilde{\varrho}}$. (The assumptions on $\mathcal{A}$ and $\mathcal{B}$ can be weakened to finite variance isotropic random walks with the additional requirement that $\sum_{x\in\mathbb{Z}^2}\mathcal{A}(0,x)|x|^\alpha<\infty$ for some $\alpha>6$.) The key to this result is a scaling limit of CBRW (Proposition 1.4) and the fact that for large T the support of ϱ_t has large holes (Proposition 1.5) combined with an adaption of a result of Harry Kesten (1995) on the range of branching random walk (Proposition 1.3).

Finally we would like to mention a situation with a striking asymmetry between the catalyst and the reactant. Consider $G=\mathbb{Z}$ and $\mathcal{B}$ Bernoulli with a drift while $\mathcal{A}=0$ is the random walk that stands still. (Apparently Greven, Klenke and Wakolbinger would have liked to consider $\mathcal{A}$ Bernoulli but could not overcome technical difficulties.) Obviously ϱ dies out locally a.s. However, the greater mobility of the reactant particles forces them to visit lots of the scattered catalyst clumps and, ironically enough, leads to local extinction (Theorem 2b)

$$\lim_{t\to\infty}\mathbb{E}_{\mathcal{H}_{i_c}}\mathbf{P}^{\varrho}_{\mathcal{H}_{i_r}}[X^{\varrho}_t(g)>0]=0,\qquad g\in\mathbb{Z}. \tag{3.21}$$

4. Mutually Catalytic Branching

Dawson and Perkins (1998) study a model of two spatial branching processes $(u_t)_{t\geq 0}$ and $(v_t)_{t\geq 0}$, where each process acts as the catalyst for the other one. In the continuous space setting u and v can be defined on $\mathbb{R}$ by an SPDE. Let $\gamma>0$ and $\dot{W}_i(t,x)$ $(i=1,2)$ be independent space-time white noises on $\mathbb{R}^+\times\mathbb{R}$. Consider the SPDE

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)&=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x)+(\gamma u(t,x)v(t,x))^{1/2}\,\dot{W}_1(t,x);\quad u(0,x)=u_0(x),\\ \frac{\partial v}{\partial t}(t,x)&=\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x)+(\gamma u(t,x)v(t,x))^{1/2}\,\dot{W}_2(t,x);\quad v(0,x)=v_0(x).\end{aligned}\tag{4.1}$$

With the aid of a duality going back to Mytnik (1996) it is shown that (for suitable u_0 and v_0) there exists a unique solution of (4.1). The question of a higher dimensional analogue was left open. There is some recent work (Dawson et al. (1999)) giving an affirmative answer, at least for two dimensions.

On $\mathbb{Z}^d$, however, the model has been constructed by Dawson and Perkins (1998) for any $d\geq 1$. Let Q be the q–matrix of a Markov chain on $\mathbb{Z}^d$ with bounded jump rate. Let $(W_i(t,k),\ t\geq 0,\ k\in\mathbb{Z}^d)$, $i=1,2$, be independent families of Brownian motions and consider the infinite system of coupled stochastic integral equations

$$\begin{aligned}u_t(k)&=u_0(k)+\int_0^t(u_sQ)(k)ds+\int_0^t(\gamma u_s(k)v_s(k))^{1/2}dW_1(s,k),\\ v_t(k)&=v_0(k)+\int_0^t(v_sQ)(k)ds+\int_0^t(\gamma u_s(k)v_s(k))^{1/2}dW_2(s,k).\end{aligned}\tag{4.2}$$

Again it is established by means of Mytnik's duality that there exists a unique weak solution to (4.2).

Henceforth let Q be the q–matrix of a random walk on $\mathbb{Z}^d$. Dawson and Perkins address the question of co-existence of types. Assume that $\langle u_0,\mathbf{1}\rangle+\langle v_0,\mathbf{1}\rangle<\infty$. Then $U_t=\langle u_t,\mathbf{1}\rangle$ and $V_t=\langle v_t,\mathbf{1}\rangle$ are nonnegative martingales and hence converge a.s. to some limit U_∞ and V_∞. We say that there is co-existence of types if $\mathbb{P}_{u_0,v_0}[U_\infty>0 \text{ and } V_\infty>0]>0$. This is the case iff Q is transient (Theorem 1.2).

Mytnik's duality connects finite initial conditions with infinite initial conditions and converts Theorem 1.2 into a statement on the longtime behaviour of (u_t, v_t) for constant initial condition $u_0 \equiv u > 0$, $v_0 \equiv v > 0$: $\mathbb{P}_{(u,v)}[(u_t, v_t) \in \bullet]$ converges to an equilibrium $\mathbb{P}_{(u,v)}[(u_\infty, v_\infty) \in \bullet]$ (Theorem 1.4) and $\mathbb{P}_{(u,v)}[u_\infty(k)v_\infty(k) > 0] = 1$ if Q is transient (Theorem 1.6). If Q is recurrent then (Theorem 1.5) $k \mapsto u_\infty(k)$ and $k \mapsto v_\infty(k)$ are a.s. constant and $\mathbb{P}_{(u,v)}[u_\infty(0)v_\infty(0) > 0] = 0$. More precisely $\mathbb{P}_{(u,v)}[(u_\infty(k), v_\infty(k)) \in \bullet]$ is the hitting distribution of the set $X := (\{0\} \times [0, \infty)) \cup ([0, \infty) \times \{0\})$ of planar Brownian motion started in $(u, v) \in \mathbb{R}^2$.

Via a duality and comparison argument Cox, Klenke and Perkins (1999) generalise Theorem 1.5 and 1.6 to a class of initial states u_0, v_0 that is preserved under time evolution. This result and an abstract new-start argument are employed by Cox and Klenke (1999) to answer the question: "If Q is recurrent, there is (local) extinction of one type. However, is it always (as time evolves) the same type that is locally predominant?" No, it changes infinitely often! In fact for any $x \in X$, δ_x is a (weak) limit point of $\mathbb{P}_{(u,v)}[(u_t(k), v_t(k)) \in \bullet]$.

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Universität Erlangen-Nürnberg, Mathematisches Institut, Bismarckstrasse $1\frac{1}{2}$, 91054 Erlangen, Germany

E-mail address: klenke@mi.uni-erlangen.de

URL: http://www.mi.uni-erlangen.de/∼klenke