# INFINITE RATE MUTUALLY CATALYTIC BRANCHING[1]

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Consider the mutually catalytic branching process with finite branching rate $\gamma$. We show that as $\gamma \to \infty$, this process converges in finite-dimensional distributions (in time) to a certain discontinuous process. We give descriptions of this process in terms of its semigroup in terms of the infinitesimal generator and as the solution of a martingale problem. We also give a strong construction in terms of a planar Brownian motion from which we infer a path property of the process.

This is the first paper in a series or three, wherein we also construct an interacting version of this process and study its long-time behavior.

## 1. Introduction and main results.

1.1. *Motivation.* In [5], Dawson and Perkins introduced a population dynamic model of two populations that live on a countable site space $S$. The individuals migrate between sites and, at any given site, perform a critical branching process with a branching rate proportional to the local size of the population of the respective other type.

More precisely, Dawson and Perkins considered the system of coupled stochastic differential equations (SDEs) (taking nonnegative values)

$$(1.1) \quad dY_{i,t}(k) = (\mathcal{A}Y_{i,t})(k)\,dt + \sqrt{\gamma Y_{1,t}(k)Y_{2,t}(k)}\,dW_{i,t}(k), \qquad i = 1, 2, k \in S.$$

Here, $\mathcal{A}(k, l) = a(k, l) - \mathbb{1}_{\{k\}}(l)$ is the $q$-matrix of a Markov chain on $S$ with symmetric jump kernel $a$, $(W_i(k), k \in S, i = 1, 2)$ is an independent family of Brownian motions and $\gamma \geq 0$ is a parameter.

Dawson and Perkins showed that there exists a unique weak solution of this SDE taking values in a suitable subspace of $([0, \infty)^2)^S$ with some growth condition. Furthermore, this process is a strong Markov process. While existence of a weak solution is rather standard due to the procedure proposed by Shiga and Shimizu [16], weak uniqueness was shown using a certain self-duality of the process established in [13]. We will describe the duality in detail below, in (2.4).

A main result of Dawson and Perkins is a dichotomy in the long-time behavior of the solutions depending on whether $\mathcal{A}$ is recurrent or transient (assuming some

mild regularity condition on $\mathcal{A}$). For recurrent $\mathcal{A}$ (fulfilling the regularity assumption), the types segregate, while for transient $\mathcal{A}$, there is coexistence of types. More precisely, let

$$M_{i,t} = \sum_{k \in S} Y_{i,t}(k)$$

denote the total mass processes ($i = 1, 2$) and assume that $M_{1,0}, M_{2,0} < \infty$. Then $M_1$ and $M_2$ are continuous orthogonal nonnegative $L^2$-martingales. Let $M_{i,\infty} = \lim_{t \to \infty} M_{i,t}$ denote the almost sure limit. Dawson and Perkins show that $\mathbf{E}[M_{1,\infty} M_{2,\infty}] = 0$ if $\mathcal{A}$ is recurrent and $\mathbf{E}[M_{1,\infty} M_{2,\infty}] = M_{1,0} M_{2,0}$ if $\mathcal{A}$ is transient. Furthermore, in the recurrent case, the joint distribution of $(M_{1,\infty}, M_{2,\infty})$ equals $Q_{(M_{1,0}, M_{2,0})}$, where, for $x \in [0, \infty)^2$, $Q_x$ is the harmonic measure of planar Brownian motion in $[0, \infty)^2$. That is, if $B = (B_1, B_2)$ is a Brownian motion in $\mathbb{R}^2$ started at $x$ and $\tau = \inf\{t > 0 : B_t \notin (0, \infty)^2\}$, then $Q_x$ is the probability measure on

$$E := [0, \infty)^2 \setminus (0, \infty)^2$$

given by

(1.2) $$Q_x = \mathbf{P}_x[B_\tau \in \cdot].$$

The explicit form of the densities of $Q_x$ can be found in (2.5).

Via the self-duality of the mutually catalytic branching process, its total mass behavior for finite initial conditions provides information on the local behavior if the initial condition is infinite and sufficiently homogeneous. For $x \in [0, \infty)^2$, let $\underline{x}$ denote the state in $([0, \infty)^2)^S$ with $\underline{x}_i(k) = x_i$ for all $k \in S$, $i = 1, 2$. Assume that $Y_0 = \underline{x}$. Then

$$\lim_{t \to \infty} \mathbf{P}_{\underline{x}}[Y_{1,t}(0) Y_{2,t}(0) > 0] > 0,$$

if $\mathcal{A}$ is transient, that is, types can coexist locally. On the other hand, for recurrent $\mathcal{A}$, the distribution of $Y_t$ converges weakly to $\int \delta_{\underline{y}} Q_x(dy)$, that is, to a spatially homogeneous point $\underline{y}$, where $y$ is sampled according to the distribution $Q_x$. Hence, in the recurrent case, the two types segregate locally and form clusters. The assumption that the initial point is constant can be weakened to an ergodic random initial condition (see [3]).

The starting point for this work was the wish to obtain a quantitative description of the cluster growth in the recurrent case. We will only briefly describe the heuristics. Dawson and Perkins also constructed a version of their process in continuous space $\mathbb{R}$ instead of $S$ as the solution of a stochastic partial differential equation

(1.3) $$\frac{dY_{i,t}(r)}{dt} = \Delta Y_{i,t}(r) + \sqrt{\gamma Y_{1,t}(r) Y_{2,t}(r)} \dot{W}_i(t, r), \qquad r \in \mathbb{R}, i = 1, 2,$$

where $\dot{W}_1$ and $\dot{W}_2$ are independent space–time white noises and $\Delta$ is the Laplace operator. As $\Delta$ on $\mathbb{R}$ is recurrent, types also segregate here. Now, due to Brownian

scaling, if we denote by $Y^\gamma$ the solution of (1.3) with that given value of $\gamma$, then we obtain

$$(1.4) \qquad \mathbf{P}_{\underline{x}}\big[(Y_T^\gamma(r\sqrt{T}))_{r\in\mathbb{R}} \in \cdot\big] = \mathbf{P}_{\underline{x}}\big[(Y_1^{\gamma T}(r))_{r\in\mathbb{R}} \in \cdot\big].$$

Equation (1.4) shows that clusters of $Y_{1,T}$ grow like $\sqrt{T}$ and that a better understanding of the precise cluster formation can be obtained by letting $\gamma \to \infty$ for fixed time. Hence, we aim to construct a model $X$, that is, in some sense, the limit of $Y^\gamma$ as $\gamma \to \infty$.

In this paper, we construct $X$ in the simple case where $S$ is a singleton and where the migration between colonies is replaced by an interaction with a time-invariant mean field. This is a first step toward the investigation of the model involving infinitely many sites. We give characterizations of the process $X$ via an infinitesimal generator, as the solution of a well-posed martingale problem and as the limit of $Y^\gamma$ as $\gamma \to \infty$. Finally, we give a strong construction of the process via a time-changed planar Brownian motion. This will also serve to derive path properties.

In two forthcoming papers, we construct the infinite rate process on a countable site space $S$ via a stochastic differential equation with jump-type noise and give a characterization via a martingale problem [9]. Furthermore, we will investigate the long-time behaviour and give conditions for segregation and for coexistence of types [10]. An alternative construction via a Trotter product approach is carried out in [11] and [14].

1.2. *Results.* We now describe the one-colony process which is the subject of investigation of this paper. Assume that $S$ is a singleton and that immigration and emigration come from and go to some colony that is thought to be infinitely big and whose effective population size (for immigration) is $\theta \in [0,\infty)^2$. Furthermore, let $c \geq 0$ be the rate of migration. Hence, we consider the solution $Y = Y^{\gamma,c,\theta}$ of the stochastic differential equation

$$(1.5) \qquad dY_{i,t} = c(\theta_i - Y_{i,t})\,dt + \sqrt{\gamma Y_{1,t} Y_{2,t}}\,dW_{i,t}, \qquad i = 1, 2.$$

This model can be thought of as a version of the model defined in (1.1) where the migration between colonies is replaced by an interaction with a time-invariant mean field $\theta$ or with an infinitely large reservoir whose types have proportions $\theta_1$ and $\theta_2$. (In fact, in [2] it was shown (Proposition 1.1) that $Y^{\gamma,c,\theta}$ arises as the McKean–Vlasov limit of solutions of (1.1) with symmetric interaction on a complete graph $S$.) More formally, the interaction term $\mathcal{A}Y$ is replaced by a drift $c(\theta_i - Y_{i,t})$. It is this simplification of the interaction that allows for a tractable exposition in this article. Note that as $t \to \infty$, the process without drift ($c = 0$) converges almost surely to some random $x \in E$. Hence, in the case $c = 0$, if we let $\gamma \to \infty$, then the limiting process would be trivial: if it starts at $x \in E$, then it stays at $x$ forever. See Section 2 for a more detailed description of the process $Y$ solving (1.5) (finite $\gamma$ process).

On a heuristic level, as the stochastic term in (1.5) defines an isotropic two-dimensional diffusion, that is, a time-transformed planar Brownian motion, if we let $\gamma \to \infty$, then we should end up with a process where the stochastic part is a planar Brownian motion at infinite speed, stopped when it reaches the boundary of the upper-right quadrant. That is, the limiting process $X$ should be a Markov process with values in $E$. When $x$ is the current state and the drift moves it to $x + c(\theta - x)\,dt$, this point should instantaneously be replaced by a random point chosen according to $Q_{x+c(\theta-x)\,dt}$. We will, in fact, be able to describe this infinitesimal dynamics both in terms of a martingale problem and in terms of a generator of Markov transition kernels. However, we first define $X$ via an explicit transition semigroup and show that it is the limit of $Y^{\gamma,c,\theta}$ as $\gamma \to \infty$. Let

$$(1.6) \quad C_l(E) := \left\{ f : E \to \mathbb{C} \text{ is cont. and } \lim_{u \to \infty} f(u,0) = \lim_{v \to \infty} f(0,v) \text{ is finite} \right\}$$

equipped with the supremum norm $\|f\|_\infty = \sup_{x \in E} |f(x)|$.

DEFINITION 1.1. Let $c \geq 0$ and $\theta \in [0, \infty)^2$. For $t \geq 0$ and $x \in E$, define the stochastic kernel $p_t$ by

$$p_t(x, \cdot) := p_t^{c,\theta}(x, \cdot) := Q_{e^{-ct}x + (1 - e^{-ct})\theta}.$$

Define the contraction semigroup $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$ on $C_l(E)$ by

$$\mathcal{S}_t f(x) = \int_E f(y) p_t(x, dy).$$

The Markov process $X = X^{c,\theta}$ with state space $E$, càdlàg paths and transition kernels $(p_t)_{t \geq 0}$ is called the infinite rate mutually catalytic branching process (IMUB) with parameters $(c, \theta)$.

