Multiple Scale Analysis of Clusters in Spatial Branching Models

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Abstract

In this paper we will investigate the long time behaviour of critical branching Brownian motion and (finite variance) super Brownian motion (the so-called Dawson-Watanabe process) on $\mathbb{R}^d$. These processes are known to be persistent if $d \geq 3$, that is there exist non-trivial equilibrium measures. If $d \leq 2$ they cluster, i.e., the processes converge to the 0 configuration while the surviving mass piles up in so-called clusters.

We study the spatial profile of the clusters in the "critical" dimension $d = 2$ via multiple space scale analysis. We will also investigate the long time behaviour of these models restricted to finite boxes in $d \geq 2$. On the way we develop coupling and comparison methods for spatial branching models.

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1 Introduction

1.1 Background

For several interacting infinite particle systems and related models there is a dichotomy between stability (i.e., non-trivial equilibrium measures exist) and clustering depending on transience or recurrence of the interaction kernel. Many infinite particle systems with site space $\mathbb{Z}^d$ or $\mathbb{R}^d$ and finite variance interaction are stable if $d \geq 3$ and cluster if $d = 1, 2$. This is well known, e.g., for the voter model, linearly interacting diffusions with compact state space, branching Brownian motion, Dawson-Watanabe process etc.

The dimension $d = 2$ is "critical" in the sense that the Green function of the interaction kernel grows only on a logarithmic scale, and is thus "almost bounded". In the critical dimension the phenomenon of "diffusive clustering" occurs. This means that clusters grow at a randomly chosen algebraic scale of order t^α, $\alpha \in [0, 1/2]$. For many models the structure of the clusters in the critical dimension is known. The voter model in $\mathbb{Z}^2$ has been investigated by Cox and Griffeath (1986). "Critical dimension" linearly interacting diffusions with compact state space on the so-called hierarchical group have been studied by Fleischmann and Greven (1994), Dawson and Greven (1993a, 1993b), Dawson, Greven and Vaillancourt (1995), and Klenke (1996). The techniques employed to describe clusters cover scaling, re-normalisation and the so-called interaction chain.

Non-compact models such as super random walk on $\mathbb{Z}^d$ and linearly interacting Brownian motions labelled by $\mathbb{Z}^d$ have been treated by Winter (1995) and Kopietz (1995).

Clusters of branching Brownian motion have been been studied by Fleischman (1978) and Lee (1991). Lee has rather precise statements for the dimension dependent rate at which the height of clusters grows conditioned on (local) non-extinction (Thm. 2.4). Lee does however not treat the question of spatial extension and profile of the clusters. His results are obtained by studying sub- and super solutions of the partial differential equation determining the Laplace functional.

The main point of this paper is to determine the spatial profile of the clusters of branching and super Brownian motion in dimension $d = 2$. Unlike Lee (1991), we will not condition on local non-extinction, but follow a different route. The compensation of the local extinction will be obtained by "blowing up" the initial configuration. This approach also enables us to to give a description of the finite system (considered next) in terms of the so-called *finite systems scheme* (introduced by Cox and Greven (1990)) that emphasises the similarities to other models.

In the theory of interacting particle systems a systematic treatment of the comparison of finite to infinite systems in high dimensions can be found in Cox and Greven (1990) and (1994). The critical dimension voter model has been studied by Cox and Greven (1991). Comparison of finite to infinite systems of linearly interacting diffusions labelled by the hierarchical group in high and critical dimensions can be found in Klenke (1996). In this paper we will also relate the behaviour of our infinite branching processes to that of their finite versions, defined on d-dimensional tori, in both the cases $d \geq 3$ and $d = 2$.

One aim of this paper is to exhibit how the clustering phenomenon can be studied with *probabilistic tools*, namely, by techniques from the theory of infinite particle systems. These will be applied to both branching particle systems and super processes. In particular we rely on moment calculations and develop coupling and comparison techniques in Section 3. Thus our approach is completely different from Lee's (1991) and coupling and comparison provide a more probabilistic understanding of these processes. These methods should allow an easy adaption to related problems.

1.2 The Models

We only give a short heuristic description of the considered models. An extensive treatment can be found in Dawson (1977) and (1993) and in Fleischman (1978). Nevertheless we have to give the basic definitions for random measures first.

Basic Definitions for Random Measures

Let E be a locally compact Polish space. By $\mathcal{B}(E)$ we denote the Borel σ-field on E. By $C_b(E)$ and $C_c(E)$ we denote the spaces of continuous real valued functions on E that are bounded resp. have compact support.

A measure μ on $\mathcal{B}(E)$ is called *locally finite* if $\mu(K) < \infty$ for all compact sets $K \subset E$. Let

$$\mathcal{M}(E) = \{\text{locally finite measures on } E\} \tag{1.1}$$

and $\mathcal{M}_\mathrm{f}(E) = \{\mu \in \mathcal{M}(E) : \mu(E) < \infty\}$.

For $\mu \in \mathcal{M}(E)$ and $f : E \to \mathbb{R}$ measurable and μ-integrable we define $\langle \mu, f\rangle := \int f \, d\mu$. $\mathcal{M}(E)$ is a Polish space with the vague topology, defined by $\mu_n \to \mu$ iff $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_c(E)$. The space $\mathcal{M}_1(\mathcal{M}(E))$ of probability measures on $\mathcal{M}(E)$, equipped with the weak topology, is also Polish (see, e.g., Kallenberg (1983)). For weak convergence of probability measures we use the symbol "$\Longrightarrow$".

Let $Q \in \mathcal{M}_1(\mathcal{M}(E))$ and $A \in \mathcal{B}(E)$. We define the restriction $Q\Big|_A \in \mathcal{M}_1(\mathcal{M}(E))$ of Q to A by

$$\int (Q\Big|_A)(d\mu)F(\langle \mu, f\rangle) = \int Q(d\mu)F(\langle \mu, f \cdot 1_A\rangle), \tag{1.2}$$

for $f \in C_c(E)$ and $F \in C_b(\mathbb{R})$.

For a signed measure μ we denote by $\|\mu\| = \sup\{\mu(B) - \mu(E \setminus B) : B \in \mathcal{B}(E)\}$ the total variation of μ.

The space of (non-negative) integer valued measures μ on $\mathcal{B}(E)$ will be denoted by

$$\mathcal{N}(E) = \{\mu \in \mathcal{M}(E) : \mu(A) \in \{0, 1, 2 \ldots, \infty\} \quad \forall A \in \mathcal{B}(E)\}. \tag{1.3}$$

The space of finite measures in $\mathcal{N}(E)$ is denoted by $\mathcal{N}_\mathrm{f}(E) = \{\mu \in \mathcal{N}(E) : \mu(E) < \infty\}$.

We use the notation $\mathcal{L}[X]$ for the distribution of a random variable X. Let $(X_t)_{t\geq 0}$ be a Markov process with values in E and $x \in E$ or $Q \in \mathcal{M}_1(E)$. By $\mathcal{L}^x[(X_t)_{t\geq 0}]$ and $\mathcal{L}^Q[(X_t)_{t\geq 0}]$ we denote the distributions of $(X_t)_{t\geq 0}$ with $\mathcal{L}^x[X_0] = \delta_x$ and $\mathcal{L}^Q[X_0] = Q$.

Branching Brownian Motion

Let $(S_t)_{t\geq 0}$ be the semigroup of a Feller process on E and let $(p_k)_{k=0,1,\ldots}$ be a probability distribution on $\mathbb{N}_0$ with $\sum_k kp_k < \infty$. We will consider a particle moving on E according to (S_t) having an exponential lifetime with mean $\frac{1}{c}$. At the time of death the particle produces an offspring of k particles with probability p_k. The offspring behave as k independent copies of the one-particle system started at the parent particle's final position. The process started with a single particle in $x \in E$ will be denoted by $(\eta_t^x)_{t\geq 0}$. Its state space is $\mathcal{N}_{\mathrm{f}}(E)$.

For initial configuration $\eta_0 = \sum_{i=1}^{\infty} \delta_{x_i}$ (δ_x = Dirac-measure on x) in $\mathcal{N}(E)$ we define

$$\eta_t = \sum_{i=1}^{\infty} \eta_t^i \, , \tag{1.4}$$

where $((\eta_t^i)_{t\geq 0}\, ,\, i \in \mathbb{N})$ are independent copies of $(\eta_t^{x_i})_{t\geq 0}$. In the case $p_0 = p_2 = \frac{1}{2}$ we will refer to (η_t) as the *critical binary branching process associated with* (S_t). One main object of consideration will be the critical binary branching Brownian motion on $\mathbb{R}^d$, abbreviated BBM($\mathbb{R}^d$).

Dawson-Watanabe Process

Next we consider the short lifetime high density limit of binary branching processes. Let $\mu \in \mathcal{M}_{\mathrm{f}}(E)$ and $\mu^N \in \mathcal{N}_{\mathrm{f}}(E)$, $N \in \mathbb{N}$, such that $\frac{1}{N}\mu^N \to \mu$, as $N \to \infty$. Let $(\eta_t^N)_{t\geq 0}$ be the branching process corresponding to $p_0 = p_2 = \frac{1}{2}$ with expected lifetime $\frac{1}{cN}$ and with initial state $\eta_0^N = \mu^N$. It is well known that there exists a continuous Markov process $(\zeta_t)_{t\geq 0}$ with values in $\mathcal{M}_{\mathrm{f}}(E)$ such that

$$\mathcal{L}^{\mu}[(\zeta_t)_{t\geq 0}] = \mathrm{w} - \lim_{N\to\infty} \mathcal{L}^{\mu^N} \left[(\frac{1}{N}\eta_t^N)_{t\geq 0} \right] \tag{1.5}$$

(see Dawson (1993), Section 4.4ff).

The process $(\zeta_t)_{t\geq 0}$ will be called the *super process associated with* (S_t). Of particular interest will be super Brownian motion on $\mathbb{R}^d$, abbreviated SBM($\mathbb{R}^d$).

Let $(Z_t)_{t\geq 0}$ be Feller's branching diffusion, this is, the diffusion on $[0,\infty[$ with generator

$$x\frac{\partial^2}{(\partial x)^2} \, . \tag{1.6}$$

It is well known that $\mathcal{L}^{\mu}[\zeta_t] = \mathcal{L}^{\|\mu\|}[Z_{t/2}]$ for $\mu \in \mathcal{M}_{\mathrm{f}}(E)$ and $t \geq 0$. Hence $\mathbf{P}[\zeta_t^x(E) = 0] \stackrel{t\to\infty}{\longrightarrow} 1$, since (Z_t) is a martingale and 0 is an absorbing boundary point.

For $\mu \in \mathcal{M}(E)$ we can define $(\zeta_t)_{t\geq 0}$ with initial configuration $\zeta_0 = \mu$ as the increasing limit of $(\zeta_t^n)_{t\geq 0}$ with initial configurations μ^n, $n \in \mathbb{N}$, such that $\mu^n \uparrow \mu$. It is known that SBM($\mathbb{R}^d$) takes values in $\mathcal{M}(E)$ if we impose a regularity condition on the initial state μ, e.g., assume $\langle \mu, (1 + \|\cdot\|^2)^{-p} \rangle < \infty$ for some $p > d/2$. This condition will always be fulfilled in this paper. The same condition also assures that $\eta_t \in \mathcal{N}(E)$, a.s., for all $t \geq 0$.

Another more analytic, though less intuitive, description is the following. We define the semigroup $(V_t)_{t\geq 0}$ of nonlinear operators on the space of bounded and measurable functions $\phi : E \longrightarrow [0, \infty[$ uniquely by the following equation

$$V_t\phi = S_t\phi - \frac{1}{2}c\int_0^t S_{t-s}((V_s\phi)^2)ds, \quad t \geq 0. \tag{1.7}$$

We can now define (ζ_t) by its *log-Laplace semigroup* (V_t), namely, by the relation

$$\langle \zeta_0, V_t\phi\rangle = -\log \mathbf{E}[\exp(-\langle \zeta_t, \phi\rangle)]. \tag{1.8}$$

A path wise construction of (ζ_t) can be found in Le Gall (1991).

From the scaling properties of Brownian motion in $\mathbb{R}^d$ and Feller's diffusion (i.e., $\mathcal{L}^{\rho/\alpha}[\alpha Z_\beta] = \mathcal{L}^{\rho}[Z_{\alpha\beta}]$) it is clear that SBM($\mathbb{R}^d$) has the following *basic scaling property*: For $K > 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ let $\mu'(\cdot) = K\mu(K^{-1/2}\,\cdot)$. Then

$$\mathcal{L}^{\mu'}\left[K^{-1}\zeta_{Kt}(K^{1/2}\,\cdot)\right] = \mathcal{L}^{\mu}[\zeta_t(\cdot)]. \tag{1.9}$$

In particular, for $d = 2$ and $\mu = \lambda$ (Lebesgue measure on $\mathbb{R}^2$) this becomes

$$\mathcal{L}^{\lambda}\left[K^{-1}\zeta_{Kt}(K^{1/2}\,\cdot)\right] = \mathcal{L}^{\lambda}[\zeta_t(\cdot)]. \tag{1.10}$$

For simplicity we will hence forward only consider (the expected lifetime) $c^{-1} = 1$.

1.3 Basic Ergodic Theory

In the following we will state the results for BBM($\mathbb{R}^d$) and SBM($\mathbb{R}^d$) simultaneously. For convenience we will thus denote by $(\psi_t)_{t\geq 0}$ either BBM($\mathbb{R}^d$) or SBM($\mathbb{R}^d$). Also let for $\rho \geq 0$

$$M(\rho) = \begin{cases} \mathcal{H}(\rho) & \text{if } (\psi_t) \text{ is BBM}(\mathbb{R}^d) \\ \delta_{\rho\cdot\lambda} & \text{if } (\psi_t) \text{ is SBM}(\mathbb{R}^d) \end{cases}, \tag{1.11}$$

where λ is the (d-dimensional) Lebesgue measure and $\mathcal{H}(\rho) \in \mathcal{M}_1(\mathcal{M}(\mathbb{R}^d))$ is the law of a Poisson point process on $\mathbb{R}^d$ with intensity measure $\rho\cdot\lambda$.