In order for this definition to make sense, in Proposition 3.2, we will show that $(\mathcal{S}_t)_{t \geq 0}$ is, in fact, a Markov semigroup.

PROPOSITION 1.2. $X^{c,\theta}$ is a Feller process and has the strong Markov property. It is ergodic and the unique invariant measure is $Q_\theta$.

PROOF. The map $x \mapsto Q_x$ is continuous, hence $x \mapsto p_t(x, \cdot)$ is also continuous, that is, $X^{c,\theta}$ is a Feller process. Since $Q_x = \delta_x$ for $x \in E$, the semigroup $\mathcal{S}$ is strongly continuous. Hence, by the general theory of Markov processes, there exists a càdlàg version of $X$ that is strong Markov (see, e.g., [15], Chapters III.7 and 8).

Ergodicity and the explicit form of the invariant measure are trivial. □

THEOREM 1.3 ($X^{c,\theta}$ as an infinite rate process). Assume that $Y_0^{\gamma,c,\theta} = X_0^{c,\theta} = x \in E$ for all $\gamma \geq 0$. As $\gamma \to \infty$, the finite-dimensional distributions of $Y^{\gamma,c,\theta}$ converge to those of $X^{c,\theta}$.

Note that in Theorem 1.3, trivially, we do not have convergence in the Skorohod path space, since continuous processes do not converge to discontinuous processes in that topology.

In addition to the convergence of the finite-dimensional distributions, we also have convergence of the $p$th moments for $p \in [1, 2)$ [but not for $p = 2$, of course, since for $x \in (0, \infty)^2$, the measure $Q_x$ does not possess finite second moments, as can be easily derived from its density formula (2.5)]. Hence, on a suitable probability space, we have $L^p$-convergence of $Y^{\gamma, c, \theta}$ to $X^{c, \theta}$.

THEOREM 1.4 ($L^p$-convergence). *Assume that* $Y_0^{\gamma, c, \theta} = X_0^{c, \theta} = x \in E$ *for all* $\gamma \geq 0$ *and let* $p \in [1, 2)$, $t \geq 0$.

  (i) *For every* $\gamma \geq 0$ *and* $i = 1, 2$, *we have*

$$\mathbf{E}_x[(Y_{i,t}^{\gamma, c, \theta})^p] \leq \mathbf{E}_x[(X_{i,t}^{c, \theta})^p] < \infty.$$

  (ii) *On a suitable probability space, for* $i = 1, 2$, *we have*

$$Y_{i,t}^{\gamma, c, \theta} \overset{\gamma \to \infty}{\longrightarrow} X_{i,t}^{c, \theta} \qquad in\ L^p.$$

It can be seen from the proofs of Theorems 1.3 and 1.4 that the statements of these theorems also hold for $Y_0^{\gamma, c, \theta} = x \in [0, \infty)^2$ and $t > 0$ if we replace $X_0^{c, \theta}$ by a random point chosen according to $Q_x$.

REMARK 1.5 (Trotter product approach). While in the one-colony case considered in this paper, it is easy to explicitly write down the semigroup for the infinite rate mutually catalytic branching process $X^{c, \theta}$, it is less obvious how to construct an interacting version of the process on a countable site space. One possibility is the Trotter product approach that is used in [11] and [14]. Here, we briefly sketch it for $X^{c, \theta}$.

In the classical setting, the Trotter product approach works as follows. In order to construct a solution $Y^{\gamma, c, \theta}$ of (1.5), in time intervals of length $\varepsilon$, one could alternate between a solution of the pure drift equation ($\gamma = 0$) and the pure stochastic noise equation ($c = 0$). As $\varepsilon \downarrow 0$, this process converges to a solution of (1.5).

If we let $\gamma \to \infty$, then the noise term results in an instantaneous jump to a point in $E$ chosen according to $Q_y$, where $y$ is the value of $Y$ at the end of the preceding "drift interval." More formally, let $(\xi(k, x), k \in \mathbb{N}, x \in [0, \infty)^2)$ be an independent family of $E$-valued random variables with distribution $\mathcal{L}[\xi(k, x)] = Q_x$. For $t \in [k\varepsilon, (k + 1)\varepsilon)$, let $X_t^\varepsilon$ be the solution of the differential equation

$$dX_t^\varepsilon = c(\theta - X_t)\, dt,$$

that is,

$$X_t^\varepsilon = e^{-c(t-k\varepsilon)} X_{k\varepsilon}^\varepsilon + (1 - e^{-c(t-k\varepsilon)})\theta.$$

Let

$$X^\varepsilon_{(k+1)\varepsilon-} := \lim_{t\uparrow(k+1)\varepsilon} X^\varepsilon_t = e^{-c\varepsilon} X^\varepsilon_{k\varepsilon} + (1 - e^{-c\varepsilon})\theta$$

and define

$$X^\varepsilon_{(k+1)\varepsilon} = \xi\big(k+1, X^\varepsilon_{(k+1)\varepsilon-}\big).$$

One can prove that $X^\varepsilon$ converges in distribution in the Skorohod topology on the space of càdlàg paths to $X^{c,\theta}$ (see [11] and [14]).

While, in Definition 1.1, we gave an explicit formula for the transition kernels of $X$, it is also interesting to characterize the process $X$ via its infinitesimal dynamics. In Section 5, we investigate the generator $\bar{\mathcal{G}}$ of the semigroup $\mathcal{S}$. For a certain class $C_l^2(E) \subset C_l(E)$ of smooth functions $f$ (see Definition 5.1), we give an explicit formula for $\bar{\mathcal{G}}f$ as an integro-differential operator. Using the classical Hille–Yoshida theorem, we show that the restricted operator $\mathcal{G} = \bar{\mathcal{G}}|_{C_l^2(E)}$ uniquely defines $(\mathcal{S}_t)_{t\geq0}$ (Theorem 5.3). Furthermore, we show that $\mathcal{G}$ restricted to an even smaller space $V$ of functions that appear in the duality for $X$ still uniquely defines the process $X$ via a martingale problem (Theorem 5.4). To define $\mathcal{G}$, it is crucial to study (for suitable functions $f$) the limit

$$\lim_{t\downarrow0} t^{-1}\big(\mathcal{S}_t f(x) - f(x)\big) = \lim_{\varepsilon\to0} \varepsilon^{-1}\bigg(\int f\,dQ_{x+\varepsilon c(\theta-x)} - f(x)\bigg),$$

which will also clarify the jump structure of the process $X$. The description of the exact form of the operator $\mathcal{G}$ and the precise statements of the theorems are a bit technical, so these are deferred to Section 5.

While, for Proposition 1.2, we used general construction principles of Markov processes, here, we provide an explicit strong construction of the process $X$ in terms of a given planar Brownian motion $B$. This construction also allows certain path properties to be investigated.

Assume $B_0 = 0$. For $z \in \mathbb{R}^2$, we write

$$[z,\infty) = [z_1,\infty) \times [z_2,\infty)$$

for the rectangular cone northeast of $z$. For $x \in [0,\infty)^2$, let

(1.7) $$\tau_x := \inf\{t > 0 : B_t \notin [-x,\infty)\}$$

and

(1.8) $$D_x := B_{\tau_x} + x \in E.$$

For $x, y \in \mathbb{R}^2$, we write $y \leq x$ if $x \in [y,\infty)$, that is, if $y_1 \leq x_1$ and $y_2 \leq x_2$. For $x \in [0,\infty)^2$, we define the $\sigma$-algebra

(1.9) $$\mathcal{F}_x^D = \sigma(D_y : y \leq x).$$

In Lemma 3.1, we will show that $D$ is a Markov process with respect to $(\mathcal{F}_x^D)_{x \in [0,\infty)^2}$.

Let $\bar{\theta} : [0,\infty) \to [0,\infty)^2$ and $\bar{c} : [0,\infty) \to [0,\infty)$ be measurable and locally integrable. For $0 \le s \le t$, define

$$(1.10) \quad C(s,t) = \exp\left( -\int_s^t \bar{c}(r)\,dr \right) \quad \text{and} \quad \Xi(s,t) = \int_s^t \frac{\bar{\theta}(r)}{C(0,r)}\,dr.$$

THEOREM 1.6.   *Let $x \in E$ and define the process $X^{\bar{c},\bar{\theta}}$ by*

$$X_t^{\bar{c},\bar{\theta}} = C(0,t)D_{x+\Xi(0,t)}, \qquad t \ge 0.$$

*Then $X^{\bar{c},\bar{\theta}}$ is a time-inhomogeneous Markov process on $E$ with càdlàg paths and with transition probabilities*

$$(1.11) \qquad p_{s,t}(z,\cdot) = Q_{C(s,t)z + C(0,t)\Xi(s,t)} \qquad \text{for } 0 \le s < t, z \in E.$$

*In particular, for $\bar{\theta} \equiv \theta \in [0,\infty)^2$ and $\bar{c} \equiv c > 0$,*

$$(1.12) \qquad\qquad X_t^{c,\theta} = e^{-ct} D_{x+(e^{ct}-1)\theta}$$

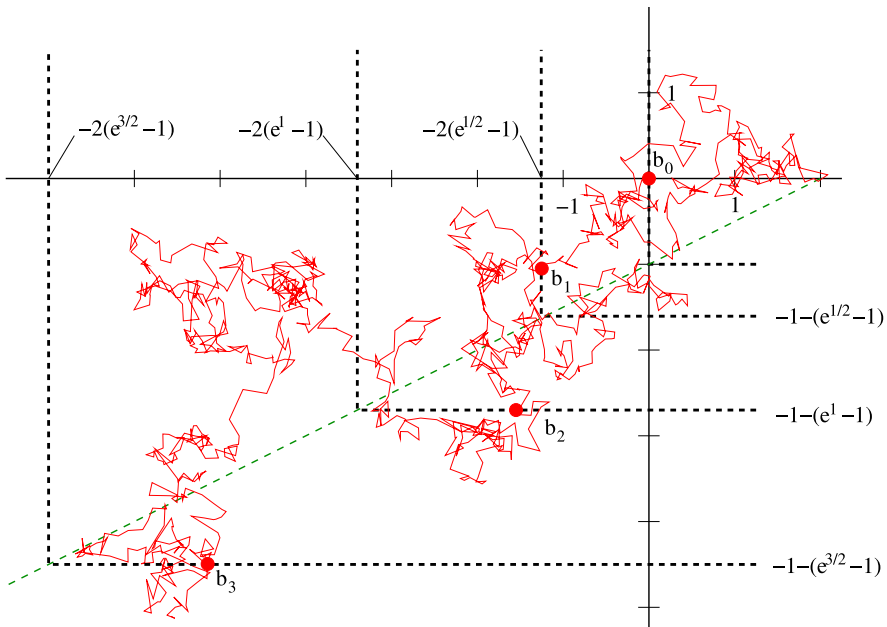*is an infinite rate mutually catalytic branching process with parameter $(c,\theta)$, see Figure 1.*



FIG. 1.   *Strong construction of $X^{1/2,(2,1)}$ with $X_0 = x = (0,1)$ via a planar Brownian motion. Here $X_t^{1/2,(2,1)} = e^{-t/2}((0,1) + b_t + (2,1)(e^{t/2}-1))$ for $t = 0,1,2,3$.*

It is tempting to use this strong construction of $X^{\bar{c},\bar{\theta}}$ in order to define an interacting version of the infinite rate mutually catalytic branching process on a countable site space $S$, where $c\theta_k(t)$ at site $k \in S$ reflects the migration from neighboring sites to $k$. However, in this paper, we do not pursue this topic. Rather, we use the strong construction in order to derive a path property of $X^{c,\theta}$ via a result of Le Gall and Meyre [12] on the cone points of planar Brownian motion.