It is well known (see Dawson (1977) and Fleischman (1978)) that if $d = 1$ or $d = 2$, then (ψ_t) clusters,

$$\mathcal{L}^{M(\rho)}[\psi_t] \overset{t\to\infty}{\Longrightarrow} \delta_{\mathbf{0}} \qquad \forall\,\rho \geq 0, \tag{1.12}$$

where $\delta_{\mathbf{0}}$ means the unit mass on $\mathbf{0} \in \mathcal{M}(\mathbb{R}^d)$.

For any $d \geq 3$, (ψ_t) is *persistent* (or stable). This means that there exists a family $(\nu_\rho\,,\,\rho \geq 0)$, $\nu_\rho \in \mathcal{M}_1(\mathcal{M}(\mathbb{R}^d))$, of non-trivial invariant (under the dynamics) measures such that

$$\mathcal{L}^{M(\rho)}[\psi_t] \overset{t\to\infty}{\Longrightarrow} \nu_\rho. \tag{1.13}$$

The ν_ρ have the following properties. ν_ρ is translation invariant and ergodic with intensity ρ,

$$\int \langle m, \phi\rangle\, \nu_\rho(dm) = \rho\cdot\langle\lambda,\phi\rangle, \tag{1.14}$$

for $\phi : \mathbb{R}^d \to [0,\infty[$ measurable. Since the particles evolve independently, the ν_ρ form a convolution semigroup $\nu_{\rho+\sigma} = \nu_\rho * \nu_\sigma,\ \rho,\sigma \geq 0$. Hence any ν_ρ is infinitely divisible and thus allows a description via its canonical measure. For details and proofs see Gorostiza and Wakolbinger (1991), Thm. 2.2, for ψ_t BBM($\mathbb{R}^d$) and Dawson (1977) for SBM($\mathbb{R}^d$). Analogous (and more detailed) results for a discrete time setting are known for a long time. See, e.g., Kallenberg (1977).

For extensions of the basic ergodic theory to more general branching mechanisms and motion semigroups see Gorostiza, Roelly and Wakolbinger (1992). Extensions to initial configurations with infinite intensity or that are not translation invariant see Bramson, Cox and Greven (1993) and (1997) for the $d = 1, 2$, resp. $d \geq 3$ case for ψ_t BBM($\mathbb{R}^d$) and SBM($\mathbb{R}^d$).

2 Results

2.1 Cluster formation for $d = 2$

Since the branching mechanism has mean 1, local extinction implies the existence of relatively small areas where more and more mass piles up. We call this phenomenon *clustering*. Our goal is to determine the spatial profile of the clusters. One way to do so is to condition on a test set B being in a cluster. The precise statement for (ψ_t) BBM($\mathbb{R}^2$) is given by Fleischman (1978),

$$\frac{\log t}{8\pi}\mathbf{P}^{M(1)}\left[\psi_t(B) > \frac{\log t}{8\pi}|B|x\right] \stackrel{t\to\infty}{\longrightarrow} e^{-x}, \quad x > 0, \tag{2.1}$$

where $B \in \mathcal{B}(\mathbb{R}^2)$ is bounded. Roughly speaking, with probability $\frac{8\pi}{\log t}$ we see a cluster of "height" $\frac{\log t}{8\pi}$ - times an exponential mean 1 random variable. For BBM($\mathbb{R}^2$), Lee (1991) has a more precise statement (Thm 2.4) due to conditioning on $\eta_t(B) > 0$. Lee studies sub- and super-solutions of Kolmogorov's equation for the Laplace functional. His methods also apply to SBM, but it is still open whether the same is true for branching random walk on the lattice or for linearly interacting Feller's diffusions (super random walk). This reflects the fact that difference equations are usually more difficult to treat than the related differential equations.

Our approach to describe the structure of clusters is based on two rescaling concepts.

(1) High density rescaling

For time $t > 1$ we define

$$\widetilde{\psi}_t = \widetilde{\psi}_t^0 := \frac{8\pi}{\log t}\psi_t \tag{2.2}$$

with

$$\mathcal{L}[\psi_0] = \widetilde{M}(t) := M\left(\frac{\log t}{8\pi}\right). \tag{2.3}$$

This serves first to obtain a non-trivial limiting probability of local non-extinction. Secondly, the height of the clusters is scaled down to have a non-trivial limit.

(2) Spatial rescaling

For (ψ_t) BBM or SBM let $I = [0,1]$ resp. $I =]-\infty, 1]$. We fix $\alpha \in I$ and define $(\widetilde{\psi}_t^\alpha)$ by

$$\widetilde{\psi}_t^\alpha(B) := \mathcal{S}_{\alpha,t}\widetilde{\psi}_t(B) := t^{-\alpha}\widetilde{\psi}_t(t^{\alpha/2}B), \tag{2.4}$$

where $\mathcal{S}_{\alpha,t} : \mathcal{M}(\mathbb{R}^2) \to \mathcal{M}(\mathbb{R}^2),\ \mu(\cdot) \mapsto t^{-\alpha}\mu(t^{\alpha/2}\cdot)$. As above we let $\widetilde{\psi}_t = \widetilde{\psi}_t^0$. This is the right notion since clusters turn out to grow spatially as $t^{\alpha/2}$ for any $\alpha \in I$.

Remark: Note that by the rescaling procedures we do not lose too much information on the family structure. This is because the high density rescaling is so smooth that by (2.1) in the limit $t \to \infty$ we get a Poisson mean 1 number of *families* in each bounded set $B \in \mathcal{B}(\mathbb{R}^2)$. On the other hand, the spatial extension of a typical family is of order $t^{\alpha/2},\ \alpha < 1$ random. Hence the rescalings do not cause an overlap of the families. The high density rescaling also proves useful to give a description of the finite versions of our branching models that underlines the similarities to other models.

A related rescaling approach to clustering phenomena in "subcritical dimensions" (and for a more general setting) has been made by Dawson and Fleischmann (1988). In the special case of SBM($\mathbb{R}^1$) their rescaling is $X_t^K(\cdot) = K^{-1}\zeta_{tK}(K\cdot)$. They obtain that $\mathrm{w}-\lim_{K\to\infty}\mathcal{L}^\lambda[(X_t^K)_{t\geq 0}]$ is the super process on $\mathbb{R}$ associated with no motion. Hence their rescaling describes the family structure of the clusters, but it is too rough to describe their spatial extension.

Now we are able to formulate the first theorem (recall that (Z_t) is Feller's branching diffusion defined in (1.6)).

Theorem 1 (Infinite System, $d = 2$)
Let (ψ_t) be either BBM($\mathbb{R}^2$) or SBM($\mathbb{R}^2$) and $I = [0, 1]$ resp. $I =]-\infty, 1]$. Fix $\alpha \in I$. Then the following holds

$$\mathcal{L}^{\widetilde{M}(t)}[\widetilde{\psi}_t^\alpha] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}^1[Z_{1-\alpha}\cdot\lambda]. \tag{2.5}$$

Theorem 1 gives a first rough description of the profile of clusters. However, the averaging procedure induced by scaling loses information about the spatial structure inside blocks of size $t^{\alpha/2}$.

The next aim is to give a more detailed description of the clusters via *multiple space scales.* That is, we want to look for different spatial scales on tuples of windows of observation (see Figure 1). To describe this properly on a formal level we introduce a rooted tree $\mathbb{T}$ (see Figure 2) and a space scale A associated with it.

Tree. We give the following representation of a (rooted) tree $\mathbb{T}$. Let $\mathbb{T}$ be a finite set of finite sequences with values in $\mathbb{N}$. The root will be denoted by $\emptyset \in \mathbb{T}$. Let $e, f \in \mathbb{T}$, $e = (e_1, \ldots, e_m)$, $f = (f_1, \ldots, f_n)$ (possibly $m = 0$ or $n = 0$) and $l = \max\{k : (e_1, \ldots, e_k) = (f_1, \ldots, f_k)\} \vee 0$. We then define the minimum $e \wedge f$ of e and f by $e \wedge f = (e_1, \ldots, e_l)$ if $l > 0$ and $e \wedge f = \emptyset$ if $l = 0$. We will assume that $(e_1, \ldots, e_k) \in \mathbb{T}\,\forall\, k \leq m$ whenever $(e_1, \ldots, e_m) \in \mathbb{T}$. In particular this implies $e \wedge f \in \mathbb{T}\,\forall\, e, f \in \mathbb{T}$. $\mathbb{T}$ allows an ordering by $e \leq f$ if and only if $e = e \wedge f$. The set of maximal elements in $\mathbb{T}$ will be denoted by $\mathbb{T}^M$. Note that we do not exclude the case in which $\mathbb{T}$ is linear, i.e., $\#\mathbb{T}^M = 1$. In order to avoid redundancy we will assume that $(e_1, \ldots, e_{m-1}, g) \in \mathbb{T}$ for $g = 1, \ldots, e_m$, whenever $(e_1, \ldots, e_m) \in \mathbb{T}$.

Space scale. A pair $\mathbb{L} = (\mathbb{T}, A)$ consisting of a tree $\mathbb{T}$ and a strictly decreasing map

$$A : \mathbb{T} \to I$$

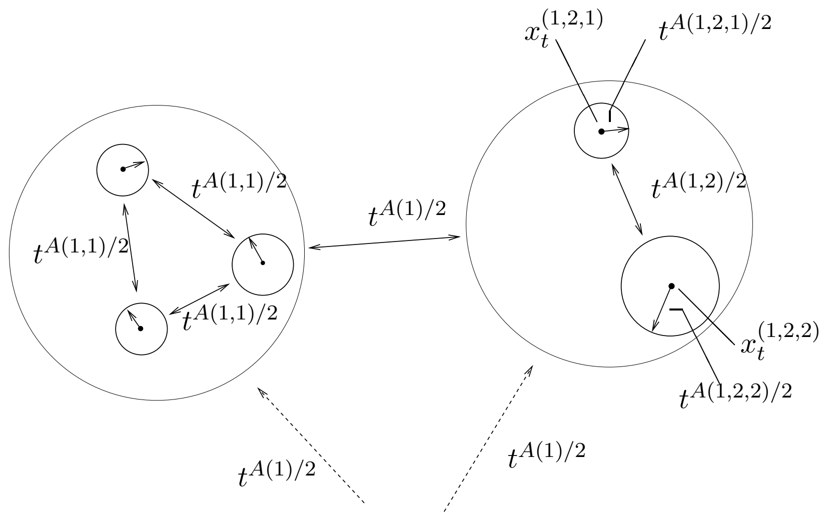

Figure 1. The points (dotted centers of the small circles) are grouped at distances growing at different scales $t^{A(\cdot)/2}$. The small circles represent the windows of observation which also grow at different scales.

(recall that $I = [0,1]$ or $I =]-\infty, 1]$ in the case of BBM resp. SBM) will be called a *multiple space scale*. Given a multiple space scale $\mathbb{L} = (\mathbb{T}, A)$, we assume that $X = (x_t^e,\, e \in \mathbb{T},\, t \geq 0)$ is a family of points $x_t^e \in \mathbb{R}^2$ such that

$$\|x_t^e - x_t^f\| \approx t^{A(e \wedge f)/2}, \quad \text{as } t \to \infty.$$

By $a_t \approx b_t$ we mean $(\log a_t)/(\log b_t) \overset{t\to\infty}{\longrightarrow} 1$. We say that X is $\mathbb{L}$-*spaced*. Our goal is to investigate the common distribution of (recall $\mathcal{S}_{\alpha,t}$ from (2.4))

$$(\mathcal{S}_{A(e),t}\mathcal{T}_{x_t^e}\widetilde{\psi}_t)_{e\in\mathbb{T}} \quad \text{as } t \to \infty,$$

where $\mathcal{T}_z : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ is the translation by z, $(\mathcal{T}_z\mu)(\cdot) = \mu(z + \cdot)$.

Feller tree. Let $(Z_t^e,\, e \in \mathbb{T})_{t\geq 0}$ be the following diffusion on $\mathbb{R}^{\mathbb{T}}$. Each $(Z_t^e)_{t\geq 0}$ is a Feller diffusion. Let $e, f \in \mathbb{T}$ with $e \neq f$. Then $Z_t^e = Z_t^f$ for $t \in [0, 1 - A(e \wedge f)]$. For $t > 1 - A(e \wedge f)$ the evolutions of Z_t^e and Z_t^f shall be independent (see Figure 3).

A similar approach to describe the age and spatial extension of clusters in a model of interacting diffusions with state space $[0,1]$ has been made by Fleischmann and Greven (1996). They describe multiple scale space-time correlations with their so-called Fisher-Wright tree. This is the analogue of our Feller tree, but with an underlying Fisher-Wright diffusion (and with only one "trunk" having branches). The similarity of their results and ours displays a close relationship between the family structures of clusters in the considered models.

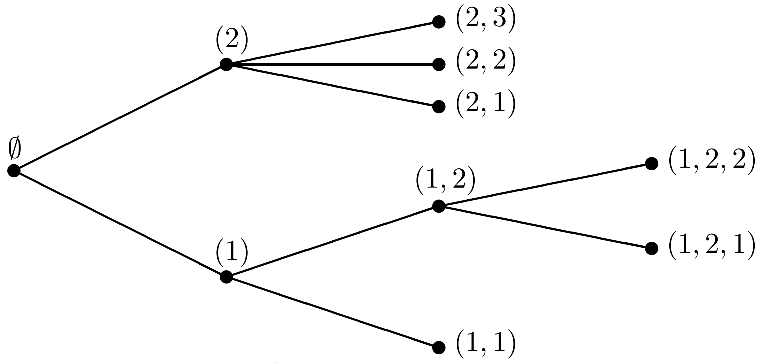

Figure 2. Diagramm of the tree
$\mathbb{T} = \{\emptyset, (1), (2), (1,1), (1,2), (1,2,1), (1,2,2), (2,1), (2,2), (2,3)\}$

Theorem 2 (Infinite System, Multiple Scale)
Let (ψ_t) be either BBM$(\mathbb{R}^2)$ or SBM$(\mathbb{R}^2)$. Then the following holds

(**a**) $$\mathcal{L}^{\widetilde{M}(t)}\left[(\mathcal{S}_{A(e),t}\mathcal{T}_{x_t^e}\widetilde{\psi}_t)_{e\in\mathbb{T}}\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}\left[\left(Z^e_{1-A(e)}\cdot\lambda\right)_{e\in\mathbb{T}}\right].$$

In particular, for $\mathbb{T}$ linear and $B\in\mathcal{B}(\mathbb{R}^2)$ bounded,

(**b**) $$\mathcal{L}^{\widetilde{M}(t)}\left[(\widetilde{\psi}^\alpha_t(B))_{\alpha\in I}\right] \underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathcal{L}^1\Big[|B|\cdot(Z_{1-\alpha})_{\alpha\in I}\Big].$$

At each scale of observation, quasi-equilibria are exhibited that are determined by their density. Observation at different scales shows a certain self-similarity of those quasi-equilibria. This is reflected by the fact that the transition between scales is determined by a homogeneous Markov process.

2.2 Finite Systems, Stable Case

Computer simulations of particle systems evidently have to be restricted to finite versions of the model. However, there are also other good reasons to study finite systems. Finite systems model a finite nature and the infinite system can be regarded as an idealisation for analytical convenience only. So the questions arise: How well do finite systems approximate the infinite system (and vice versa)? How long can a finite system be observed until it "feels" its finiteness and which effects of finiteness do occur?

We start with the definition of the finite versions of the d-dimensional BBM and SBM. Fix $d\in\mathbb{N}$ and let Λ^d_ℓ, $\ell>0$, be the torus of size ℓ,

$$\Lambda^d_\ell := \mathbb{R}^d/(\ell\mathbb{Z}^d). \tag{2.6}$$

We will regard Λ^d_ℓ as the cube $[0,\ell[^d$ with periodic boundary conditions. Λ^d_ℓ inherits the Brownian motion $(X_{\ell,t})_{t\ge 0}$ from $\mathbb{R}^d$. That is, $(X_{\ell,t})$ has transition densities

$$p_{\ell,t}(x,y) = \sum_{k\in\mathbb{Z}^d} p_t(x, y+\ell k), \tag{2.7}$$

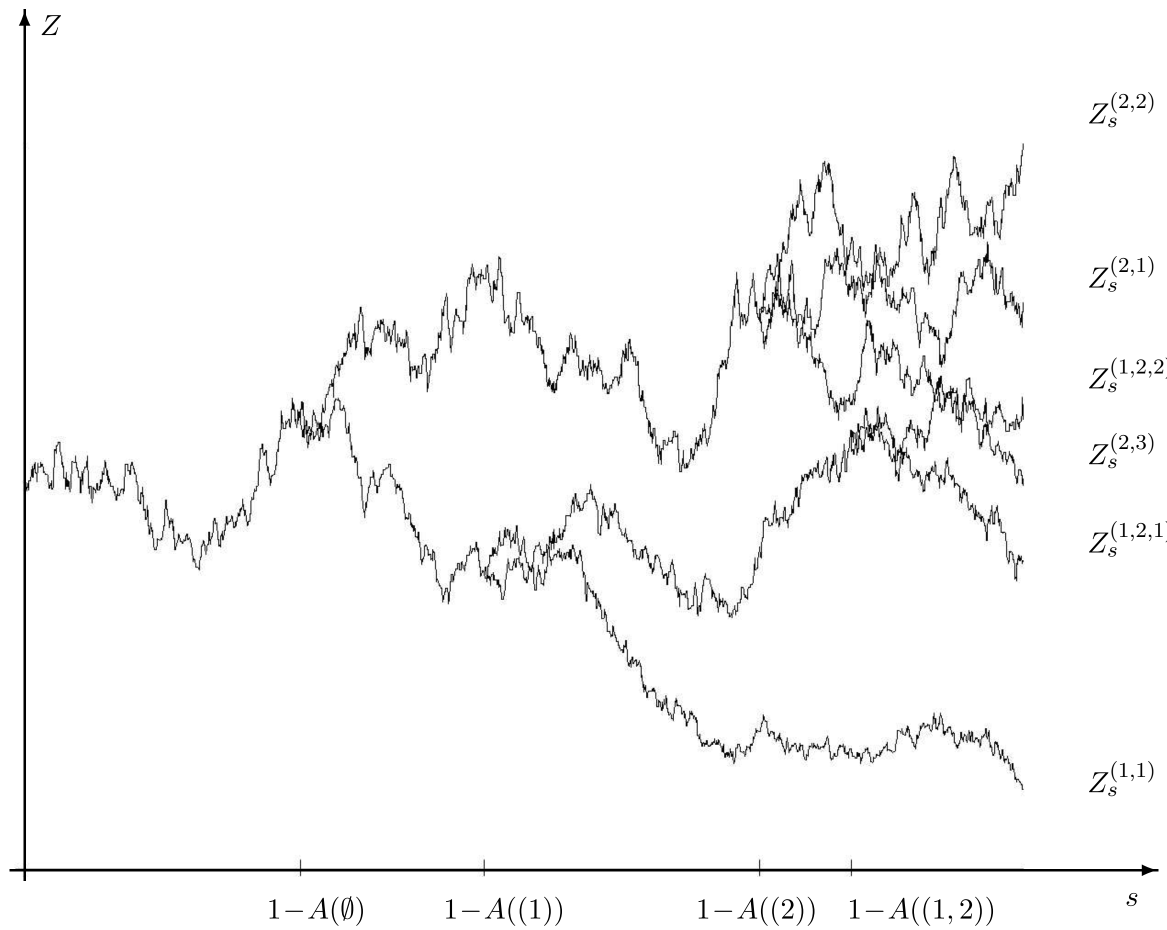

Figure 3. A sample of $(Z_s^e)_{s\geq 0}$, $e \in \mathbb{T}^M$ *for*
$\mathbb{T} = \{\emptyset, (1), (2), (1,1), (1,2), (1,2,1), (1,2,2), (2,1), (2,2), (2,3)\}$

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where

$$p_t(x,y) = (2\pi t)^{-d/2} \exp\left(-\frac{\|x-y\|^2}{2t}\right) \tag{2.8}$$

is the transition density of d-dimensional Brownian motion. Finally, denote by $M_\ell(\rho)$, $\mathcal{H}_\ell(\rho)$ etc. the restrictions of $M(\rho)$, $\mathcal{H}(\rho)$ etc. to Λ_ℓ^d.

The objects of interest will be critical binary branching Brownian motion $(\eta_{\ell,t})_{t\geq 0}$ on Λ_ℓ^d, abbreviated BBM(Λ_ℓ^d), and super Brownian motion $(\zeta_{\ell,t})_{t\geq 0}$ on Λ_ℓ^d, abbreviated SBM(Λ_ℓ^d). Again let $(\psi_{\ell,t})_{t\geq 0}$ be either BBM(Λ_ℓ^d) or SBM(Λ_ℓ^d). The behaviour of the system is dictated by the *empirical population density* of the finite system,

$$\ell^{-d}\psi_{\ell,t}(\Lambda_\ell^d).$$

Note that we have

$$\mathcal{L}^{M_\ell(\rho)}\left[\ell^{-d}\psi_{\ell,T(\ell)}(\Lambda_\ell^d)\right] \overset{\ell\to\infty}{\Longrightarrow} \mathcal{L}^\rho[Z_{\sigma/2}], \tag{2.9}$$

if the observation time $T(\ell)$ satisfies

$$\ell^{-d}T(\ell) \stackrel{\ell\to\infty}{\longrightarrow} \sigma, \qquad \sigma \in [0,\infty]. \tag{2.10}$$

The idea of how to describe stable, i.e., $d \geq 3$, finite systems is suggested by Cox and Greven (1990) and (1994): The system is dominated by the macroscopic variable of the empirical population density. Roughly speaking, it relaxes to the "equilibrium state" ν_θ with intensity θ, given that the empirical population density is θ. This relaxation takes place faster than the fluctuation of the empirical population density.

Thus, by (2.9), ℓ^d is the right time scale to look at the finite system. At this scale the empirical population density becomes random.

With these heuristics we are prepared for (recall ν_ρ from (1.13))

Theorem 3 (Finite System, Stable Case)
Let $d \geq 3$ and $(\psi_{\ell,t})_{t\geq 0}$ be either BBM(Λ_ℓ^d) or SBM(Λ_ℓ^d). Fix $\sigma \in [0,\infty]$ and $T(\ell)$ such that $\ell^{-d}T(\ell) \stackrel{\ell\to\infty}{\longrightarrow} \sigma$. Then the following holds

$$\mathcal{L}^{M_\ell(\rho)}\left[\psi_{\ell,T(\ell)}\right] \stackrel{\ell\to\infty}{\Longrightarrow} \int_0^1 \mathbf{P}^\rho[Z_{\sigma/2} \in d\theta]\nu_\theta. \tag{2.11}$$

2.3 Finite Systems, Critical Dimension

In dimension $d = 2$ we have to modify the ideas developed above in the fashion of rescaling presented in Subsection 2.1.

Fix $\alpha \in I$ and let for $t, \ell > 1$,

$$\widetilde{\psi}^\alpha_{\ell,t}(B) = \frac{8\pi}{\log t} t^{-\alpha}\psi_{\ell,t}\left((t^{\alpha/2}B) \cap \Lambda_\ell^2\right), \qquad B \in \mathcal{B}(\mathbb{R}^2). \tag{2.12}$$

Denote by $\widetilde{M}_\ell(t)$ the restriction of $\widetilde{M}(t)$ to Λ_ℓ^2. Then

$$\mathcal{L}^{\widetilde{M}_\ell(T(\ell))}\left[\widetilde{\psi}_{\ell,T(\ell)}(\Lambda_\ell^2)\right] \stackrel{\ell\to\infty}{\Longrightarrow} \mathcal{L}^1[Z_{4\pi\sigma}], \tag{2.13}$$

if the observation time $T(\ell)$ satisfies

$$\frac{T(\ell)}{\beta(\ell)} \stackrel{\ell\to\infty}{\longrightarrow} \sigma, \qquad \sigma \in [0,\infty]. \tag{2.14}$$

Here

$$\beta(\ell) = \ell^2 \log \ell. \tag{2.15}$$

It is due to the high density rescaling that $\beta(\ell) = \ell^2 \log \ell$ is the right time scale to be used in the critical dimension. Many models in the critical dimension show a behaviour similar to (2.13). Namely, linearly interacting diffusions with compact state space (Fisher-Wright, Fleming-Viot etc.), the voter model, etc. Interacting diffusions have been investigated in the "critical dimension" on the so-called hierarchical group by Fleischmann and Greven (1994), Dawson and Greven (1993a, 1993b), Dawson, Greven and Vaillancourt (1995), and Klenke (1996). Cox

(1989) and Cox and Greven (1991) treat the voter model on $\mathbb{Z}^2$. The point seems to be that the Green function of the interaction kernel is growing so slowly that taking the block averages is asymptotically the same as re-normalisation. Thus the role of the limiting diffusion (here Feller's diffusion in (2.13)) is played by the fixed point of the re-normalisation (see also Baillon et al. (1995)). The appropriate time scale in these models is the volume of the finite box times the recurrent potential kernel of the interaction kernel, maximised over the box. For an extensive treatment of this latter point see Theorem 1 of Klenke (1996).

Having in mind the proceeding of Subsection 2.1, the finite versions of Theorem 1 and 2 are easy to guess.

Theorem 4 (Finite System, $d = 2$)
Let $(\psi_{\ell,t})_{\ell,t}$ be either BBM(Λ^2_ℓ) or SBM(Λ^2_ℓ) and $I = [0,1]$ resp. $]-\infty,1]$. Fix $\sigma \in [0,\infty]$ and $T(\ell)$ such that $T(\ell)/\beta(\ell) \overset{\ell\to\infty}{\longrightarrow} \sigma$. Then the following holds

$$\mathcal{L}^{\widetilde{M}_\ell(T(\ell))}\left(\widetilde{\psi}^\alpha_{\ell,T(\ell)}\right) \overset{\ell\to\infty}{\Longrightarrow} \int_0^\infty \mathbf{P}^1[Z_{2\pi\sigma}\in d\rho]\mathcal{L}^\rho[Z_{1-\alpha}] = \mathcal{L}^1[Z_{2\pi\sigma+1-\alpha}], \qquad \alpha\in I. \tag{2.16}$$

Remark: Cox and Greven (1991) suggested to study the asymptotics of occupation times for the related model of branching random walk on $\mathbb{Z}^2$. Note that our result is more detailed than a description of the occupation time in that a time average is not made.

Let $\mathbb{L} = (\mathbb{T}, A)$ be a multiple space scale and let $X = (x^e_\ell, e\in\mathbb{T}, \ell\geq 0)$ be $\mathbb{L}$-scaled.