Recall that a measurable set $A \subset E$ is called *polar* for $X^{c,\theta}$ if for all $x \in E$, we have

$$\mathbf{P}_x[X_t^{c,\theta} \in A \text{ for some } t > 0] = 0.$$

THEOREM 1.7. *The point $0 \in E$ is polar for $X^{c,\theta}$.*

1.3. *Organization of the paper.* In Section 2, we give a detailed description of the duality for the process with finite branching rate. In Section 3, we establish a similar duality for the infinite rate process and use it in order to show the convergence in Theorems 1.3 and 1.4. In Section 4, we justify the strong construction of Theorem 1.6 and also prove Theorem 1.7. Finally, in Section 5, we describe the infinite rate process in terms of its infinitesimal dynamics and state and prove the theorem on the construction via the Hille–Yoshida theory (Theorem 5.3) and via a martingale problem (Theorem 5.4).

**2. Duality of the finite $\gamma$ process.** A major tool for the investigation of mutually catalytic branching processes is a self-duality for the process. As it turns out to be crucial also for the limiting case of infinite branching rate ($\gamma = \infty$), we describe this duality here in more detail. For $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathbb{R}^2$, we introduce the *lozenge product*

$$(2.1) \qquad x \diamond y := -(x_1 + x_2)(y_1 + y_2) + i(x_1 - x_2)(y_1 - y_2)$$

(with $i = \sqrt{-1}$) and define

$$(2.2) \qquad F(x, y) = \exp(x \diamond y).$$

Note that $x \diamond y = y \diamond x$. Furthermore, define the "scalar product"

$$(2.3) \qquad \langle x, y \rangle = x_1 y_1 + x_2 y_2 \qquad \text{for } x, y \in [0, \infty)^2.$$

For $x = (x(k))_{k \in S}$ and $y = (y(k))_{k \in S}$, we write

$$H(x, y) = \exp\left(\sum_{k \in S} x(k) \diamond y(k)\right).$$

If $Y$ is the process defined in (1.1) started in state $y$ and $\tilde{Y}$ is the process started in some suitable $\tilde{y}$ (such that all sums are finite), then the duality reads (see [13], equation (2.5))

$$(2.4) \qquad \mathbf{E}_y[H(Y_t, \tilde{y})] = \mathbf{E}_{\tilde{y}}[H(y, \tilde{Y}_t)].$$

In fact, this duality also holds for asymmetric $\mathcal{A}$ if $\tilde{Y}$ is a solution of (1.1) with $\mathcal{A}$ replaced by its transpose $\mathcal{A}^*$. As this mixed Laplace and Fourier transform $H$ is measure determining ([13], Lemma 2.5), the duality yields uniqueness of the solutions of (1.1). Furthermore, it provides a tool for translating local properties of the solutions into global properties and vice versa. If $x = (u, v) \in (0, \infty)^2$, then the harmonic measure $Q_x$ [recall (1.2)] has a one-dimensional Lebesgue density on

$$E := \big([0, \infty) \times \{0\}\big) \cup \big(\{0\} \times [0, \infty)\big)$$

that can be computed explicitly

$$(2.5) \quad Q_{(u,v)}(d(\bar{u}, \bar{v})) = \begin{cases} \dfrac{4}{\pi} \dfrac{uv\bar{u}}{4u^2v^2 + (\bar{u}^2 + v^2 - u^2)^2}\, d\bar{u}, & \text{if } \bar{v} = 0, \\[3mm] \dfrac{4}{\pi} \dfrac{uv\bar{v}}{4u^2v^2 + (\bar{v}^2 + u^2 - v^2)^2}\, d\bar{v}, & \text{if } \bar{u} = 0. \end{cases}$$

Furthermore, trivially we have

$$(2.6) \qquad\qquad Q_x = \delta_x \qquad \text{if } x \in E.$$

We now turn to the situation of only one colony. We consider the solution $Z = (Z_1, Z_2)$ of

$$(2.7) \quad dZ_{i,t} = \sqrt{\gamma Z_{1,t} Z_{2,t}}\, dW_{i,t}, \qquad i = 1, 2, \qquad Z_0 = z \in [0, \infty)^2.$$

By Theorem 1 of [4], there is the unique strong solution to the above equation.

Clearly, $Z_1$ and $Z_2$ are orthogonal $L^2$-martingales and hence they converge almost surely to some random variable $Z_\infty = (Z_{1,\infty}, Z_{2,\infty})$. As $Z$ is an isotropic diffusion on $[0, \infty)^2$, it is a time-transformed Brownian motion. Thus $Z_\infty$ has the same distribution as a planar Brownian motion $B$ started at $z$ and stopped (at time $\tau$) upon leaving $(0, \infty)^2$, that is [see (2.5)],

$$\mathcal{L}_z[Z_\infty] = \mathcal{L}_z[B_\tau] = Q_z.$$

(We denote by $\mathcal{L}_x[X_t] = \mathbf{P}_x[X_t \in \cdot] = \mathbf{P}[X_t \in \cdot \mid X_0 = x]$ the distribution of the process $X$ at time $t$ when started at $x$.) It is easy to see that, in fact,

$$\tau^Z := \inf\{t > 0 : Z_t \in E\} < \infty \qquad \text{almost surely,}$$

and that

$$Z_t = Z_{\tau^Z} \qquad \text{for all } t > \tau^Z.$$

Clearly, increasing $\gamma$ amounts to speeding up the process. Hence, in the limit, we would have a process that instantaneously jumps from $z$ to a random point (picked according to $Q_z$) and then stays there. In order to obtain a more interesting limiting process, and with a view toward interacting colonies, we introduce a

drift term and consider the equation (which was analyzed in more detail in [2], Propositions 1.1 and 1.2)

$$(2.8) \qquad dY_{i,t} = c(\theta_i - Y_{i,t})\,dt + \sqrt{\gamma Y_{1,t} Y_{2,t}}\,dW_{i,t}, \qquad i = 1, 2.$$

Here, $c \geq 0$ and $\theta \in [0, \infty)^2$ are parameters of the process. It is standard to show that (2.8) has a weak solution. Weak uniqueness can be obtained via duality. We first outline the general picture for the duality that comes from the interacting colonies case and then give an explicit computation for our special situation.

Let us consider a two-colonies model with site space $S = \{1, 2\}$, where $Y$ is the size of the population at site 1 and the size of the population at site 2 is constant and equals $\theta$. This amounts to a migration matrix

$$(2.9) \qquad \mathcal{A} = \begin{pmatrix} -c & c \\ 0 & 0 \end{pmatrix}$$

and to branching rates $\gamma(1) = \gamma$ (at site 1) and $\gamma(2) = 0$ (at site 2). Note that the approach of Dawson and Perkins does not require that the branching rate be constant; neither does it require that the migration matrix be symmetric or a $q$-matrix. (At least if $S$ is finite—otherwise, certain regularity conditions have to be imposed.) Dawson and Perkins use a duality with respect to a process $\tilde{Y}$ with migration matrix $\mathcal{A}^*$ (the transpose of $\mathcal{A}$) and with the same branching rates as $Y$ to show weak uniqueness of $Y$.

Let us now construct the dual process explicitly. We will later use this approach in order to construct a dual for the $\gamma = \infty$ limiting process. Let $\tilde{y} = (\tilde{y}(1), \tilde{y}(2)) \in ([0, \infty)^2)^2$ and let $Z$ be the unique strong (by Theorem 1 of [4]) $[0, \infty)^2$-valued solution of

$$(2.10) \qquad dZ_{i,t} = \sqrt{\gamma Z_{1,t} Z_{2,t}}\,dW_{i,t}, \qquad i = 1, 2, \qquad Z_0 = \tilde{y}(1).$$

Define a process $\tilde{Y}$ on $([0, \infty)^2)^2$ by

$$(2.11) \qquad \tilde{Y}_t(1) = e^{-ct} Z_t \quad \text{and} \quad \tilde{Y}_t(2) = \tilde{y}(2) + \int_0^t ce^{-cr} Z_r\,dr.$$

Note that this $\tilde{Y}$ is a solution of (1.1) with $S = \{1, 2\}$, with site-dependent branching rate $\gamma(1) = \gamma$, $\gamma(2) = 0$ and with $\mathcal{A}$ from (2.9) replaced by $\mathcal{A}^*$. In particular, $\tilde{Y}$ is a time-homogeneous Markov process. We also get the time-homogeneous Markov property via an explicit computation:

$$\tilde{Y}_{t+s} = \left( e^{-c(t+s)} Z_{t+s}, \tilde{y}(2) + \int_0^{t+s} ce^{-cr} Z_r\,dr \right)$$

$$= \left( e^{-cs}(e^{-ct} Z_{t+s}), \tilde{y}(2) + \int_0^t ce^{-cr} Z_r\,dr + \int_0^s ce^{-cr}(e^{-ct} Z_{t+r})\,dr \right)$$

$$= \left( e^{-cs} Z'_s, \tilde{y}'(2) + \int_0^s ce^{-cr} Z'_r\,dr \right),$$

where $Z'_r = e^{-ct} Z_{t+r}$ and $\tilde{y}'(2) = \tilde{Y}_t(2) = \tilde{y}(2) + \int_0^t ce^{-cr} Z_r \, dr$. Clearly, $Z'$ has the distribution of a solution of (2.7) with $\tilde{y}'(1) := Z'_0 = \tilde{Y}_t(1)$.

For $x, x', y, y' \in [0, \infty)^2$, recall that

$$(2.12) \qquad H((x, x'), (y, y')) = F(x, y)F(x', y').$$

PROPOSITION 2.1 (Duality). *Let $Y$ and $\tilde{Y}$ be defined by (2.8) and (2.11), respectively. Then, for all $y \in [0, \infty)^2$, $\tilde{y} \in ([0, \infty)^2)^2$ and $t \geq 0$, we have*

$$(2.13) \qquad \mathbf{E}_y[H((Y_t, \theta), \tilde{y})] = \mathbf{E}_{\tilde{y}}[H((y, \theta), \tilde{Y}_t)].$$

*In particular, if $Z$ is a solution of (2.10) with $Z_0 = z \in [0, \infty)^2$, then*

$$(2.14) \qquad \mathbf{E}_y[F(Y_t, z)] = \mathbf{E}_z\left[F(y, e^{-ct} Z_t)F\left(\theta, \int_0^t ce^{-cr} Z_r \, dr\right)\right].$$

A similar duality was derived in [2], Lemma 4.2. Before we prove the proposition, we have to collect some properties of the derivatives of $F$. We omit the proof of the following lemma.

LEMMA 2.2 (Derivatives of the duality function). *Denote the partial derivatives of $F$ by*

$$\nabla_1 F(x, y) := \frac{d}{dx} F(x, y), \qquad \nabla_2 F(x, y) := \frac{d}{dy} F(x, y)$$

*and define the Laplace operators $\Delta_1$ and $\Delta_2$ by*

$$\Delta_1 F(x, y) := \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] F(x, y), \qquad \Delta_2 F(x, y) := \left[\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right] F(x, y).$$

*Then, for all $x, y, z \in [0, \infty)^2$, we have [recall (2.1) and (2.3)]*

$$\langle z, \nabla_1 F(x, y)\rangle = (z \diamond y)F(x, y),$$
$$\langle z, \nabla_2 F(x, y)\rangle = (z \diamond x)F(x, y),$$
$$\Delta_1 F(x, y) = 8y_1 y_2 F(x, y),$$
$$\Delta_2 F(x, y) = 8x_1 x_2 F(x, y).$$

PROOF OF PROPOSITION 2.1. We use Itô's formula and Lemma 2.2 to compute the derivatives of both sides of (2.13) at $t = 0$:

$$
\begin{aligned}
\frac{d}{dt} &\mathbf{E}_y[H((Y_t, \theta), \tilde{y})]|_{t=0} \\
&= \langle c(\theta - y), \nabla_1 F(y, \tilde{y}(1))\rangle F(\theta, \tilde{y}(2)) \\
&\quad + \frac{1}{2}\gamma y_1 y_2 \Delta_1 F(y, \tilde{y}(1))F(\theta, \tilde{y}(2)) \\
&= H((y, \theta), \tilde{y})[c(\theta - y) \diamond \tilde{y}(1) + 4\gamma y_1 y_2 \tilde{y}_1(1)\tilde{y}_2(1)]
\end{aligned}
$$

(2.15)

and

$$\frac{d}{dt}\mathbf{E}_{\tilde{y}}[H((y, \theta), \tilde{Y}_t)]|_{t=0}$$

$$= F(\theta, \tilde{y}(2))\Big(\langle -c\tilde{y}(1), \nabla_2 F(y, \tilde{y}(1))\rangle$$

(2.16)
$$+ \frac{\gamma}{2}\tilde{y}_1(1)\tilde{y}_2(1)\Delta_2 F(y, \tilde{y}(1))\Big)$$

$$+ F(y, \tilde{y}(1))\langle c\tilde{y}(1), \nabla_2 F(\theta, \tilde{y}(2))\rangle$$

$$= H((y, \theta), \tilde{y})[c(\theta - y) \diamond \tilde{y}(1) + 4\gamma y_1 y_2 \tilde{y}_1(1)\tilde{y}_2(1)].$$

Since the two derivatives coincide, (2.13) holds (see Corollary 4.4.13 of [6] with $\alpha = \beta = 0$). Equation (2.14) is a direct consequence of (2.13). $\square$

COROLLARY 2.3. *Recall Z defined by* (2.10).

(i) *Taking $c = 0$, Proposition* 2.1 *implies that Z is self-dual*:

$$\mathbf{E}_x[F(Z_t, y)] = \mathbf{E}_y[F(x, Z_t)] \qquad \text{for all } x, y \in [0, \infty)^2, t \geq 0.$$

(ii) *Letting $t \to \infty$ in (i) and recalling that $\mathcal{L}_x[Z_t] \overset{t\to\infty}{\Longrightarrow} Q_x$, we get, by dominated convergence, the duality relation for the harmonic measure*:

$$\int_E F(z, y)Q_x(dz) = \int_E F(x, z)Q_y(dz) \qquad \text{for all } x, y \in [0, \infty)^2.$$

(iii) *In particular (since $Q_x = \delta_x$ for $x \in E$ and due to the symmetry of F), for all $x \in E$ and $y \in [0, \infty)^2$, we have*

$$\int_E F(x, z)Q_y(dz) = F(x, y) = F(y, x) = \int_E F(z, x)Q_y(dz).$$

COROLLARY 2.4. (i) *The family of functions*

$$\mathcal{F}_0 = \{[0, \infty)^2 \to \mathbb{C} : x \mapsto F(x, y), y \in [0, \infty)^2\}$$

*is measure determining for $[0, \infty)^2$.*

(ii) *The vector space*

(2.17)    $$V := \left\{ \sum_{m=1}^{n} \lambda_m F(\cdot, z_m) : n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in \mathbb{C}, z_1, \ldots, z_n \in E \right\}$$

*spanned by $\mathcal{F} := \{E \to \mathbb{C} : x \mapsto F(x, z), z \in E\}$ is dense in $C_l(E)$. In particular, $\mathcal{F}$ is measure determining for $E$.*

PROOF.    Let $\mathcal{D}_0$ be the algebra generated by $\mathcal{F}_0$. Clearly, $\mathcal{F}_0$ separates points of $[0, \infty)^2$, contains $1 = F(\cdot, 0)$ and is closed under multiplication and under complex conjugation since $\overline{F(x, (y_1, y_2))} = F(x, (y_2, y_1))$. Hence, by the Stone–Weierstrass theorem, $\mathcal{D}_0$ is dense in the space $C_l([0, \infty)^2)$ of functions $[0, \infty)^2 \to \mathbb{C}$ that are continuous and have a limit at infinity. As $\mathcal{F}_0$ is closed under multiplication, $\mathcal{D}_0$ is the vector space spanned by $\mathcal{F}_0$ and thus $\mathcal{F}_0$ is measure determining on $[0, \infty)^2$.

Let $\mathcal{F}_E = \{f|_E : f \in \mathcal{F}_0\} \supset \mathcal{F}$ and let $\mathcal{D}_E = \{f|_E : f \in \mathcal{D}_0\}$ denote the algebra generated by $\mathcal{F}_E$. By the above argument, $\mathcal{D}_E \subset C_l(E)$ is dense. Now, by Corollary 2.3(iii), an element $F(\cdot, y) \in \mathcal{F}_E$ can be written as the integral $F(x, y) = \int F(x, z) Q_y(dz)$, where the integrand functions are in $\mathcal{F}$. The integral can be approximated (uniformly in $x$) by finite sums, that is, by elements of $V$. Hence, $V$ is dense in $\mathcal{D}_E$ and thus also in $C_l(E)$.    □

Apparently, $Y$ is ergodic and has a unique invariant distribution with a Lebesgue density on $(0, \infty)^2$. Unlike for the analogous one-dimensional equation

$$dU_t = c(b - U_t)\, dt + \sqrt{\gamma U_t}\, dW_t,$$

where the invariant distribution is known to be the Gamma distribution $\Gamma_{2c/\gamma, 2cb/\gamma}$, here, the explicit form of the density is unknown. It is known (see, e.g., [7], Example IV.8.2, page 237) that $U$ hits 0 if and only if $2cb/\gamma < 1$. Hence, we may expect that $Y = Y^{\gamma, c, \theta}$ hits $E$ only at $((2c\theta_2/\gamma, \infty) \times \{0\}) \cup (\{0\} \times (2c\theta_1/\gamma, \infty))$. Compare this with the fact that $0 \in E$ is not hit by the infinite $\gamma$ process $X^{c, \theta}$ (see Theorem 1.7).

## 3. Convergence as $\gamma \to \infty$: Proofs of Theorems 1.3, 1.4.

3.1. *Construction of the process.*    Recall the definitions of $p_t$, $\mathcal{S}$ and $X^{c, \theta}$ in Definition 1.1. In order for the definition to make sense, we still have to show, in Proposition 3.2 below, that $p_t$ is indeed a Markov kernel and that the Chapman–Kolmogorov equation holds. We prepare for Proposition 3.2 with a lemma.

Recall the definitions of $C$, $\Xi$, $D$ and $\mathcal{F}^D$ in (1.8), (1.9) and (1.10).

LEMMA 3.1.    (i) *$D$ has the Markov property, that is, for $x, y \in [0, \infty)^2$ and $A \subset E$ measurable, we have*

$$\mathbf{P}[D_{x+y} \in A \mid \mathcal{F}_x^D] = Q_{y+D_x}(A).$$

(ii) *For $f : E \to \mathbb{C}$ bounded and measurable and $r \geq 0$, we have*

$$\int_E f(rz) Q_x(dz) = \int_E f\, dQ_{rx}.$$

(iii) *Furthermore,*

$$\int_E Q_x(dz) Q_{rz+y} = Q_{rx+y}.$$

PROOF. (i) Let $\mathcal{F}^B$ denote the filtration generated by the Brownian motion $B$ and let $\mathcal{F}^B_{\tau_x}$ denote the $\sigma$-algebra of the $\tau_x$ past of $B$ [recall (1.7)]. Note that $\mathcal{F}^B_{\tau_x} \supset \mathcal{F}^D_x$.