Theorem 5 (Finite System, Multiple Scale)
Under the conditions of Theorem 4 the following holds

(a) $$\mathcal{L}^{\widetilde{M}(\ell)}\left[\left(\mathcal{S}_{A(e),T(\ell)}\mathcal{T}_{x^e_\ell}\widetilde{\psi}_{\ell,T(\ell)}\right)_{e\in\mathbb{T}}\right] \overset{\ell\to\infty}{\Longrightarrow} \int_0^\infty \mathbf{P}^1[Z_{2\pi\sigma}\in d\rho]\,\mathcal{L}^\rho\left[\left(Z^e_{1-A(e)}\cdot\lambda\right)_{e\in\mathbb{T}}\right].$$

In particular, for $\mathbb{T}$ linear and $B\in\mathcal{B}(\mathbb{R}^2)$ bounded,

(b) $$\mathcal{L}^{\widetilde{M}_\ell(T(\ell))}\left[(\widetilde{\psi}^\alpha_{\ell,T(\ell)}(B))_{\alpha\in I}\right] \underset{\text{fdd}}{\overset{\ell\to\infty}{\Longrightarrow}} \mathcal{L}^1\Big[|B|\cdot(Z_{2\pi\sigma+1-\alpha})_{\alpha\in I}\Big].$$

2.4 Outline

The rest of this paper is organised as follows. In Section 3 we will provide some tools needed later. This includes moment formulas, coupling techniques and comparison techniques. In Section 4, we prepare for the proof of Theorem 1 with an admittedly rather tedious moment calculation. Theorem 1 will be proved in Section 5. There we also apply the refined coupling methods in order to prove Theorem 2. In Section 6, the finite version theorems are proved with the comparison techniques from Section 3.

3 Basic Tools

In this section we develop the following tools for the investigation of the long time behaviour of our branching processes:

- We give a general basic *coupling* lemma and then give its applications to the special setting of an underlying Brownian motion. A further refinement will be obtained by the so-called local coupling (Lemma 3.5). This is the main result of this section. It serves to speed up the coupling. Hence it overcomes the difficulty that the subsequently given comparison technique works only for times $L(t)$ of order $L(t) \ll t^2$.
- We give a simple *comparison* technique
- We give $n-$th *moment* (recursion) *formulas*

For logical reasons we start with the presentation of the moment formulas.

3.1 Moment Formulas

Let E be either $\mathbb{R}^d$ or Λ_ℓ^d. We will develop recursion formulas for the moments of BBM(E) and SBM(E).

We start with $(\eta_t)_{t\geq 0}$ BBM(E).

Lemma 3.1 (Moment Formula, BBM) *Let $(\eta_t)_{t\geq 0}$ be a BBM(E), where E is Λ_ℓ^d or $\mathbb{R}^d$. Denote by $(S_t)_{t\geq 0}$ the semigroup of Brownian motion on E.*

(a) *For $n \in \mathbb{N}$, $x \in E$ and $\phi : E \to \mathbb{R}$ measurable and bounded or non-negative the n-th moment fulfills the following recursion formula*

$$\mathbf{E}^x[\langle \eta_t, \phi\rangle^n] = \langle \delta_x, S_t(\phi^n)\rangle + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\int_0^t S_{t-s}\left(\mathbf{E}^{\cdot}[\langle \eta_s, \phi\rangle^k]\mathbf{E}^{\cdot}[\langle \eta_s, \phi\rangle^{n-k}]\right)(x)\, ds. \quad (3.1)$$

In particular, the first and second moments are

$$\mathbf{E}^x\left[\langle \eta_t, \phi\rangle\right] = \langle \delta_x, S_t\phi\rangle, \quad (3.2)$$

$$\mathbf{E}^x\left[\langle \eta_t, \phi\rangle^2\right] = \langle \delta_x, S_t(\phi^2)\rangle + \left\langle \delta_x, \int_0^t S_{t-s}((S_s\phi)^2)\, ds\right\rangle. \quad (3.3)$$

(b) *For $\mu \in \mathcal{N}_{\mathrm{f}}(E)$, or $\mu \in \mathcal{N}(E)$ and ϕ bounded with compact support, the first and second moments are*

$$\mathbf{E}^\mu\left[\langle \eta_t, \phi\rangle\right] = \langle \mu, S_t\phi\rangle, \quad (3.4)$$

$$\mathbf{E}^\mu\left[\langle \eta_t, \phi\rangle^2\right] = \langle \mu, S_t\phi\rangle^2 + \left\langle \mu, \int_0^t S_{t-s}((S_s\phi)^2)\, ds\right\rangle + \left\langle \mu, S_t(\phi^2) - (S_t\phi)^2\right\rangle. \quad (3.5)$$

Proof For $f : \mathcal{N}_{\mathrm{f}} \to \mathbb{R}$ in the domain of the generator of BBM($\mathbb{R}^d$), $f(\eta_t)$ fulfills the following Kolmogorov backward equation

$$\frac{\partial}{\partial t}\mathbf{E}^{\delta_x}\left[f(\eta_t)\right] = \frac{1}{2}\Delta\mathbf{E}^{\delta_x}\left[f(\eta_t)\right] + \frac{1}{2}\mathbf{E}^{2\delta_x}\left[f(\eta_t)\right] + \frac{1}{2}\mathbf{E}^{\mathbf{0}}\left[f(\eta_t)\right] - \mathbf{E}^{\delta_x}\left[f(\eta_t)\right], \quad (3.6)$$

where Δ denotes the Laplace operator with respect to x and $\mathbf{0} \in \mathcal{N}_{\mathrm{f}}(E)$ means the zero measure. In particular, for $\phi : E \to [0,\infty[$ twice continuously differentiable, $n \in \mathbb{N}$ and $f(\mu) = \langle \mu, \phi\rangle^n$, equation (3.6) becomes (using the independence of the particles)

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta\right) \mathbf{E}^x\left[\langle \eta_t, \phi\rangle^n\right] = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\mathbf{E}^x[\langle \eta_t, \phi\rangle^k]\mathbf{E}^x[\langle \eta_t, \phi\rangle^{n-k}]. \tag{3.7}$$

Integrating this yields (3.1). By an approximation argument, (3.7) holds for $\phi : E \to \mathbb{R}$ measurable and bounded or non-negative.

For part (b) note that by the independence of the particles we have

$$\mathbf{E}^\mu[\langle \eta_t, \phi\rangle^2] = \langle \mu, S_t\phi\rangle^2 + \int \mu(dx)\mathrm{Var}^x[\langle \eta_t, \phi\rangle] \tag{3.8}$$

and use part (a). □

We continue with a moment recursion formula for SBM(E).

Lemma 3.2 (Moment Formula, SBM) *Let $(\zeta_t)_{t\geq 0}$ be a SBM(E), where E is Λ_ℓ^d or $\mathbb{R}^d$. Recall that $(S_t)_{t\geq 0}$ is the semigroup of Brownian motion on E. Let $\phi : E \to [0,\infty[$ be bounded, measurable and with compact support and let $\mu \in \mathcal{M}(E)$. Then, for $t \geq 0$ and $n \in \mathbb{N}$,*

$$\mathbf{E}^\mu[\langle \zeta_t, \phi\rangle^n] = \sum_{k=0}^{n-1}\binom{n-1}{k}\langle \mu, u^{(n-k)}(t)\rangle\, \mathbf{E}^\mu[\langle \zeta_t, \phi\rangle^k], \tag{3.9}$$

where $u^{(n)}(t) : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$u^{(n)}(t) = \begin{cases} S_t\phi & , \quad n = 1 \\ \dfrac{1}{2}\displaystyle\sum_{k=1}^{n-1}\binom{n}{k}\int_0^t S_{t-s}\left(u^{(k)}(s)u^{(n-k)}(s)\right) ds & , \quad n \geq 2. \end{cases} \tag{3.10}$$

In particular (for ϕ not necessarily non-negative),

$$\mathbf{E}^\mu[\langle \zeta_t, \phi\rangle] = \langle \mu, S_t\phi\rangle, \tag{3.11}$$

$$\mathbf{E}^\mu[\langle \zeta_t, \phi\rangle^2] = \langle \mu, S_t\phi\rangle^2 + \left\langle \mu, \int_0^t S_{t-s}((S_s\phi)^2)\, ds\right\rangle. \tag{3.12}$$

Note that the first moment coincides with that of BBM while the second moment of BBM is greater than that of SBM. This reflects the fact that the "motion part" of SBM is deterministic while that of BBM is random.

The result and the idea of the proof can be found in Dawson (1993), Lemma 4.7.1. Unfortunately there are some misprints. So we give the proof in detail. **Proof** Recall from (1.8) that (V_t) is the log-Laplace semigroup of (ζ_t). Also recall that we assumed $c = 1$ in (1.7). For $\theta \geq 0$ and $n \in \mathbb{N}$ let

$$u^{(n)}(t,\theta) = (-1)^{n-1}\frac{\partial^n}{(\partial\theta)^n}V_t(\theta\phi) \tag{3.13}$$

and

$$u^{(0)}(t,\theta) = -V_t(\theta\phi).$$

We can calculate $u^{(n)}(t,\theta)$ recursively with (1.7),

$$u^{(n)}(t,\theta) = \begin{cases} S_t\phi & , \quad n=1 \\ \frac{1}{2}c\int_0^t S_{t-s}\left(\sum_{k=0}^{n}\binom{n}{k}u^{(k)}(s,\theta)u^{(n-k)}(s,\theta)\right)ds & , \quad n\geq 2. \end{cases} \tag{3.14}$$

Differentiating (1.8) w.r.t. θ yields

$$\langle\mu, u^{(1)}(t,\theta)\rangle\,\mathbf{E}^\mu\left[\langle\zeta_t,\phi\rangle\exp(-\theta\langle\zeta_t,\phi\rangle)\right] = \mathbf{E}^\mu\left[\exp(-\theta\langle\zeta_t,\phi\rangle)\right]. \tag{3.15}$$

Differentiate (3.15) $(n-1)$-times w.r.t. θ to obtain

$$\mathbf{E}^\mu\left[\langle\zeta_t,\phi\rangle^n\exp(-\theta\langle\zeta_t,\phi\rangle)\right] = \sum_{k=0}^{n-1}\binom{n-1}{k}\langle\mu, u^{(n-k)}(t,\theta)\rangle\,\mathbf{E}^\mu\left[\langle\zeta_t,\phi\rangle^k\exp(-\theta\langle\zeta_t,\phi\rangle)\right]. \tag{3.16}$$

Evaluating (3.16) at $\theta=0$ yields the assertion.

To see that the second moment formula still holds for ϕ, assuming also negative values, let $\phi=\phi^+-\phi^-$, where $\phi^+=\phi\vee 0$ and $\phi^-=(-\phi)\vee 0$. Now use

$$\mathbf{E}^\mu[\langle\zeta_t,\phi\rangle^2] = 2\mathbf{E}^\mu[\langle\zeta_t,\phi^+\rangle^2] + 2\mathbf{E}^\mu[\langle\zeta_t,\phi^-\rangle^2] - \mathbf{E}^\mu[\langle\zeta_t,\phi^++\phi^-\rangle^2].$$

□

3.2 Coupling

In this subsection we shall construct two different couplings for our processes, the so-called basic coupling lemma (Lemma 3.3) and the local coupling (Lemma 3.5). On the way we recall in Lemma 3.4 the usual coupling for Brownian motions. We start by explaining the notion of coupling in general.

Let $(S_t)_{t\geq 0}$ be the semigroup of a Feller process on the locally compact Polish space E. By a *coupling* we mean a bivariate Feller process $(X_t,Y_t)_{t\geq 0}$ with càdlàg paths such that (X_t) and (Y_t) are each copies of a Feller process with semigroup (S_t). Note that in general these copies are not independent. This definition is more general than the usual definition. In particular, our coupling does not need to be successful. In fact, we will use different notions of the "success" of a coupling.

Define the coupling time τ by

$$\tau = \inf\{t\geq 0 : X_t = Y_t\}. \tag{3.17}$$

We say that the coupling is *successful* for $(x,y)\in E\times E$ if $\mathbf{P}^{(x,y)}[\tau<\infty]=1$ and

$$\mathbf{P}^{(x,y)}[\{X_t\neq Y_t\}\cap\{\tau<t\}] = 0 \qquad \forall\, t\geq 0. \tag{3.18}$$

We come to the first coupling (basic coupling). It deals with the coupling of two deterministic initial configurations μ^1 and μ^2.

Let $\mu^1, \mu^2 \in \mathcal{M}_\mathrm{f}(E)$ and define $\mu \in \mathcal{M}_\mathrm{f}(E \times E)$ by $\mu = \mu^1 \otimes \mu^2$. We need that the coupling time is (stochastically) uniformly bounded for all starting points in the support of μ. Thus we assume that there exists a non-negative random variable H such that

$$\mathcal{L}^{(x,y)}[\tau] \leq \mathcal{L}[H] \text{ stochastically for } \mu\text{-almost all } (x,y) \in E \times E. \tag{3.19}$$

Also we assume that (3.18) holds. For $A \in \mathcal{B}(E)$ let

$$C_t(A) = \sup\{(S_t \mathbb{1}_A)(x),\ x \in \operatorname{supp}(\mu^1 + \mu^2)\}.$$

Let $(\gamma_t^1)_{t\geq 0}$ and $(\gamma_t^2)_{t\geq 0}$ be binary branching processes or super processes associated with (S_t). In the former case we will also assume that $\mu^1, \mu^2 \in \mathcal{N}_\mathrm{f}(E)$.