For $x' \in [0, \infty)^2$, denote by $\mathbf{P}_{-x'}$ the law of $B$ when started at $B_0 = -x'$. Hence, by spatial homogeneity, for $x' \leq x$, we have

$$\mathbf{P}_{-x'}[B_{\tau_{x+y}} + (x + y) \in A] = Q_{y+(x-x')}(A).$$

Choosing $x' = -B_{\tau_x}$, we infer that

$$\mathbf{P}_{B_{\tau_x}}[B_{\tau_{x+y}} + (x + y) \in A] = Q_{y+D_x}(A).$$

We now apply the strong Markov property of $B$ to obtain

$$\mathbf{P}[D_{x+y} \in A \mid \mathcal{F}^D_x] = \mathbf{E}\big[\mathbf{P}_0[B_{\tau_{x+y}} + (x + y) \in A \mid \mathcal{F}^B_{\tau_x}] \mid \mathcal{F}^D_x\big]$$

$$= \mathbf{E}\big[\mathbf{P}_{B_{\tau_x}}[B_{\tau_{x+y}} + (x + y) \in A] \mid \mathcal{F}^D_x\big]$$

$$= \mathbf{E}[Q_{y+D_x}(A) \mid \mathcal{F}^D_x] = Q_{y+D_x}(A).$$

(ii) This follows from the spatial homogeneity of $B$.

(iii) Recall that $D_{rx}$ has distribution $Q_{rx}$. Hence, by (ii) and (i), we get

$$\int_E Q_x(dz) Q_{rz+y}(A) = \int_E Q_{rx}(dz) Q_{z+y}(A)$$

$$= \mathbf{E}[Q_{y+D_{rx}}(A)] = \mathbf{P}[D_{rx+y} \in A]$$

$$= Q_{rx+y}(A). \qquad \square$$

PROPOSITION 3.2. $(\mathcal{S}_t)_{t\geq 0}$ defined in Definition 1.1 is a Markov semigroup.

PROOF. Recall that $x \mapsto Q_x$ is a continuous map. Hence, for open sets $A \subset E$, the map $x \mapsto Q_x(A)$ is lower semicontinuous, by the portmanteau theorem (see, e.g., [8], Theorem 13.16), and is hence measurable. Hence, $x \mapsto Q_x(A)$ is measurable for all Borel sets $A \subset E$. It remains to check the Chapman–Kolmogorov equation for $(p_t)$. By Lemma 3.1(iii), we infer that

$$\int_E p_t(x, dy) p_s(y, \cdot) = \int_E Q_{e^{-ct}x+(1-e^{-ct})\theta}(dy) Q_{e^{-cs}y+(1-e^{-cs})\theta}$$

$$= Q_{e^{-c(t+s)}x+e^{-cs}(1-e^{-ct})\theta+(1-e^{-cs})\theta}$$

$$= Q_{e^{-c(t+s)}x+(1-e^{-c(t+s)})\theta}$$

$$= p_{t+s}(x, \cdot). \qquad \square$$

3.2. *Duality and proof of finite-dimensional distributions convergence* (*Theorem* 1.3). In this section, we prove the convergence of the finite-dimensional distributions of $Y^{\gamma,c,\theta}$ to those of $X = X^{c,\theta}$ by means of a duality relation. For $Y^{\gamma,c,\theta}$, we have already established the duality, in Proposition 2.1. We now come to the duality for $X$. Recall the definition of $\tilde{Y}$ from (2.11). We will need as initial values only $\tilde{y} \in E \times [0, \infty)^2$. Note that, in this case, the process $Z$ is constant in time and the process $\tilde{Y}$ is given by the deterministic equation

$$(3.1) \qquad \tilde{Y}_t = \left(e^{-ct}\tilde{y}(1), (1 - e^{-ct})\tilde{y}(1) + \tilde{y}(2)\right).$$

Hence, $\tilde{Y}$ can be understood as a deterministic Markov process with state space $E \times [0, \infty)^2$. Recall $H$ from (2.12) and $F$ from (2.2).

PROPOSITION 3.3. *$X$ and $\tilde{Y}$ are dual in the sense that for all initial conditions $X_0 = x \in E$, $\tilde{Y}_0 = \tilde{y} \in E \times [0, \infty)^2$ and for all $t \geq 0$, we have*

$$(3.2) \qquad \mathbf{E}_x[H((X_t, \theta), \tilde{y})] = \mathbf{E}_{\tilde{y}}[H((x, \theta), \tilde{Y}_t)].$$

*In particular, we get*

$$(3.3) \quad \mathbf{E}_x[F(X_t, z)] = F(x, e^{-ct}z)F\left(\theta, (1 - e^{-ct})z\right) \qquad \text{for } x \in [0, \infty)^2, z \in E,$$

*and the distribution of $X_t$ is determined by* (3.3).

PROOF. As $\tilde{Y}$ is deterministic, (3.2) and (3.3) are equivalent and so we only need to show (3.3). Since $z \in E$, by Corollary 2.3(iii), the left-hand side of (3.3) equals

$$\int_E F(y, z) Q_{e^{-ct}x + (1-e^{-ct})\theta}(dy) = F\left(e^{-ct}x + (1 - e^{-ct})\theta, z\right)$$

$$= F(x, e^{-ct}z)F\left(\theta, (1 - e^{-ct})z\right).$$

By Corollary 2.4, equation (3.3) determines the distribution of $X_t$. □

We are now ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. As both $X^{c,\theta}$ and $Y^{\gamma,c,\theta}$ are Markov processes, it is easy to see that for convergence of finite-dimensional distributions, it is enough to show that for any $t \geq 0$, $x \in E$ and $(x_\gamma)_{\gamma \geq 0}$ in $[0, \infty)^2$ such that $\lim_{\gamma \to \infty} x_\gamma \to x$, we have

$$(3.4) \qquad \mathcal{L}_{x_\gamma}[Y_t^{\gamma,c,\theta}] \overset{\gamma \to \infty}{\longrightarrow} \mathcal{L}_x[X_t^{c,\theta}] \qquad \text{weakly.}$$

As shown in the proof of Corollary 2.4(i), $\mathcal{D}_0$ is dense in $C_l([0, \infty)^2)$. Hence, it is enough to consider $F(\cdot, z)$, $z \in [0, \infty)^2$, as test functions. Denote by $Z^\gamma$ the process defined in (2.10) started at $Z_0^\gamma = z$. For $\gamma = 1$, we drop the superscript, that is, $Z := Z^1$. Denote by $Z_\infty$ the almost sure limit of $Z_t$ as $t \to \infty$ and recall

that its distribution is $Q_z$. Note that, due to Brownian scaling, $(Z_t^\gamma)_{t\geq 0} \stackrel{\mathcal{D}}{=} (Z_{\gamma t})_{t\geq 0}$. Hence, by Proposition 2.1, we have

$$
\begin{aligned}
\mathbf{E}_{x_\gamma}[F(Y_t^{\gamma,c,\theta}, z)] &= \mathbf{E}_z\left[F(x_\gamma, e^{-ct}Z_{\gamma t})F\left(\theta, \int_0^t ce^{-cr}Z_{\gamma r}\,dr\right)\right] \\
&\stackrel{\gamma\to\infty}{\longrightarrow} \mathbf{E}_z[F(x, e^{-ct}Z_\infty)F(\theta, (1-e^{-ct})Z_\infty)] \\
&= \int_E F(x, e^{-ct}y)F(\theta, (1-e^{-ct})y)Q_z(dy) \\
&= \int_E \mathbf{E}_x[F(X_t, y)]Q_z(dy) \\
&= \mathbf{E}_x\left[\int_E F(X_t, y)Q_z(dy)\right] \\
&= \mathbf{E}_x[F(X_t, z)],
\end{aligned}
$$

where the fourth line follows by (3.3) and the last equality follows by Corollary 2.3(iii). □

REMARK 3.4. We could also define $X^{c,\theta}$ in Definition 1.1 for initial values $x \in [0, \infty)^2$ (instead of $E$ only). This means that $X^{c,\theta}$ starts life with a jump from $x$ to a random point on $E$ chosen according to $Q_x$ and then continues with the usual dynamics. Clearly, this process does not have a càdlàg version (due to the jump at time 0) and its transition semigroup is not strongly continuous at 0. Nevertheless, the proof of Theorem 1.3 shows that that theorem also holds for this process and hence for $Y_0^{\gamma,c,\theta} = X_0^{c,\theta} = x \in [0, \infty)^2$.

3.3. *Proof of the $L^p$-convergence* (*Theorem* 1.4). We prepare for the proof of Theorem 1.4 with two lemmas.

LEMMA 3.5. *Let* $B = (B_1, B_2)$ *be a planar Brownian motion started at* $(B_{1,0}, B_{2,0}) = (u, v) \in [0, \infty)^2$ *and let*

$$
\tau = \inf\{t > 0 : B_t \notin (0, \infty)^2\}.
$$

*Then, for any* $p \in [1, 2)$, *we have*

$$
\mathbf{E}[\tau^{p/2}] \leq \frac{2}{2-p}\left(\frac{2}{\pi}\right)^{p/2}(uv)^{p/2} < \infty.
$$

More generally, one could show for the exit time of a cone with angle $2\alpha$ (here, $\alpha = \pi/4$) that $\mathbf{E}[\tau^{p/2}] < \infty$ if and only if $p\alpha < \pi/2$ (see [1], equation (3.8)). We give the short proof here in order to be self-contained.

PROOF.    By the reflection principle and independence of $B_1$ and $B_2$, we get

$$\mathbf{P}[\tau > t] = 4\mathcal{N}_{0,t}(0, u)\mathcal{N}_{0,t}(0, v),$$

where $\mathcal{N}_{0,t}(a, b) = (2\pi t)^{-1/2} \int_a^b e^{-r^2/2t}\, dr$ is the centred normal distribution with variance $t$. Hence,

$$\begin{aligned}
\mathbf{E}[\tau^{p/2}] &= \int_0^\infty \mathbf{P}[\tau > t^{2/p}]\, dt \\
&\leq \int_0^\infty 1 \wedge \left(\frac{2}{\pi} uv t^{-2/p}\right) dt = \frac{2}{2-p}\left(\frac{2}{\pi}\right)^{p/2} (uv)^{p/2}. \qquad \square
\end{aligned}$$

LEMMA 3.6.    *For every* $(u, v) \in [0, \infty)^2$, *every* $p \in [1, 2)$ *and every* $i = 1, 2$, *we have*

$$\int_E x_i^p Q_{(u,v)}(dx) \leq |u^2 - v^2|^{p/2} + \frac{2^{p/2}(uv)^{p/2}}{\cos(p\pi/4)} < \infty.$$

PROOF.    This can be verified by means of an explicit computation using the density formula of $Q_{(u,v)}$ in (2.5).   $\square$

Note that finiteness of the expression on the left-hand side in Lemma 3.6 (which is what we need in the proof of Theorem 1.4) could also be inferred without computations by the Burkholder–Davis–Gundy inequality and Lemma 3.5.