Lemma 3.3 (Basic Coupling)
There exists a coupling $(\gamma_t^1, \gamma_t^2)_{t\geq 0}$ *with* $\gamma_0 = (\mu^1, \mu^2)$ *that is successful in the sense that*

$$\mathbf{E}\left[\left\|(\gamma_t^1 - \gamma_t^2)\Big|_A\right\|\right] \leq C_t(A) \cdot \Big|\|\mu^1\| - \|\mu^2\|\Big| + 2\min(\|\mu^1\|, \|\mu^2\|) \cdot \mathbf{P}[H > t]. \tag{3.20}$$

In particular, for $\|\mu^1\| = \|\mu^2\|$,

$$\mathbf{E}\left[\|(\gamma_t^1 - \gamma_t^2)\|\right] \leq 2\|\mu^1\| \cdot \mathbf{P}[H > t]. \tag{3.21}$$

Proof W.l.o.g. we may assume $\|\mu^1\| \leq \|\mu^2\|$. Let $\mu^2 = \bar{\mu}^2 + \widetilde{\mu}^2$ be a decomposition of μ^2 such that $\|\bar{\mu}^2\| = \|\mu^1\|$. Then (3.19) holds with μ^2 replaced by either $\bar{\mu}^2$ or $\widetilde{\mu}^2$. It is clear (by the first moment formulas of the previous subsection) that (3.20) holds for any coupling $\widetilde{\gamma}_t = (\widetilde{\gamma}_t^1, \widetilde{\gamma}_t^2)$ with $\widetilde{\gamma}_0 = (0, \widetilde{\mu}^2)$. Thus if we can show (3.21) for $(\bar{\gamma}_t)$ with $\bar{\gamma}_0 = (\mu^1, \bar{\mu}^2)$, we are done by setting $\gamma_t^i = \bar{\gamma}_t^i + \widetilde{\gamma}_t^i,\ i = 1, 2$.

Thus we will now assume $\|\mu^1\| = \|\mu^2\|$. Let $\mu \in \mathcal{M}_\mathrm{f}(E \times E)$ (resp. $\mu \in \mathcal{N}_\mathrm{f}(E \times E)$) with marginals $\mu^1(\cdot) = \mu(\cdot \times E)$ and $\mu^2(\cdot) = \mu(E \times \cdot)$. Let $(X_t, Y_t)_{t\geq 0}$ and τ be as above. Then we have by assumption

$$\mathbf{P}^{(x,y)}[X_t \neq Y_t] \leq \mathbf{P}[H > t] \qquad \text{for } \mu\text{-almost all } (x,y). \tag{3.22}$$

Define $(\gamma_t)_{t\geq 0}$ to be the critical branching (or super) process on $E \times E$ associated with the bivariate process $(X_t, Y_t)_{t\geq 0}$ on $E \times E$. For $t \geq 0$ we have that γ_t is in $\mathcal{M}_\mathrm{f}(E \times E)$, resp. $\mathcal{N}_\mathrm{f}(E \times E)$, almost surely. Let $\gamma_t^1(\cdot) = \gamma_t(\cdot \times E)$ and $\gamma_t^2(\cdot) = \gamma_t(E \times \cdot)$ be its marginals. Since the branching mechanism is spatially homogeneous, $(\gamma_t^1)_{t\geq 0}$ and $(\gamma_t^2)_{t\geq 0}$ are critical branching (resp. super) processes associated with (X_t) and (Y_t). Thus (γ_t^1) and (γ_t^2) are both associated with (S_t). E.g., we show that (γ_t^1) is an (S_t)-super process. Let $q_t(x, y, A, B) = \mathbf{P}^{(x,y)}[X_t \in A, Y_t \in B]$ denote the transition kernel of (X_t, Y_t) and let $p_t(x, A) = \mathbf{P}^x[X_t \in A] = q_t(x, y, A, E)$. Let $\phi \in C_b(E),\ \phi \geq 0$, and let $\phi'(x, y) = \phi(x),\ x, y \in E$. Then

$$u_t(x,y) := -\log \mathbf{E}^{(x,y)}[\exp(-\langle \gamma_t, \phi' \rangle)] = -\log \mathbf{E}^{(x,y)}[\exp(-\langle \gamma_t^1, \phi \rangle)] \tag{3.23}$$

is the unique solution (see (1.7)) of $u_0(x,y) = \phi(x)$ and

$$u_t(x,y) = \int_{E\times E} q_t(x,y,dx',dy')\phi'(x',y') - \frac{1}{2}\int_0^t ds \int_{E\times E} q_{t-s}(x,y,dx',dy')u_s(x',y')^2. \quad (3.24)$$

Let (ζ_t) be an (S_t)-super process and let $v_t(x) = -\log \mathbf{E}^x[\exp(-\langle\zeta_t,\phi\rangle)]$. Then $v_0(x) = \phi(x)$ and

$$v_t(x) = \int_E p_t(x,dx')\phi(x') \ - \frac{1}{2}\int_0^t ds \int_E p_{t-s}(x,dx')v_s(x')^2. \quad (3.25)$$

Note that $v_t(x)$ solves (3.24). Thus $u_t(x,y) = v_t(x),\ x,y\in E$, and (γ_t^1) is an (S_t)-super process as claimed.

Denote by $D = \{(x,x) : x\in E\}$ the diagonal in $E\times E$. Then

$$\mathbf{E}^\mu\left[\left\|\gamma_t^1 - \gamma_t^2\right\|\right] \le \mathbf{E}^\mu[\gamma_t((E\times E)\setminus D)] \le 2\|\mu\|\cdot\mathbf{P}[H>t]. \quad (3.26)$$

□

We come back to the special situation $E=\mathbb{R}^d$ or $E=\Lambda_\ell^d$ and $(S_t)_{t\ge 0}$ the semigroup of Brownian motion on E. In this case there exists a successful coupling:

Lemma 3.4 *Let E be either Λ_ℓ^d or $\mathbb{R}^d$ and let $R>0$. For $x,y\in E$ with $\|x-y\|\le R$ there exists a coupling $(W_t^1,W_t^2)_{t\ge0}$ for the (standard) Brownian motion on E such that*

$$\mathbf{P}^{(x,y)}\left[W_t^1 \ne W_t^2\right] \le \sqrt{\frac{1}{\pi}}R\cdot t^{-1/2}. \quad (3.27)$$

Proof We may assume $E=\mathbb{R}^d$ since on Λ_ℓ^d the coupling works even better. By translation and orthogonal transformation, we may also assume $x=0$ and $y=(r,0,\dots,0)$ with $r=\|x-y\|\le R$.

If $d\ge 2$ we let

$$W_t^i = (Y_t^i, Z_t), \qquad i=1,2. \quad (3.28)$$

Here $(Z_t)_{t\ge0}$ is a Brownian motion on $\mathbb{R}^{d-1}$ with $Z_0=0$. The processes $(Y_t^1)_{t\ge0}$ and $(Y_t^2)_{t\ge0}$ are Brownian motions on $\mathbb{R}$ that move independently until they first meet and then move together. The initial points are $Y_0^1=0$ and $Y_0^2=r$. In the case $d=1$ we simply let $(W_t^i)=(Y_t^i),\ i=1,2$.

Let $H=\frac{1}{2}\inf\{t\ge0: Y_t^2=0\}$. Then (since $Y_t^2-Y_t^1$ is a Brownian motion running at double speed) $\mathcal{L}[\inf\{t\ge0: W_t^1=W_t^2\}] = \mathcal{L}^r[H]$. By the reflection principle,

$$\mathbf{P}^r[H>t] = \sqrt{\frac{2}{\pi}}\int_0^{r/\sqrt{2t}} e^{-u^2/2}\,du \le \sqrt{\frac{1}{\pi}}Rt^{-1/2}. \quad (3.29)$$

□

The aim is now to couple the evolutions of $(\psi_t)_{t\ge0}$ started from two different (random) configurations. In the context of our problem one of those laws is only vaguely known since it will be the result of long-time evolution of a (ψ_t)-type process. The other law will be better known. Typically it will be $M(\rho')$, where the (random) value ρ' is obtained by some averaging over the first configuration. The details follow in the subsequent sections.

Since $\mathrm{supp}(\gamma^1 + \gamma^2)$ will typically be too large to apply Lemma 3.4 directly, we have to construct a local coupling. The idea is the following.

We start with a translation invariant initial configuration. Thus the support *is* large. In order to apply Lemma 3.4 successfully, we divide E into boxes of length $R > 0$. We do the coupling independently in each box according to Lemma 3.4. Finally we have to shift the pattern of boxes by a random offset $z \in [0, R[^d$ in order to obtain a translation invariant coupling.

Let $Q = Q(d\gamma^1, d\gamma^2) \in \mathcal{M}_1(\mathcal{M}(E) \times \mathcal{M}(E))$ be translation invariant. That is, $\mathcal{T}_x Q = Q \;\forall\, x \in E$, where the translation $\mathcal{T}_x Q \in \mathcal{M}_1(\mathcal{M}(E) \times \mathcal{M}(E))$ is defined by

$$\begin{aligned} \int \mathcal{T}_x Q(d\gamma^1, d\gamma^2) e^{-\langle \gamma^1, f\rangle - \langle \gamma^2, g\rangle} &= \int Q(d\gamma^1, d\gamma^2) e^{-\langle \gamma^1, \mathcal{T}_x f\rangle - \langle \gamma^2, \mathcal{T}_x g\rangle} \\ &= \int Q(d\gamma^1, d\gamma^2) e^{-\langle \gamma^1, f(x+\cdot)\rangle - \langle \gamma^2, g(x+\cdot)\rangle}, \end{aligned} \tag{3.30}$$

for $f, g : E \to [0, \infty[$ measurable.

Fix $R > 0$. In the case $E = \Lambda_\ell^d$ we will assume that $t/R =: N \in \mathbb{N}$.

Lemma 3.5 (Local Coupling)
There exists a (translation invariant) coupling $(\psi_t^1, \psi_t^2)_{t \geq 0}$ of BBM(E) or SBM(E) with

$$\mathcal{L}[(\psi_0^1, \psi_0^2)] = Q \tag{3.31}$$

and such that

$$\begin{aligned} &\mathbf{E}[|(\psi_t^1 - \psi_t^2)\Big|_A|] \\ &\quad\leq |A| \cdot R^{-d} \Big[\mathbf{E}[|(\psi_0^1 - \psi_0^2)([0, R[^d)|] + \mathbf{E}[(\psi_0^1 + \psi_0^2)([0, R[^d)] \cdot \sqrt{d/\pi}\; R \cdot t^{-1/2} \Big]. \end{aligned} \tag{3.32}$$

Proof Fix an initial configuration $(\mu^1, \mu^2) \in \mathcal{M}(E) \times \mathcal{M}(E)$. Let

$$C_k = kR + [0, R[^d, \tag{3.33}$$

for $k \in \mathbb{Z}^d$ (or $k \in \{0, \dots, N-1\}^d$ if $E = \Lambda_\ell^d$). Let

$$\mu_k^i = \mu^i \mathbb{1}_{C_k}, \qquad i = 1, 2, \text{ for each } k. \tag{3.34}$$

We want to use the independence in the branching systems to obtain a coupling $(\gamma_{k,t}^1, \gamma_{k,t}^2)_{t \geq 0}$ for μ_k^1 and μ_k^2, for each k separately. Fix k. We apply Lemma 3.3 and Lemma 3.4 with $A = E$ (note that two points in C_k have distance at most $R\sqrt{d}\,$) to get

$$\mathbf{E}^{(\mu_k^1, \mu_k^2)}\left[\left\|\gamma_{k,t}^1 - \gamma_{k,t}^2\right\|\right] \leq \Big| \|\mu_k^1\| - \|\mu_k^2\| \Big| + 2 \min(\|\mu_k^1\|, \|\mu_k^2\|) \cdot \sqrt{d/\pi}\; R \cdot t^{-1/2}. \tag{3.35}$$

Integrating (3.35) with respect to $Q(d\mu^1, d\mu^2)$ and using translation invariance we get

$$\mathbf{E}[\|\gamma_{k,t}^1 - \gamma_{k,t}^2\|] \leq \mathbf{E}[|(\psi_0^1 - \psi_0^2)(C_0)|] + \mathbf{E}[(\psi_0^1 + \psi_0^2)(C_0)] \cdot \sqrt{d/\pi}\; R \cdot t^{-1/2} =: \varepsilon. \tag{3.36}$$

If we let $\gamma_t^i = \sum_k \gamma_{k,t}^i$, $i = 1, 2$, then $\mathcal{L}[(\gamma_0^1, \gamma_0^2)] = Q$ and (by translation invariance)

$$\mathbf{E}[\|(\gamma_t^1 - \gamma_t^2)\Big|_{C_k}\|] \leq \varepsilon \qquad \forall\, k. \tag{3.37}$$

Note that in the last step we have used the σ-additivity of $\|(\gamma_t^1 - \gamma_t^2)\Big|_C\|$ as a function of $C \in \mathcal{B}(E)$. In order to get a translation invariant coupling, we pick $z \in C_0$ at random and shift the "grid" $R\mathbb{Z}^d$ by z: For $z \in C_0$ define $(\gamma_t^i(z))_{t\geq 0}$, $i = 1, 2$, as above with C_k replaced by $C_k(z) = z + C_k$. Let

$$\mathcal{L}[\psi_t^i] = \frac{1}{R^d}\int_{C_0} \mathcal{L}[\gamma_t^i(z)]\,dz, \qquad i = 1, 2. \tag{3.38}$$

Then (ψ_t^1, ψ_t^2) is a coupling with the asserted properties: (3.31) holds because it holds for each $(\psi_0^1(z), \psi_0^2(z))$, $z \in C_0$. By construction, $\mathbf{E}\left[\left\|(\psi_t^1 - \psi_t^2)\Big|_B\right\|\right]$ is translation invariant on E as a measure in B. Hence it is a multiple of the Lebesgue measure on E. By (3.37), its density is $\leq \varepsilon/R^d$.

□

Corollary 3.6 *Let $Q \in \mathcal{M}_1(\mathcal{M}(\Lambda_\ell^d)\times\mathcal{M}(\Lambda_\ell^d))$ or $\mathcal{M}_1(\mathcal{N}(\Lambda_\ell^d)\times\mathcal{N}(\Lambda_\ell^d))$ be translation invariant with*

$$\rho := t^{-d}\int \gamma^1(\Lambda_\ell^d)\,Q(d\gamma^1, d\gamma^2) < \infty. \tag{3.39}$$

Given γ^1, under $Q(d\gamma^1, d\gamma^2)$, the distribution of γ^2 shall be $M_t(\rho')$ with $\rho' := t^{-d}\gamma^1(\Lambda_\ell^d)$.

Let further $N \in \mathbb{N}$, $R = t/N$ and $\varepsilon > 0$ such that

$$\mathbf{E}[|\gamma^1(\Lambda_\ell^d) - N^d\gamma^1([0, R[^d)|] < \varepsilon t^d. \tag{3.40}$$

Then there exists a coupling $(\psi_{\ell,t}^1, \psi_{\ell,t}^2)_{t\geq 0}$ of BBM(Λ_ℓ^d) or SBM(Λ_ℓ^d) with $\mathcal{L}[(\psi_{\ell,0}^1, \psi_{\ell,0}^2)] = Q$ and such that for $B \in \mathcal{B}(\Lambda_\ell^d)$ and $t \geq 0$,

$$\mathbf{E}\left[\left\|(\psi_{\ell,t}^1 - \psi_{\ell,t}^2)\Big|_B\right\|\right] \leq |B|\cdot\left[\varepsilon + 2\sqrt{\rho R^{-d}} + 2\sqrt{d/\pi}\,\rho R\cdot t^{-1/2}\right]. \tag{3.41}$$

If (ψ_t) is SBM(Λ_ℓ^d), the term $2\sqrt{\rho R^{-d}}$ on the r.h.s. of (3.41) can be dropped.

Proof In the case of SBM clearly $\mathbf{E}[|(\psi_{\ell,0}^1 - \psi_{\ell,0}^2)([0, R[^d)|] \leq \varepsilon R^d$. Consider now the case of BBM. Note that for a Poisson random variable X with mean $\theta > 0$, $\mathbf{E}[|X-\theta|] \leq \sqrt{\theta} + \frac{1}{\sqrt{\theta}}\mathrm{Var}[X] = 2\sqrt{\theta}$. By this and Jensen's inequality we obtain

$$\begin{aligned}\mathbf{E}\left[\left|(\psi_{\ell,0}^1 - \psi_{\ell,0}^2)([0, R[^d)\right|\right] &\leq \varepsilon R^d + \mathbf{E}\left[\left|\gamma^2([0, R[^d) - N^{-d}\gamma^1(\Lambda_\ell^d)\right|\right] \\ &\leq \varepsilon R^d + 2\mathbf{E}\left[\sqrt{N^{-d}\gamma^1(\Lambda_\ell^d)}\right] \\ &\leq \varepsilon R^d + 2\sqrt{\rho R^d}\,.\end{aligned} \tag{3.42}$$

Now apply Lemma 3.5. □

Corollary 3.7 *Let $S > R > 0$ and $E = \mathbb{R}^d$. Consider $(\psi_t^1)_{t\geq 0}$ BBM($\mathbb{R}^d$) or SBM($\mathbb{R}^d$). Assume that $\mathcal{L}[\psi_0^1]$ is translation invariant and that $\varepsilon, \delta > 0$ and $0 < \rho < \infty$ are chosen such that*

$$\begin{aligned} \mathbf{E}[\psi_0^1([0,1[^d)] &= \rho \\ \mathbf{E}[|R^{-d}\psi_0^1([0,R[^d) - S^{-d}\psi_0^1([0,S[^d)|] &< \varepsilon \qquad (3.43) \\ \mathbf{E}[|\psi_0^1([0,S[^d) - \psi_0^1(S(z+[0,1[^d))|] &< \delta S^d \quad \forall\, z \in [-1,1]^d. \qquad (3.44) \end{aligned}$$

Then there exists a coupling $(\psi_t^1, \psi_t^2)_{t\geq 0}$ such that

$$\mathcal{L}[\psi_0^2 \Big| \psi_0^1] = M(S^{-d}\psi_0^1([0,S[^d)) \qquad (3.45)$$

and for $t > 0$,

$$\mathbf{E}[\|(\psi_t^1 - \psi_t^2)\Big|_B\|] \leq |B| \cdot \left[\varepsilon + 3\delta + d\, e^{-D^2/2t} + 2\sqrt{\rho R^{-d}} + 2\sqrt{d/\pi}\, \rho R t^{-1/2}\right], \qquad (3.46)$$

where $B \in \mathcal{B}(\mathbb{R}^d)$, $B \subset [0,S[^d$ and $D = \operatorname{dist}(B, \mathbb{R}^d \setminus [0,S[^d)$. If (ψ_t) is SBM($\mathbb{R}^d$), the term $2\sqrt{\rho R^{-d}}$ on the r.h.s. of (3.46) can be dropped.

Remark The coupling takes place at scale R while the averaging takes place at scale S. The conditions (3.43) and (3.44) make sure that ψ_0^1 does not vary too much on these scales. **Proof** If the common distribution of ψ_0^1 and ψ_0^2 was translation invariant we could argue as in Corollary 3.6. However, in general it is not. So we have to work a little more. The aim is to construct a third process $(\psi_t^3)_{t\geq 0}$ such that $\mathcal{L}[\psi_0^1, \psi_0^3]$ is translation invariant while ψ_t^2 and ψ_t^3 are close. Here are the details.

Recall that $*$ denotes the convolution in $\mathcal{M}_1(\mathcal{M}(\mathbb{R}^d))$ and that $Q\Big|_A$ is the restriction of $Q \in \mathcal{M}_1(\mathcal{M}(\mathbb{R}^d))$ to $A \in \mathcal{B}(\mathbb{R}^d)$ (see (1.2)). For $\gamma \in \mathcal{M}(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$ define

$$\Gamma(z,\gamma) = \mathop{\Huge *}_{k\in\mathbb{Z}^d} \Big(M(S^{-d}\gamma(S(k+z+[0,1[^d))) \Big|_{S(z+k+[0,1[^d)} \Big). \qquad (3.47)$$

Define ψ_0^1 and ψ_0^3 on one probability space such that

$$\mathcal{L}[\psi_0^3 | \psi_0^1] = \int_{[0,1[^d} \Gamma(z, \psi_0^1)\, dz.$$

We show that $\mathcal{L}[(\psi_0^1, \psi_0^3)]$ is translation invariant. Since $\mathcal{T}_x\Gamma(z,\mu) = \Gamma(z+x, \mathcal{T}_x\mu)$, $x \in \mathbb{R}^d$, we have $\int_{[0,1[^d} \mathcal{T}_x\Gamma(z,\mu)\, dz = \int_{[0,1[^d} \Gamma(z, \mathcal{T}_x\mu)\, dz$. Hence, for $f, g : \mathbb{R}^d \to [0,\infty[$ measurable, we have

$$\begin{aligned} &\mathbf{E}[\exp(-\langle \psi_0^1, \mathcal{T}_x f\rangle - \langle \psi_0^3, \mathcal{T}_x g\rangle)] \qquad (3.48) \\ &= \mathbf{E}\left[\exp(-\langle \psi_0^1, \mathcal{T}_x f\rangle) E\left[\exp\left(-\langle \psi_0^3, \mathcal{T}_x g\rangle\right) \Big| \psi_0^1\right]\right] \\ &= \mathbf{E}\left[\exp\left(-\langle \psi_0^1, \mathcal{T}_x f\rangle\right) \int_{[0,1[^d} dz \int_{\mathcal{M}(\mathbb{R}^d)} \Gamma(z, \psi_0^1)(dm) \exp\left(-\langle m, \mathcal{T}_x g\rangle\right)\right] \\ &= \mathbf{E}\left[\exp\left(-\langle \mathcal{T}_x\psi_0^1, f\rangle\right) \int_{[0,1[^d} dz \int_{\mathcal{M}(\mathbb{R}^d)} \mathcal{T}_x\Gamma(z, \psi_0^1)(dm) \exp\left(-\langle m, g\rangle\right)\right] \end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}\left[\exp\left(-\langle \mathcal{T}_x\psi_0^1, f\rangle\right)\int_{[0,1[^d} dz \int_{\mathcal{M}(\mathbb{R}^d)} \Gamma(z, \mathcal{T}_x\psi_0^1)(dm)\exp\left(-\langle m, g\rangle\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\langle \psi_0^1, f\rangle\right)\int_{[0,1[^d} dz \int_{\mathcal{M}(\mathbb{R}^d)} \Gamma(z, \psi_0^1)(dm)\exp\left(-\langle m, g\rangle\right)\right] \\
&= \mathbf{E}\left[\exp\left(-\langle \psi_0^1, f\rangle - \langle \psi_0^3, g\rangle\right)\right].
\end{aligned}$$

Then clearly (by a suitable coupling of the Poisson processes in (3.47) and (3.45) in the case of BBM) we can assume

$$\mathbf{E}[\|(\psi_0^3 - \psi_0^2)\Big|_A\|] \le \delta|A|, \quad A \subset [0, S[^d, \tag{3.49}$$

which implies that we can couple (ψ_t^2) and (ψ_t^3) such that

$$\begin{aligned}
\mathbf{E}[|(\psi_t^3 - \psi_t^2)(B)|] &\le \delta|B| + 2\rho \int_{\mathbb{R}^d\setminus[0,S[^d} dx \int_B dy\, p_t(x,y) \\
&\le |B|(\delta + 2\rho d\, e^{-D^2/2t}).
\end{aligned} \tag{3.50}$$

(This coupling is done by defining three independent processes with initial configurations $\psi_0^2 \wedge \psi_0^3$, $(\psi_0^2 - \psi_0^3)^+$, $(\psi_0^2 - \psi_0^3)^-$.) As in (3.42) we get

$$\begin{aligned}
\mathbf{E}[|(\psi_0^3 - \psi_0^1)([0,R[^d)|] &\le \mathbf{E}\left[\left|\psi_0^3([0,R[^d) - \mathbf{E}[\psi_0^3([0,R[^d)]\right|\right] \\
&\quad + \mathbf{E}\left[|\psi_0^1([0,R[^d) - \mathbf{E}[\psi_0^3([0,R[^d)\Big|\psi_0^1]|\right] \\
&\le 2\sqrt{\rho R^d} + (\varepsilon+\delta)R^d.
\end{aligned} \tag{3.51}$$

Now apply Lemma 3.5 to (ψ_0^1, ψ_0^3). □

3.3 Comparison

In this subsection we compare the finite versions of our branching processes to their infinite versions. We show that the finite system is not "too far off" from its infinite counterpart if the time $L(t)$ of observation is not too large. Unfortunately "not too large" here means $L(t) \ll t^2$. Hence the obtained comparison result is not at all surprising. However, with the strong tool of local coupling this will be sufficient for our purposes.

Lemma 3.8 (Comparison) *Let $\ell > 0$ and $A \in \mathcal{B}(\Lambda_\ell^d)$, $|A| > 0$, such that $D = \frac{1}{2}(\ell - \mathrm{diam}(A)) > 0$. There exist two BBM or SBM, $(\psi_t^1)_{t\ge0}$ on $\mathbb{R}^d$ and $(\psi_{\ell,t}^2)_{t\ge0}$ on Λ_ℓ^d, on one probability space such that for $t > 0$,*

$$\psi_0^1 = M(\rho) \qquad \textit{and} \qquad \psi_{\ell,0}^2 = M_\ell(\rho) \tag{3.52}$$

and

$$\mathbf{E}\left[\left|\psi_t^1(A) - \psi_{\ell,t}^2(A)\right|\right] \le 2d\exp\left(-\frac{D^2}{2t}\right)\cdot\rho|A|\frac{\sqrt{t}}{D}. \tag{3.53}$$

In particular, for a sequence $L(\ell) \ll \ell^2$ and $A_\ell = \ell^{\alpha/2}A$, $\alpha \in [0,2[$, we get uniformly in $\rho > 0$

$$\frac{\ell^{-d\alpha/2}}{\rho|A|}\mathbf{E}\left[\left|\psi^1_{L(\ell)}(\ell^{\alpha/2}A) - \psi^2_{\ell,L(\ell)}(\ell^{\alpha/2}A)\right|\right] \overset{\ell\to\infty}{\longrightarrow} 0. \tag{3.54}$$

Proof W.l.o.g. we may assume that A is centered in Λ^d_ℓ such that

$$\inf\{\|x-y\|,\, x \in A, y \in \mathbb{R}^d \setminus \Lambda^d_\ell\} \geq \frac{1}{2}(\ell - \mathrm{diam}(A)).$$

For $m \in \mathbb{Z}^d$ let $(\gamma^m_t)_{t\geq 0}$ be independent BBM($\mathbb{R}^d$) or SBM($\mathbb{R}^d$) with (independent) initial configurations

$$\mathcal{L}[\gamma^m_0] = M(\rho)\Big|_{\ell(m+[0,1[^d)}. \tag{3.55}$$

Let

$$\psi^1_t(\cdot) = \sum_{m\in\mathbb{Z}^d}\gamma^m_t(\cdot) \quad \text{and} \quad \psi^2_{\ell,t}(\cdot) = \sum_{m\in\mathbb{Z}^d}\gamma^0_t(m\ell + \cdot). \tag{3.56}$$

Then (ψ^1_t) and $(\psi^2_{\ell,t})$ are as asserted and we have to show (3.53). By construction,

$$\begin{aligned}
\mathbf{E}\left[\left|\psi^1_t(A) - \psi^2_{\ell,t}(A)\right|\right] &\leq \sum_{m\in\mathbb{Z}^d\setminus\{0\}}\mathbf{E}[\gamma^m_t(A)] + \mathbf{E}[\gamma^0_t(m\ell + A)] \\
&= 2\sum_{m\in\mathbb{Z}^d\setminus\{0\}}\mathbf{E}\left[\gamma^0_t(m\ell+A)\right] \\
&= 2\rho\int_{\mathbb{R}^d\setminus\Lambda^d_\ell}dx\int_A dy\, p_t(x,y) \\
&\leq 2\rho|A|\mathbf{P}^0\Big[\|W_t\| \geq D\Big],
\end{aligned} \tag{3.57}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$. The proof of (3.53) is now a standard estimate while (3.54) is an immediate consequence of (3.53). □

4 Moment Calculations in the Critical Dimension

In this section we give the asymptotics of the moments of BBM($\mathbb{R}^2$) and SBM($\mathbb{R}^2$). We will obtain bounds for the moments as well. These allow us to express the Laplace transform in terms of the moments in the next section.