PROOF OF THEOREM 1.4.    (i) By Lemma 3.6, we have

$$\mathbf{E}[(X_{i,t}^{c,\theta})^p] = \int_E y_i^p Q_{e^{-ct}x + (1-e^{-ct})\theta}(dy) < \infty.$$

Fix $t > 0$ and define

$$M_{i,s}^t := e^{-ct}x_i + (1 - e^{-ct})\theta_i + \int_0^s e^{c(r-t)}\sqrt{\gamma Y_{1,r}^{\gamma,c,\theta} Y_{2,r}^{\gamma,c,\theta}}\, dW_{i,r}.$$

Let $\langle M_1^t \rangle = \langle M_2^t \rangle$ denote the square variation process of both $M_1^t$ and $M_2^t$. Note that $M_{i,t}^t = Y_{i,t}^{\gamma,c,\theta} \geq 0$ and that $M_i^t$ is a martingale and thus

(3.5)                    $M_{i,s}^t = \mathbf{E}[M_{i,t}^t \mid M_{i,s}^t] \geq 0$        for all $s \in [0, t]$.

Now, $(M_s^t)_{s \geq 0}$ is an isotropic diffusion in $\mathbb{R}^2$ and is hence a time-transformed planar Brownian motion. That is, there exists a planar Brownian motion $B$ (with respect to some right-continuous complete filtration $\mathcal{F}$) started at $B_0 = e^{-ct}x + (1 - e^{-ct})\theta$ such that each $\langle M_{1,\cdot}^t \rangle_s$ is an $\mathcal{F}$ stopping time and such that $B_{\langle M_{1,\cdot}^t \rangle_s} = M_s^t$ for all $s \geq 0$.

Define the $\mathcal{F}$ stopping times

$$\tau := \inf\{s > 0 : B_s \notin (0, \infty)^2\} \quad \text{and} \quad \tau_0 := \inf\{s > 0 : B_s \notin [0, \infty)^2\}.$$

Clearly, we have $\tau = \tau_0$ almost surely and, hence, by (3.5),

$$\langle M_{1,\cdot}^t \rangle_t \leq \tau_0 = \tau \qquad \text{a.s.}$$

Using the Burkholder–Davis–Gundy inequality for the martingale $(B_{i,s})_{s \geq 0}$ yields (see Lemma 3.5)

$$\mathbf{E}\Big[\sup_{s \leq \tau} B_{i,s}^p\Big] \leq 2^{p-1}\big(B_{i,0}^p + (4p)^p \mathbf{E}[\tau^{p/2}]\big) < \infty.$$

Hence, $(|B_{i,\tau \wedge s}|^p)_{s \geq 0}$ is uniformly integrable and we can apply the optional sampling theorem to obtain

$$\mathbf{E}[(Y_{i,t}^{\gamma,c,\theta})^p] = \mathbf{E}\big[\big(B_{i,\langle M_{1,\cdot}^t \rangle_t}\big)^p\big] \leq \mathbf{E}[(B_{i,\tau})^p] = \mathbf{E}[(X_{i,t}^{c,\theta})^p].$$

(ii) By Theorem 1.3 and the Skorohod embedding theorem, we may construct all processes on one probability space such that $Y_t^{\gamma,c,\theta} \to X_t^{c,\theta}$ almost surely as $\gamma \to \infty$. By part (i), the $p$th moments of $Y_{i,t}^{\gamma,c,\theta}$, $\gamma \geq 0$, are uniformly integrable and so we have the desired $L^p$-convergence. $\square$

**4. The strong construction (proofs of Theorems 1.6 and 1.7).** Recall the definitions of $C$, $\Xi$ and $D$ in (1.8) and (1.10).

LEMMA 4.1. *The map $x \mapsto D_x$ is càdlàg.*

PROOF. This follows from continuity of $B$ and the definition of $\tau_x$. $\square$

PROOF OF THEOREM 1.6. From Lemmas 3.1 and 4.1, we infer that $X^{\bar{c},\bar{\theta}}$ has the Markov property and càdlàg paths. It remains to show (1.11).

By Lemma 3.1, for $x, z \in E$, $A \subset E$ measurable and $0 \leq s < t$, we have (with $\mathbf{P}_x$ denoting the probability law of $X_t^{\bar{c},\bar{\theta}}$, as defined in Theorem 1.6)

$$\begin{aligned}
p_{s,t}(z, A) &= \mathbf{P}_x[X_t^{\bar{c},\bar{\theta}} \in A \mid X_s^{\bar{c},\bar{\theta}} = z] \\
&= \mathbf{P}\big[C(0,t)D_{x+\Xi(0,t)} \in A \mid D_{x+\Xi(0,s)} = C(0,s)^{-1}z\big] \\
&= Q_{C(0,s)^{-1}z+\Xi(s,t)}(C(0,t)^{-1}A) \\
&= Q_{C(s,t)z+C(0,t)\Xi(s,t)}(A). \qquad\qquad \square
\end{aligned}$$

PROOF OF THEOREM 1.7. If $c\theta = 0$, then $X^{c,\theta}$ is the deterministic process $X_t^{c,\theta} = e^{-ct}X_0^{c,\theta}$ and hence 0 is polar.

Now, assume that $c\theta \neq 0$. Le Gall and Meyre [12] show that almost surely, for all $z \in (0,\infty)^2$, the planar Brownian motion $B$ does not leave the cone $[-z, \infty)$ first at $-z$. More formally, consider the event

$$A := \{B_{\tau_z} \neq -z \text{ for all } z \in (0,\infty)^2\}.$$

Theorem 1 of [12] then implies that $\mathbf{P}[A] = 1$ (in fact, they show that *no* rectangular cone is first left at its vertex, not only northeast cones $[z, \infty)$). Now, by (1.12), we have

$$\{X_t^{c,\theta} \neq 0 \text{ for all } t > 0\} = \{D_{x+r\theta} \neq 0 \text{ for all } r > 0\}$$

$$= \{B_{\tau_{x+r\theta}} \neq x + r\theta \text{ for all } r > 0\} \supset A.$$

This shows the claim of Theorem 1.7. $\square$

**5. The infinitesimal dynamics of $X^{c,\theta}$.** In this section, we give a description and construction of the infinite rate mutually catalytic branching process $X$ in terms of its infinitesimal characteristics. To this end, we will define a linear operator $\mathcal{G}^{c,\theta}$ that:

(i) defines the contraction semigroup of $X$ in the sense of the Hille–Yoshida theorem (Theorem 5.3);

(ii) defines a well-posed martingale problem whose unique solution is $X$ (Theorem 5.4).

5.1. *Results.* Recall, from Definition 1.1, that the linear operator $\mathcal{S}_t$ on $C_l(E)$ is defined by

$$\mathcal{S}_t f(x) := \int_E f(y) p_t(x, dy) = \int_E f(y) Q_{e^{-ct}x+(1-e^{-ct})\theta}(x, dy).$$

In order to define the generator of $\mathcal{S} = (\mathcal{S}_t)_{t \geq 0}$, we will need to study (for suitable functions $f$) the limit

$$(5.1) \quad \lim_{t \downarrow 0} t^{-1}\big(\mathcal{S}_t f(x) - f(x)\big) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\bigg(\int f \, dQ_{x+\varepsilon c(\theta-x)} - f(x)\bigg).$$

In the sequel, we will use the shorthand notation

$$\partial_1 f(u, v) := \frac{\partial}{\partial u} f(u, v) \quad \text{and} \quad \partial_2 f(u, v) := \frac{\partial}{\partial v} f(u, v).$$

In order to define what we mean by a suitable function, we introduce the subspace $C_l^2(E) \subset C_l(E)$.

DEFINITION 5.1. Let $C_l^2(E) \subset C_l(E)$ be the subspace of such functions $f \in C_l(E)$:

(i) whose partial derivatives $\partial_1 f$ and $\partial_2 f$ exist on $(0, \infty) \times \{0\}$ and $\{0\} \times (0, \infty)$, respectively, are continuous, can be continuously extended to $\{0\} \times [0, \infty)$ and fulfill

$$(5.2) \quad \lim_{u \to \infty} u \, \partial_1 f(u, 0) = \lim_{v \to \infty} v \, \partial_2 f(0, v) = 0;$$

(ii) whose partial second derivatives $\partial_1^2 f$ and $\partial_2^2 f$ exist on $(0, \infty) \times \{0\}$ and $\{0\} \times (0, \infty)$, respectively, and are such that

$$(5.3) \qquad \|f\|_{2,\infty} := \sup_{r \in (0,\infty)} r\left(|\partial_1^2 f(r, 0)| + |\partial_2^2 f(0, r)|\right) < \infty.$$

Note that, for $f \in C_l^2(E)$, we have

$$(5.4) \qquad \|f\|_{1,\infty} := \sup_{r \in [0,\infty)} \left(|\partial_1 f(r, 0)| + |\partial_2 f(0, r)|\right) < \infty.$$

In order to get an explicit formula for the limit in (5.1), we define the vague limits (for $u, v > 0$)

$$\nu_{(0,v)} := \text{v-}\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} Q_{(\varepsilon, v)} \quad \text{and} \quad \nu_{(u,0)} := \text{v-}\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} Q_{(u, \varepsilon)}.$$

$\nu_{(u,0)}$ can be thought of as the "Lévy measure" of the next jump when the actual position is $(u, 0)$ and the drift is in direction of $(0, 1)$. In order to formalize this, for the drift in direction $(0, 1)$, we define the linear operator $\mathcal{G}_2$ on $C_l^2(E)$ by $\mathcal{G}_2 f(x) = \partial_2 f(x)$ if $x_1 = 0$ and

$$\mathcal{G}_2 f(x) = \int_E [f(y) - f(x) - (y_1 - x_1) \, \partial_1 f(x)] \nu_x(dy) \qquad \text{if } x_1 > 0.$$

For the drift in direction $(1, 0)$, we define $\mathcal{G}_1$ similarly. Note that $\nu_x$ is not a finite measure and that the integral of $y_1 - u$ with respect to $\nu_{(u,0)}$ is well defined only as a Cauchy principal value and, as such, equals zero. Hence, this term in the integral is needed in order for the integral to be well defined in the usual sense. In Lemma 5.5 below, we will show that $\mathcal{G}_1 f$ and $\mathcal{G}_2 f$ are, in fact, well defined and are in $C_l(E)$.

Due to spatial homogeneity of planar Brownian motion, we have a scaling relation that helps to get rid of the many different $\nu_x$ in the definition of $\mathcal{G}_1$ and $\mathcal{G}_2$:

$$\int_E f(x) \nu_{(u,0)}(dx) = \frac{1}{u} \int_E f(ux) \nu_{(1,0)}(dx).$$

Furthermore, letting $f^\dagger((x_1, x_2)) := f((x_2, x_1))$, by symmetry, we have

$$\int_E f(x) \nu_{(0,v)}(dx) = \int_E f^\dagger(x) \nu_{(v,0)}(dx) = \frac{1}{v} \int_E f^\dagger(vx) \nu_{(1,0)}(dx).$$

Hence, we can express $\mathcal{G}_1$ and $\mathcal{G}_2$ in terms of

$$(5.5) \qquad \nu := \nu_{(1,0)}.$$

Using the explicit form of the density of $Q_{(1,\varepsilon)}$ in (2.5) and letting $\varepsilon \to 0$, we get that the $\sigma$-finite measure $\nu$ on $E$ has a one-dimensional Lebesgue density given by

$$(5.6) \qquad \nu(d(u, v)) = \begin{cases} \dfrac{4}{\pi} \dfrac{u}{(1 - u)^2 (1 + u)^2} \, du, & \text{if } v = 0, \\ \dfrac{4}{\pi} \dfrac{v}{(1 + v^2)^2} \, dv, & \text{if } u = 0. \end{cases}$$

$\mathcal{G}_1$ and $\mathcal{G}_2$ can now be written as

$$(5.7) \qquad \mathcal{G}_2 f(x) = \begin{cases} \partial_2 f(x), & \text{if } x_1 = 0, \\ \dfrac{1}{x_1} \displaystyle\int_E [f(x_1 y) - f(x) \\ \qquad\qquad - x_1(y_1 - 1)\, \partial_1 f(x)]\nu(dy), & \text{if } x_1 > 0, \end{cases}$$

and

$$(5.8) \qquad\qquad\qquad \mathcal{G}_1 f = (\mathcal{G}_2 f^\dagger)^\dagger.$$

Finally, we define the operator $\mathcal{G}^{c,\theta}$ on $C_l^2(E)$ with domain $\mathcal{D}(\mathcal{G}^{c,\theta}) = C_l^2(E)$ that determines the infinitesimal characteristics of the process $X = X^{c,\theta}$:

$$(5.9) \qquad\qquad\qquad \mathcal{G}^{c,\theta} f(x) = \sum_{i=1}^{2} c(\theta_i - x_i) \mathcal{G}_i f(x).$$

LEMMA 5.2. *The operator $\mathcal{G}^{c,\theta}$ is well defined. That is, for $f \in C_l^2(E)$, the expressions in (5.9) and (5.7) are well defined and we have $\mathcal{G}^{c,\theta} f \in C_l(E)$.*

This lemma will be proven in Section 5.2.

THEOREM 5.3 ($X^{c,\theta}$ via its generator). (i) *For every $f \in C_l^2(E)$, we have, pointwise, for all $x \in E$, that*

$$(5.10) \quad \mathcal{G}^{c,\theta} f(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\left(\int_E f\, dQ_{x+\varepsilon c(\theta - x)} - f(x)\right) = \lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t}.$$

(ii) *The operator $\mathcal{G}^{c,\theta}$ on $C_l(E)$ is closable and its closure generates the contraction semigroup $S$ of the process $X^{c,\theta}$.*

The theorem will be proven in Section 5.2 using the classical Hille–Yoshida theorem.

A different, and more modern, approach to constructing Markov processes from their infinitesimal dynamics is the martingale problem technique due to Stroock and Varadhan.

Recall from (2.17) that $V \subset C_l^2(E)$ is the vector space spanned by $\{F(\cdot, z), z \in E\}$. Define the linear operator $\mathcal{G}^{c,\theta}$ on $V$ by (5.9) and (5.7). By Theorem 5.3(i), we obtain for $z \in E$ [using Corollary 2.3(iii) in the second line and Lemma 2.2 in the last line] that

$$\mathcal{G}^{c,\theta} F(\cdot, z)(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\left(\int_E F(y, z)\, dQ_{x+\varepsilon c(\theta - x)}(dy) - F(x, z)\right)$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1}\left(F(x + \varepsilon c(\theta - x), z) - F(x, z)\right)$$

$$(5.11) \qquad\qquad = \langle c(\theta - x), \nabla_1 F(x, z)\rangle$$

$$= F(x, y)[c(\theta - x) \diamond z].$$

Hence, (5.11) is enough to define $\mathcal{G}^{c,\theta}$ on $V$ and we do not really need the measure $\nu$ from (5.7) here.

A solution of the $(\mathcal{G}^{c,\theta}, V)$ martingale problem is an $E$-valued measurable stochastic process $X$ such that

$$M_t := F(X_t, z) - \int_0^t (c(\theta - X_s) \diamond z) F(X_s, z) \, ds$$

is a ($\mathbb{C}$-valued) martingale. A martingale problem is said to be *well posed* if, for every probability measure $\mu$ on $E$, there exists a solution $X$ with $\mathcal{L}[X_0] = \mu$ (existence) and any two solutions have the same finite-dimensional distributions (uniqueness). In this case, $X$ is a Markov process (see [6], Theorem 4.4.2(a)).

THEOREM 5.4 (Martingale problem characterization of $X^{c,\theta}$). *The martingale problem $(\mathcal{G}^{c,\theta}, V)$ is well posed and its unique solution is $X^{c,\theta}$.*

This theorem will be proven in Section 5.3.

5.2. *The Hille–Yoshida approach* (*proof of Theorem* 5.3). Lemma 5.2 and part (i) of Theorem 5.3 are direct consequences of the following two lemmas.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

LEMMA 5.5. *For $f \in C_l^2(E)$, $x \in E$ and $i = 1, 2$, the expression $\mathcal{G}_i f(x)$ from* (5.7) *and* (5.8) *is well defined and we have*

$$(5.12) \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left( \int_E f \, dQ_{x+\varepsilon e_i} - f(x) \right) = \mathcal{G}_i f(x).$$

LEMMA 5.6. *For $f \in C_l^2(E)$, we have $\mathcal{G}^{c,\theta} f \in C_l(E)$.*

PROOF OF LEMMA 5.5. For $x = (0, 0)$, since $Q_{\varepsilon e_i} = \delta_{\varepsilon e_i}$, this is the very definition of $\mathcal{G}_i$. For $u \neq (0, 0)$, by linear scaling and symmetry, it is enough to consider the case $x = (1, 0)$. If $i = 1$, then the left-hand side of (5.12) equals

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (f(1 + \varepsilon, 0) - f(1, 0)) = \partial_1 f(1, 0) = (\mathcal{G}_1 f)(1, 0).$$

Now, consider $i = 2$. It is a simple exercise to compute that for every $\varepsilon > 0$,

$$\frac{4}{\pi} \int_0^\infty \frac{r(r-1)}{4\varepsilon^2 + (r^2 + \varepsilon^2 - 1)^2} \, dr = \frac{2}{\pi} \varepsilon^{-1} \arctan(\varepsilon)$$

$$= \frac{4}{\pi} \int_0^\infty \frac{s}{4\varepsilon^2 + (s^2 - \varepsilon^2 + 1)^2} \, ds.$$

Hence, if we let $g(y) := (y_1 - 1) \partial_1 f(1, 0)$, then, for every $\varepsilon > 0$,

$$\int_E (g(y) - g(1, 0)) Q_{(1,\varepsilon)}(dy) = 0.$$

Hence, we can replace $f$ by $f - g$. Now, $f - g$ is twice differentiable, has at most linear growth and $\partial_1 (f - g)(1, 0) = 0$. Hence,

$$\sup_{u \geq 0, u \neq 1} \frac{|(f - g)(u, 0) - f(1, 0)|}{(u - 1)^2} < \infty.$$

This allows us to use dominated convergence in the following computation to obtain

$$
\begin{aligned}
\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} & \left( \int_E f \, dQ_{(1,0)+\varepsilon e_2} - f(1, 0) \right) \\
&= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int [f(x) - f(1, 0) - (x_1 - 1) \, \partial_1 f(1, 0)] Q_{(1,\varepsilon)}(dx) \\
&= \lim_{\varepsilon \downarrow 0} \left( \frac{4}{\pi} \int_0^\infty \frac{u[(f - g)(u, 0) - f(1, 0)]}{4\varepsilon^2 + (u^2 + \varepsilon^2 - 1)^2} \, du \right. \\
&\qquad \left. + \frac{4}{\pi} \int_0^\infty \frac{v[(f - g)(0, v) - f(1, 0)]}{4\varepsilon^2 + (v^2 - \varepsilon^2 + 1)^2} \, dv \right) \\
&= \lim_{\varepsilon \downarrow 0} \left( \frac{4}{\pi} \int_0^\infty \frac{u[(f - g)(u, 0) - f(1, 0)]}{4\varepsilon^2 + \varepsilon^4 + 2\varepsilon^2(u + 1)(u - 1) + (u + 1)^2 (u - 1)^2} \, du \right. \\
&\qquad \left. + \frac{4}{\pi} \int_0^\infty \frac{v[(f - g)(0, v) - f(1, 0)]}{4\varepsilon^2 + (v^2 - \varepsilon^2 + 1)^2} \, dv \right) \\
&= \frac{4}{\pi} \int_0^\infty \frac{u[(f - g)(u, 0) - f(1, 0)]}{(u^2 - 1)^2} \, du \\
&\qquad + \frac{4}{\pi} \int_0^\infty \frac{v[(f - g)(0, v) - f(1, 0)]}{(v^2 + 1)^2} \, dv \\
&= \int_E [f(y) - f(1, 0) - (y_1 - 1) \, \partial_1 f(1, 0)] \nu(dy) \\
&= \mathcal{G}_2 f(1, 0). \qquad \qquad \square
\end{aligned}
$$

PROOF OF LEMMA 5.6. We have to show that for any $f \in C_l^2(E)$, $\mathcal{G}^{c,\theta} f(x)$ is continuous in $x \in E$ and has a limit at $\infty$. By (5.9), it is enough to derive these properties for $G_i(x) := (\theta_i - x_i) \mathcal{G}_i f(x)$, $i = 1, 2$. We will give the proof only for the case $i = 2$ since the case $i = 1$ is analogous.