Fix $B \in \mathcal{B}(\mathbb{R}^2)$ and $\alpha \in [0,1]$. For $t \geq 0$, let

$$B_t = B_{\alpha,t} = t^{\alpha/2}B. \tag{4.1}$$

For $n \in \mathbb{N}$, $x \in \mathbb{R}^2, s \geq 0$ and $t > 1$ we define

$$\begin{aligned}
m_n(x,s,t) &= m_n(x,s,t,\alpha) = \mathbf{E}^x\left[(\psi_s(B_{\alpha,t}))^n\right], &(4.2)\\
\widetilde{m}_n(x,s,t) &= \widetilde{m}_n(x,s,t,\alpha) = \frac{s}{(\log s)^{n-1}}t^{-n\alpha}\mathbf{E}^x\left[(\psi_s(B_{\alpha,t}))^n\right], &(4.3)
\end{aligned}$$

and

$$\varphi(x) = \frac{1}{2\pi} \exp\left\{-\|x\|^2/2\right\}, \quad x \in \mathbb{R}^2. \tag{4.4}$$

The proof of the following lemma relies on a recursion that requires some uniformity in the statements. This forces us to a somewhat cumbersome formulation.

Fix $x \in \mathbb{R}^2$ and three non-negative sequences $(a_t) \downarrow 0, (b_t) \downarrow 0$ and $(c_t) \uparrow \infty$.

Lemma 4.1 *Let $B \in \mathcal{B}(\mathbb{R}^2)$ be bounded and $\alpha \in [0,1]$.*

(a) Uniformly in β such that $1 \geq \beta \geq \alpha$ and uniformly in the sequences $(x_t)_{t\geq 0}$ and $(s_t)_{t\geq 0}$ such that $\left\|x_t/\sqrt{s_t} - x\right\| < a_t$ and $\left|\frac{\log s_t}{\log t} - \beta\right| < b_t$, and such that $s_t > t^\alpha c_t$, the following holds

$$\lim_{t\to\infty} \widetilde{m}_n(x_t, s_t, t, \alpha) = \varphi(x)\left(1 - \frac{\alpha}{\beta}\right)^{n-1} \frac{|B|^n n!}{(8\pi)^{n-1}} \tag{4.5}$$

and

$$\lim_{t\to\infty} \frac{1}{s_t}\int_{\mathbb{R}^2} \widetilde{m}_n(y, s_t, t, \alpha)\, dy = \left(1 - \frac{\alpha}{\beta}\right)^{n-1} \frac{|B|^n n!}{(8\pi)^{n-1}}. \tag{4.6}$$

(b) There exists $\Gamma < \infty$ such that

$$\sup_{t: t\geq s_t \geq 3}\ \sup_{n\in\mathbb{N}} \frac{1}{n!\Gamma^n} \widetilde{m}_n(x_t, s_t, t, \alpha) < \infty \tag{4.7}$$

and

$$\sup_{t: t\geq s_t \geq 3}\ \sup_{n\in\mathbb{N}} \frac{1}{n!\Gamma^n}\frac{1}{s_t}\int_{\mathbb{R}^2} \widetilde{m}_n(y, s_t, t, \alpha)\, dy < \infty. \tag{4.8}$$

Remark: We use the convention $(1-\alpha/\beta)^{n-1} = 1$ if $\alpha = \beta = 0$. This case is actually covered in Fleischman (1978). **Proof** Throughout this proof we will suppress the α where no ambiguities may occur.

Our main tool is the moment recursion formula for BBM($\mathbb{R}^d$) (recall p_t from (2.8))

$$\begin{aligned} \mathbf{E}^x[(\eta_s(A))^n] &= \mathbf{E}^x[\eta_s(A)] \\ &+ \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\int_0^s du \int_{\mathbb{R}^2} dy\, p_{s-u}(x,y)\mathbf{E}^y[(\eta_u(A))^k]\mathbf{E}^y[(\eta_u(A))^{n-k}] \quad \forall\, A \in \mathcal{B}(\mathbb{R}^2), \end{aligned} \tag{4.9}$$

(this is (3.1) with $\phi = 1\!\!1_A$). In particular, for $A = B_{\alpha,t}$, (4.9) becomes

$$m_n(x,s,t) = m_1(x,s,t) + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\int_0^s du \int_{\mathbb{R}^2} dy\, p_{s-u}(x,y) m_k(y,u,t) m_{n-k}(y,u,t). \tag{4.10}$$

Compare this with the moment formula for SBM($\mathbb{R}^d$) given in Lemma 3.2. The main contribution turns out to come from the $k = 0$ term in (3.9). Since the leading terms coincide, it suffices to prove the assertion for the case $(\psi_t) = (\eta_t)$ is BBM($\mathbb{R}^2$). Note that for the case (ψ_t) SBM($\mathbb{R}^d$), also the existence of Γ with the asserted properties follows easily from the existence in the case considered here.

We start with the proof of part (a). The proof follows an idea of Durrett (1979) (proof of Thm. 8.1). We proceed by induction over n using (4.10). To do so, we cut the left and right side of the domain $[0, s_t]$ of integration. In the remaining term we may use the asymptotics (4.5) and (4.6). On the other hand, the error terms resulting from the truncation of the domain of integration will be estimated by the following bounds. These will be proved successively in the course of the induction.

We show the existence of constants C_n, D_n and E_n (depending on B) with

$$\sup_{\substack{t\geq s\geq u\geq 3\\ y\in\mathbb{R}^2}} \frac{1}{u}\int_{\mathbb{R}^2} (s-u)p_{s-u}(y,z)\widetilde{m}_n(z,u,t)dz \leq C_n, \tag{4.11}$$

$$\sup_{\substack{t\geq u\geq 3\\ y\in\mathbb{R}^2}} \widetilde{m}_n(y,u,t) \leq D_n, \tag{4.12}$$

and

$$\sup_{t\geq s\geq 3} \frac{1}{s}\int_{\mathbb{R}^2} \widetilde{m}_n(y,s,t)\,dy \leq E_n. \tag{4.13}$$

For $n=1$ the assertions clearly hold because

$$\widetilde{m}_1(x_t,s_t,t) = t^{-\alpha}s_t\int_{B_t} p_{s_t}(x_t,y)dy \overset{t\to\infty}{\longrightarrow} \varphi(x)|B|, \tag{4.14}$$

$$\frac{1}{s_t}\int_{\mathbb{R}^2} \widetilde{m}_1(y,s_t,t)\,dy = t^{-\alpha}\int_{B_t} dy\int_{\mathbb{R}^2} dz\, p_{s_t}(z,y) = t^{-\alpha}\int_{B_t} dy = |B|, \tag{4.15}$$

$$\frac{1}{s}\int_{\mathbb{R}^2} (s-u)p_{s-u}(y,z)\widetilde{m}_1(z,u,t)\,dz = \frac{s-u}{s}ut^{-\alpha}\int_{B_t} p_s(y,z)\,dz \leq \frac{s-u}{s}\frac{u}{s}|B| \leq |B|, \tag{4.16}$$

$$\widetilde{m}_1(y,u,t) = ut^{-\alpha}\int_{B_t} p_u(y,z)\,dz \leq |B|, \tag{4.17}$$

and

$$\frac{1}{s}\int_{\mathbb{R}^2} \widetilde{m}_1(y,s,t)\,dy = t^{-\alpha}\int_{\mathbb{R}^2} dy\int_{B_t} dz\, p_s(y,z) = |B|. \tag{4.18}$$

We will also need the following bound for the moments of the total mass,

$$\mathbf{E}^x[(\eta_t(\mathbb{R}^2))^n] \leq F_n\cdot(t+1)^{n-1}, \tag{4.19}$$

where $F_n = n!$. For $n=1$ this is clear since the l.h.s. of (4.19) equals 1. For $n\geq 2$ this is easily shown by induction using (4.9),

$$\begin{aligned} \mathbf{E}^x[(\eta_t(\mathbb{R}^2))^n] &\leq F_1(t+1) + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}F_kF_{n-k}\int_0^t (s+1)^{n-2}\,ds \\ &\leq F_1(t+1) + \frac{1}{2}\frac{1}{n-1}\sum_{k=1}^{n-1}\binom{n}{k}F_kF_{n-k}(t+1)^{n-1} \\ &\leq n!(t+1)^{n-1}. \end{aligned} \tag{4.20}$$

The uniformity of the claim in terms of the sequences (a_t), (b_t) and (c_t) will be needed to do the induction properly. Following the lines of the proof it can easily be established. We omit the details to avoid an unnecessary blowup of the proof.

Now let $n \geq 2$. In the sequel we will assume that the validity of (4.5), (4.6) and (4.11)-(4.13) is already shown for all $n' < n$.

We start with providing an inequality needed in some places. Assume that $X_1, \dots, X_{\|\eta_u\|}$ are the positions of the particles of η_u at time u, i.e., $\eta_u = \sum_{k=1}^{\|\eta_u\|} \delta_{X_k}$. Further, let $Y_k = 1_{B_t}(X_k)$. Each Y_k is independent of $\|\eta_u\|$ and has expectation $\mathbf{E}^{x_t}[Y_k] = \int_{B_t} p_u(x_t, y)\, dy$. Thus (by (4.19))

$$\begin{aligned} m_n(x_t, u, t) &= \mathbf{E}[\mathbf{E}[\left(\sum_{k=1}^{\|\eta_u\|} Y_k\right)^n \Big| \|\eta\|]] \qquad (4.21) \\ &\leq \mathbf{E}[\|\eta_u\|^{n-1}\mathbf{E}^{x_t}[\sum_{k=1}^{\|\eta_u\|} Y_k \Big| \|\eta_u\|]] \\ &= \mathbf{E}^{x_t}[\|\eta_u\|^n] \int_{B_t} p_u(x_t, y)\, dy \\ &\leq F_n (u+1)^{n-1} \int_{B_t} p_u(x_t, y)\, dy. \end{aligned}$$

Note that

$$m_1(x_t, s_t, t) \ll \frac{t^{n\alpha}(\log t)^{n-1}}{s_t}, \qquad (4.22)$$

i.e., the l.h.s. in (4.22) is negligible compared with the expected main term of $m_n(X_t, s_t, t)$. We thus calculate now

$$h_{n,k}(x_t, s, v, w) := \int_v^w du \int_{\mathbb{R}^2} dy\, p_{s-u}(x_t, y) m_k(y, u, t) m_{n-k}(y, u, t). \qquad (4.23)$$

Let $(\delta_t)_{t \geq 0}$ be a sequence with $\delta_t \uparrow \infty$ so slowly that $\frac{\delta_t}{\log t} \stackrel{t\to\infty}{\longrightarrow} 0$. By (4.19) and (4.21),

$$\begin{aligned} h_{n,k}(x_t, s_t, 0, \delta_t t^\alpha) &\leq F_k F_{n-k} \int_0^{\delta_t t^\alpha} du\, (u+1)^{n-2} \int_{\mathbb{R}^2} dy\, p_{s_t-u}(x_t, y) \int_{B_t} dz\, p_u(y, z) \qquad (4.24) \\ &\leq \frac{F_k F_{n-k}}{n-1} (\delta_t t^\alpha + 1)^{n-1} \frac{t^\alpha}{s_t} |B| \\ &\ll \frac{t^{n\alpha}}{s_t} (\log s_t)^{n-1} \end{aligned}$$

is small. The other side of the integration interval will be estimated as follows. Let $(\varepsilon_t)_{t \geq 0}$ be a sequence such that $\varepsilon_t \downarrow 0$ and such that $\frac{\log \varepsilon_t}{\log t} \stackrel{t\to\infty}{\longrightarrow} 0$. Then

$$\begin{aligned} h_{n,k}(x_t, s_t, \varepsilon_t s_t, s_t) &\leq 2(C_k + D_k) D_{n-k} \frac{t^{n\alpha}}{s_t} \int_{\varepsilon_t s_t}^{s_t} \frac{(\log u)^{n-2}}{u}\, du \qquad (4.25) \\ &= 2(C_k + D_k) D_{n-k} \frac{1}{n-1} \frac{t^{n\alpha}}{s_t} (\log s_t)^{n-1} \left[1 - \left(1 - \frac{\log \varepsilon_t}{\log s_t}\right)^{n-1}\right] \end{aligned}$$

$$\ll \frac{t^{n\alpha}}{s_t}(\log s_t)^{n-1}.$$

Hence the main term results from the integration over $[\delta_t t^\alpha, \varepsilon_t s_t]$. To evaluate this integral we split the spatial integral into the integral over the disc $D_u = \{y \in \mathbb{R}^2 : \|y\| \le K_u\sqrt{u}\}$ and its complement $D_u^c = \mathbb{R}^2 \setminus D_u$, where $K_u \uparrow \infty$ as $u \to \infty$ will be fixed later. By the induction hypotheses (4.5), (4.11) and (4.12) we get

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{s_t t^{-n\alpha}}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{1}{u} \int_{D_u^c} dy\, p_{s_t-u}(x_t,y)\ u\ m_k(y,u,t) m_{n-k}(y,u,t) \qquad (4.26)\\
&\le D_{n-k} \limsup_{t\to\infty} \frac{1}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{D_u^c} dy\, s_t\ p_{s_t-u}(x_t,y) \frac{1}{u}\widetilde{m}_k(y,u,t)\\
&\le \frac{1}{\pi} D_{n-k} \limsup_{t\to\infty} \frac{1}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{D_u^c} dy\, \frac{1}{u}\widetilde{m}_k(y,u,t).
\end{aligned}$$

The last inequality holds since $s_t p_{s_t-u}(x_t,y) \le 1/\pi$ for $\varepsilon_t < \frac{1}{2}$. Fix $\beta' \ge 0$ and let (u_t) be a sequence such that $\frac{\log u_t}{\log t} \stackrel{t\to\infty}{\longrightarrow} \beta'$. Then by Fatou's lemma

$$\begin{aligned}
\liminf_{t\to\infty} \int_{D_{u_t}} \frac{1}{u_t}\, \widetilde{m}_k(y,u_t,t)\, dy &= \liminf_{t\to\infty} \int_{\|y\|\le K_{u_t}} \widetilde{m}_k(y\sqrt{u_t},u_t,t)\, dy \qquad (4.27)\\
&\ge \left(1-\frac{\alpha}{\beta'}\right)^{k-1} \frac{|B|^k k!}{(8\pi)^{k-1}} \cdot \int_{\mathbb{R}^2} \varphi(y)\, dy\\
&= \left(1-\frac{\alpha}{\beta'}\right)^{k-1} \frac{|B|^k k!}{(8\pi)^{k-1}}.
\end{aligned}$$

Let (u_t) be a sequence with $u_t \gg t^\alpha$ and let $\delta > 0$. Then by (4.6) for t sufficiently large

$$\int_{D_{u_t}^c} \frac{1}{u_t}\widetilde{m}_k(y,u_t,t)\, dy < \delta. \qquad (4.28)$$

Thus the expression in (4.26) is less than or equal too

$$\frac{1}{\pi}\frac{\delta D_{n-k}}{n-1} \limsup_{t\to\infty} \frac{(\log \varepsilon_t s_t)^{n-1} - (\log \delta_t t^\alpha)^{n-1}}{(\log s_t)^{n-1}} = \frac{1}{\pi}\frac{\delta D_{n-k}}{n-1}\left(1-\left(\frac{\alpha}{\beta}\right)^{n-1}\right). \qquad (4.29)$$

Since $\delta > 0$ was arbitrary the three expressions in (4.26) are equal and equal to zero.