For $x_1 = 0$, we have

(5.13) $$G_2(x) = G_2(0, x_2) = (\theta_2 - x_2) \, \partial_2 f(0, x_2).$$

This expression is clearly continuous in $x_2 \in [0, \infty)$ and, by (5.2), we have

(5.14) $$\lim_{x_2 \to \infty} G_2(x) = 0.$$

Now, consider the case $x_1 > 0$. Hence, by (5.7),

$$G_2(x) = \int g(x, y)\nu(dy),$$

where

$$g(x, y) := \frac{\theta_2}{x_1}[f(x_1 y) - f(x) - x_1(y_1 - 1)\,\partial_1 f(x)].$$

Since $f \in C_l^2(E)$, for all $y \in E$, we have:

  (i)  $x \mapsto g(x, y)$ is continuous on $(0, \infty) \times \{0\}$;
  (ii)  $\lim_{x_1 \to \infty} g(x, y) = 0$;
  (iii)  $\lim_{x_1 \downarrow 0} g(x, y) = \theta_2\,\partial_2 f(0, 0)y_2$.

In order to find an integrable dominating function for $g$, define $h : E \to [0, \infty)$ by [recall (5.3) and (5.4)]

$$h(y) := \begin{cases} \theta_2 \|f\|_{2,\infty}(y_1 - 1)^2, & \text{if } y_1 \in (\tfrac{1}{2}, \tfrac{3}{2}), \\ 2\theta_2 \|f\|_{1,\infty}(y_1 + y_2 + 1), & \text{otherwise.} \end{cases}$$

Note that the density of $\nu(dy)$ decays like $1/(y_1 + y_2)^3$ as $y \to \infty$. Furthermore,

$$(y_1 - 1)^2 \frac{\nu(dy)}{dy_1} = \frac{4}{\pi} \frac{y_1}{(1 + y_1)^2}$$

is bounded on $(1/2, 3/2) \times \{0\}$. Hence, we have $\int h \, d\nu < \infty$.

For all $y \in E$ and $x_1 > 0$, we have

$$|g(x, y)| \leq \frac{\theta_2}{x_1}\big(|f(x_1 y) - f(0, 0)| + |f(x) - f(0, 0)| + x_1(y_1 + 1)|\partial_1 f(x)|\big)$$

$$\leq 2\theta_2(y_1 + y_2 + 1)\|f\|_{1,\infty}.$$

Furthermore, recalling (5.3), for $y_1 \in (1/2, 3/2)$, by Taylor's formula, we get that

$$|g(x, y)| = \frac{\theta_2}{x_1}\big|f\big((y_1 - 1)x + x\big) - f(x) - x_1(y_1 - 1)\,\partial_1 f(x)\big|$$

$$\leq \frac{\theta_2}{2}(y_1 - 1)^2 \sup_{u \geq x_1/2} x_1|\partial_1^2 f(u, 0)|$$

$$\leq \theta_2 \|f\|_{2,\infty}(y_1 - 1)^2.$$

Hence, in fact, $|g(x, y)| \leq h(y)$ for all $y \in E$, $x \in (0, \infty) \times \{0\}$ and the dominated convergence theorem yields that $G_2$ shares the properties (i) and (ii) of $g(x, \cdot)$ and that

$$\lim_{x_1 \downarrow 0} G_2(x) = \theta_2\,\partial_2 f(0, 0) \int y_2 \nu(dy) = \theta_2\,\partial_2 f(0, 0) = G_2(0, 0).$$

Combining this with (5.13) and (5.14), we have $G_2 \in C_l(E)$.  $\square$

In order to show part (ii) of Theorem 5.3, we will apply the Hille–Yoshida theorem for generators of contraction semigroups. Recall, from Corollary 2.4, that $V$ is dense in $C_l(E)$. Also, by Lemma 2.2, one can easily check that

$$V \subset C_l^2(E).$$

For each $z \in E$, define the map $u_y : [0, \infty) \to C_l(E)$ by $u_y(t) := \mathcal{S}_t F(\cdot, y)$.

By [6], Proposition 1.3.4, the operator $\mathcal{G}^{c,\theta}$ on $C_l(E)$ with domain $\mathcal{D}(\mathcal{G}^{c,\theta}) = C_l^2(E)$ is closable and its closure generates (uniquely) the semigroup $(\mathcal{S}_t)_{t \geq 0}$ on $C_l(E)$ if the following conditions are all fulfilled:

(a) $\mathcal{G}^{c,\theta}$ is dissipative;
(b) $u_y(t) \in \mathcal{D}(\mathcal{G}^{c,\theta})$ for all $t > 0$;
(c) the map $(0, \infty) \to C_l(E)$, $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is continuous;
(d) for all $t > 0$,

$$(5.15) \qquad u_y(t) - u_y(0) = \int_0^t \mathcal{G}^{c,\theta} u_y(s) \, ds.$$

Hence, in order to prove part (ii) of Theorem 5.3, it remains to check (a)–(d).

(a) Let $f \in C_l^2(E)$ and assume that $f$ assumes its maximum at $x \in E \cup \{\infty\}$. Since $\mathcal{S}_t f(x) \leq f(x)$ for all $t \geq 0$, equation (5.10) implies that $\mathcal{G}^{c,\theta} f(x) \leq 0$. Hence, $\mathcal{G}^{c,\theta}$ fulfills the positive maximum principle and is thus dissipative (see, e.g., [6], Lemma 4.2.1).

(b) By Proposition 3.3, for any $y \in E$, $x \in E$ and $t > 0$, we have

$$(5.16) \qquad u_y(t)(x) = \mathcal{S}_t F(\cdot, y)(x) = F(x, e^{-ct} y) F(\theta, (1 - e^{-ct}) y).$$

As $F(\cdot, e^{-ct} y)$ is in $C_l^2(E)$, so is $\mathcal{S}_t F(\cdot, y)$.

(c) By (5.10), we have

$$\mathcal{G}^{c,\theta} u_y(t)(x) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (\mathcal{S}_{t+\varepsilon} - \mathcal{S}_t) F(\cdot, y)(x) = \frac{d}{dt} (u_y(t)(x)).$$

Using (5.16) and Lemma 2.2, for every $x \in E$, we get

$$\mathcal{G}^{c,\theta} \mathcal{S}_t F(\cdot, y)(x) = \langle -ce^{-ct} y, \nabla_2 F(c, e^{-ct} y) \rangle F(\theta, (1 - e^{-ct}) y)$$
$$+ F(x, e^{-ct} y) \langle ce^{-ct} y, \nabla_2 F(\theta, (1 - e^{-ct}) y) \rangle$$
$$= [ce^{-ct} (\theta - x) \diamond y] F(x, e^{-ct} y) F(\theta, (1 - e^{-ct}) y).$$

Hence, $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is clearly continuous [in $C_l(E)$].

(d) As $t \mapsto \mathcal{G}^{c,\theta} u_y(t)$ is continuous, it is integrable, and

$$\left( \int_0^t \mathcal{G}^{c,\theta} u_y(s) \, ds \right)(x) = \int_0^t \mathcal{G}^{c,\theta} u_y(s)(x) \, ds = u_y(t)(x) - u_y(0)(x)$$

implies (5.15).

5.3. *The martingale problem* (*proof of Theorem* 5.4). Before we prove this theorem, we derive a duality relation for processes satisfying the martingale problem $(\mathcal{G}^{c,\theta}, V)$. Recall the definition of $\tilde{Y}$ from (2.11).

LEMMA 5.7. *Let $\mu$ be a probability measure on $E$. Let $X$ be any solution of the martingale problem $(\mathcal{G}^{c,\theta}, V)$ with $\mathcal{L}[X_0] = \mu$. Then $X$ and $\tilde{Y}$ are dual, in the sense that for any $\tilde{y} \in E \times [0, \infty)^2$, we have*

$$\mathbf{E}_\mu[H((X_t, \theta), \tilde{y})] = \int_E \mathbf{E}_{\tilde{y}}[H((x, \theta), \tilde{Y}_t)]\mu(dx) \qquad \text{for all } t \geq 0.$$

PROOF. As $X$ is a solution of the martingale problem, we have that

$$H((X_t, \theta), \tilde{y}) - \int_0^t H((X_s, \theta), \tilde{y})[c(\theta - X_s) \diamond \tilde{y}(1)]\,ds$$

$$= F(\theta, \tilde{y}(2))\left(F(X_t, \tilde{y}(1)) - \int_0^t F(X_s, \tilde{y}(1))[c(\theta - X_s) \diamond \tilde{y}(1)]\,ds\right)$$

is a martingale. On the other hand, by (2.16) [since $\tilde{y}(1) \in E$, one term vanishes],

$$\frac{d}{dt}\mathbf{E}_{\tilde{y}}[H((x, \theta), \tilde{Y}_t)]|_{t=0} = \langle -c\tilde{y}(1), \nabla_2 F(x, \tilde{y}(1))\rangle F(\theta, \tilde{y}(2))$$

(5.17)
$$+ \langle c\tilde{y}(1), \nabla_2 F(\theta, \tilde{y}(2))\rangle F(x, \tilde{y}(1))$$

$$= H((\theta, x), \tilde{y})[c(\theta - x) \diamond \tilde{y}(1)].$$

Since $\tilde{Y}$ is deterministic, we get that

$$H((x, \theta), \tilde{Y}_t) - \int_0^t H((\theta, x), \tilde{Y}_s)c(\theta - x) \diamond \tilde{Y}_s(1)\,ds = H((x, \theta), \tilde{y})$$

is the trivial martingale. By [6], Corollary 4.4.13, this implies that

$$\mathbf{E}_\mu[H((X_t, \theta), \tilde{y})] = \int \mathbf{E}_y[H((x, \theta), \tilde{Y}_t)]\mu(dx)$$

and we are done. $\square$

PROOF OF THEOREM 5.4. By Theorem 5.3(ii) and (5.11), and since $V \subset C_l(E)$, by definition of $X^{c,\theta}$, the process $X^{c,\theta}$ is, in fact, a solution of the martingale problem $(\mathcal{G}^{c,\theta}, V)$.

Now, assume that $X$ and $X'$ are two solutions with $\mathcal{L}[X_0] = \mathcal{L}[X_0'] = \mu$. By Lemma 5.7, we get

$$\mathbf{E}_\mu[F(X_t, y)] = \mathbf{E}_\mu[F(X_t', y)] \qquad \text{for all } t \geq 0 \text{ and } y \in E.$$

By Corollary 2.4, $\{F(\cdot, y), y \in E\}$ is measure determining on $E$. Hence, $\mathcal{L}_\mu[X_t] = \mathcal{L}_\mu[X_t']$ for all $t \geq 0$. By [6], Theorem 4.4.2, this implies that the finite-dimensional distributions of $X$ and $X'$ coincide. $\square$

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