Our task is now to determine the main term. By (4.5), (4.12) and the theorem of dominated convergence, we may let $K_u \uparrow \infty$ so slowly that (uniformly in $\beta' \le 1$)

$$\begin{aligned}
\frac{1}{u_t}\int_{D_{u_t}} \widetilde{m}_k(y,u_t,t)\widetilde{m}_{n-k}(y,u_t,t)\, dy &= \int_{\|y\|\le K_{\sqrt{u_t}u_t}} \widetilde{m}_k(\sqrt{u_t}y,u_t,t)\widetilde{m}_{n-k}(y,u_t,t)\, dy \qquad (4.30)\\
&\stackrel{t\to\infty}{\longrightarrow} \left(1-\frac{\alpha}{\beta'}\right)^{n-2} \frac{|B|^n k!(n-k)!}{(8\pi)^{n-2}} \int_{\mathbb{R}^2} \varphi(y)^2\, dy\\
&= 2\left(1-\frac{\alpha}{\beta'}\right)^{n-2} \frac{|B|^n k!(n-k)!}{(8\pi)^{n-1}}.
\end{aligned}$$

Assuming further $K_{\varepsilon_t s_t}\sqrt{\varepsilon_t} \stackrel{t\to\infty}{\longrightarrow} 0$ we get uniformly in $u \le \varepsilon_t s_t$ and $y \in D_u$ that

$$s_t p_{s_t-u}(x_t, y) \stackrel{t\to\infty}{\longrightarrow} \varphi(x). \tag{4.31}$$

We are now in the position to calculate

$$\lim_{t\to\infty} \frac{s_t t^{-n\alpha}}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du \int_{D_u} dy\, p_{s_t-u}(x_t, y) m_k(y, u, t) m_{n-k}(y, u, t) \tag{4.32}$$

$$= \lim_{t\to\infty} \frac{s_t}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{D_u} dy\, p_{s_t-u}(x_t, y) \frac{1}{u} \widetilde{m}_k(y, u, t) \widetilde{m}_{n-k}(y, u, t)$$

$$= \lim_{t\to\infty} \frac{\varphi(x)}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{D_u} dy\, \frac{1}{u} \widetilde{m}_k(y, u, t) \widetilde{m}_{n-k}(y, u, t) \tag{4.33}$$

$$= \varphi(x) 2 \frac{|B|^n k!(n-k)!}{(8\pi)^{n-1}} \lim_{t\to\infty} \frac{1}{(\log s_t)^{n-1}} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} \frac{(\log u)^{n-2}}{u} \left(1 - \alpha \frac{\log t}{\log u}\right)^{n-2} du$$

$$= \varphi(x) \frac{2}{n-1} \frac{|B|^n k!(n-k)!}{(8\pi)^{n-1}} \lim_{t\to\infty} \frac{(\log(\varepsilon_t s_t) - \alpha \log t)^{n-1} - (\log(\delta_t t^\alpha) - \alpha \log t)^{n-1}}{(\log s_t)^{n-1}}$$

$$= \varphi(x) \frac{2}{n-1} \left(1 - \frac{\alpha}{\beta}\right)^{n-1} \frac{|B|^n k!(n-k)!}{(8\pi)^{n-1}}.$$

Summation over k in (4.10) now yields (4.5).

To show (4.6) we integrate (4.10)

$$\int_{\mathbb{R}^2} m_n(x, s, t)\, dx = \int_{\mathbb{R}^2} m_1(x, s, t)\, dx + \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \int_0^s du \int_{\mathbb{R}^2} dy\, m_k(y, u, t) m_{n-k}(y, u, t). \tag{4.34}$$

As above the first term is small and we have to evaluate

$$g_{n,k}(v, w) := \int_v^w du \int_{\mathbb{R}^2} dy\, m_k(y, u, t) m_{n-k}(y, u, t). \tag{4.35}$$

For (δ_t) as above we get from (4.13) and (4.19) that

$$\begin{aligned} g_{n,k}(3, \delta_t t^\alpha) \quad &\le \quad t^{(n-k)\alpha} F_k \int_3^{\delta_t t^\alpha} du\, \frac{(\log u)^{n-k-1}}{u} (u+1)^{k-1} \int_{\mathbb{R}^2} dy\, \widetilde{m}_{n-k}(y, u, t) \qquad (4.36) \\ &\le \quad \left(\frac{4}{3}\right)^{k-1} \frac{F_k E_{n-k}}{n-k} t^{(n-1)\alpha} (t\delta_t)^{k-1} \left(\log(\delta_t t^\alpha)\right)^{n-k} \ll t^\alpha \end{aligned}$$

(note that $\frac{u+1}{u} \le \frac{4}{3}$ on the domain of integration). Let $\Delta = \operatorname{diam}(B)$. By assumption, $\Delta < \infty$ which serves to show that

$$g_{n,k}(0, 3) \quad \le \quad \int_0^3 du \int_{\mathbb{R}^2} dy\, m_n(y, u, t) \tag{4.37}$$

$$
\begin{aligned}
&= \int_0^3 du \int_{[0,\Delta t^{\alpha/2}[^2} dy \sum_{l\in\mathbb{Z}^d} m_n(y + l\Delta t^{\alpha/2}, u, t) \\
&\le \int_0^3 du \int_{[0,\Delta t^{\alpha/2}[^2} dy\, \mathbf{E}^y[(\eta_u(\mathbb{R}^2))^n] \\
&\le \Delta^2 t^\alpha \frac{4^n - 1}{n} F_n.
\end{aligned}
$$

Since the expected main term is of order $t^{n\alpha}(\log t)^{n-1}$, we have got that $g_{n,k}(0, \delta_t t^\alpha)$ is negligible. Let also (ε_t) be as above to obtain by (4.12) and (4.13) that $g_{n,k}(\varepsilon_t s_t, s_t)$ is small,

$$
\begin{aligned}
g_{n,k}(\varepsilon_t s_t, s_t) &= t^{n\alpha} \int_{\varepsilon_t s_t}^{s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{\mathbb{R}^2} dy\, \widetilde{m}_k(y,u,t) \frac{1}{u} \widetilde{m}_{n-k}(y,u,t) \qquad (4.38) \\
&\le \frac{D_k E_{n-k}}{n-1} t^{n\alpha} \left[(\log s_t)^{n-1} - (\log(\varepsilon_t s_t))^{n-1}\right] \ll t^{n\alpha} (\log s_t)^{n-1}.
\end{aligned}
$$

We split up $g_{n,k}(\delta_t t^\alpha, \varepsilon_t s_t)$ as above. The integral over D_u has already been determined in (4.33) and the the integral over D_u^c is small since

$$
\begin{aligned}
&\int_{\varepsilon_t t^\alpha}^{\varepsilon^t s_t} du \int_{D_u^c} dy\, m_k(y,u,t) m_{n-k}(y,u,t) \qquad (4.39) \\
&\qquad \le D_{n-k} t^{n\alpha} \int_{\delta_t t^\alpha}^{\varepsilon_t s_t} du\, \frac{(\log u)^{n-2}}{u} \int_{D_u^c} dy\, \frac{1}{u} \widetilde{m}_k(y,u,t) \\
&\qquad \ll t^{n\alpha} (\log s_t)^{n-1}.
\end{aligned}
$$

So far we have shown part (a) of the lemma. To prove part (b) we still have to show that (4.11)-(4.13) hold and that the size of the constants can be controlled. We will do this by means of recursion formulas for C_n, D_n and E_n.

By (4.21) we have

$$
\begin{aligned}
&\int_0^3 du \int_{\mathbb{R}^2} dy\, (s-u) p_{s-u}(y,z) m_k(z,u,t) m_{n-k}(z,u,t) \qquad (4.40) \\
&\qquad \le F_k F_{n-k} \int_0^3 du\, (u+1)^{n-2} \int_{\mathbb{R}^2} dz \int_{B_t} dw\, (s-u) p_{s-u}(y,z) p_u(z,w) \\
&\qquad \le F_k F_{n-k} \int_0^3 du\, (u+1)^{n-2} \int_{B_t} dw\, (s-u) p_s(z,w) \\
&\qquad \le \frac{F_k F_{n-k}}{n-1} 4^{n-1} |B| t^\alpha.
\end{aligned}
$$

Putting this into the recursion formula (4.10) we get

$$
\begin{aligned}
\frac{s-u}{u} \int_{\mathbb{R}^2} p_{s-u}(y,z) \widetilde{m}_n(z,u,t)\, dz &\le \frac{1}{(\log u)^{n-1}} \Bigg[C_1 + \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} \Bigg(\frac{F_k F_{n-k}}{n-1} 4^{n-1} |B| \qquad (4.41) \\
&+ \int_{\mathbb{R}^2} dz \int_3^u dv\, \frac{(\log v)^{n-2}}{v} \int_{\mathbb{R}^2} dz'\, (s-u) p_{s-u}(y,z) p_{u-v}(z,z') \frac{1}{v} \widetilde{m}_k(z',v,t) \widetilde{m}_{n-k}(z',v,t) \Bigg) \Bigg].
\end{aligned}
$$

Doing the integration the summands equal

$$\int_3^u dv\,\frac{(\log v)^{n-2}}{v}\int_{\mathbb{R}^2} dz'\,(s-u)p_{s-v}(y,z')\frac{1}{v}\widetilde{m}_k(z',v,t)\widetilde{m}_{n-k}(z',v,t) \tag{4.42}$$
$$\leq C_k D_{n-k}\int_3^u \frac{(\log v)^{n-2}}{v}\,dv \leq \frac{C_k D_{n-k}}{n-1}(\log u)^{n-1}.$$

We have shown that (4.11) holds with

$$C_n \leq C_1 + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{C_k D_{n-k}}{n-1} + \frac{F_k F_{n-k}}{n-1}4^{n-1}|B|\right). \tag{4.43}$$

We now turn to the D_n. By the recursion formula (4.10) we get for $t \geq s \geq 3$ and $y \in \mathbb{R}^2$

$$\widetilde{m}_n(y,s,t) \leq t^{-n\alpha}\frac{s}{(\log s)^{n-1}}\Big(m_1(y,s,t) + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}h_{n,k}(y,s,0,t)\Big). \tag{4.44}$$

Now

$$\begin{aligned} h_{n,k}(y,s,3,t) &\leq D_{n-k}\int_0^2 du\,\frac{1}{u}\int_{\mathbb{R}^2} dz\,p_{u-s}(y,z)m_k(z,s,t)(\log su)^{n-k-1} \\ &\leq 2(C_k+D_k)D_{n-k}\int_3^s \frac{(\log u)^{n-2}}{u}du \\ &\leq \frac{2(C_k+D_k)D_{n-k}}{n-1}\frac{(\log s)^{n-1}}{s}. \end{aligned} \tag{4.45}$$

From this and (4.24) we get that D_n can be chosen to be

$$D_n \leq D_1 + \frac{1}{2(n-1)}\sum_{k=1}^{n-1}\binom{n}{k}\left[F_k F_{n-k}4^{n-1}|B| + 2(C_k+D_k)D_{n-k}\right]. \tag{4.46}$$

Finally, the E_n will be determined as follows

$$\frac{1}{s}\int_{\mathbb{R}^2}\widetilde{m}_n(y,s,t)\,dy \leq E_1 + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\frac{1}{(\log s)^{n-1}}g_{n,k}(0,s). \tag{4.47}$$

Now

$$\begin{aligned} g_{n,k}(3,s) &\leq D_k\int_3^s \frac{1}{u}(\log u)^{k-1}\int_{\mathbb{R}^2} dz\,m_{n-k}(z,u,t) \\ &\leq D_k E_{n-k}\int_3^s \frac{(\log u)^{k-2}}{u}du \\ &\leq \frac{D_k E_{n-k}}{n-1}(\log s)^{n-1}. \end{aligned} \tag{4.48}$$

Together with (4.37) this yields that we can choose E_n to be

$$E_n \le E_1 + \frac{1}{2(n-1)} \sum_{k=1}^{n-1} \binom{n}{k} \left[D_k E_{n-k} + \Delta^2 4^n F_n \right]. \tag{4.49}$$

Putting together (4.19),(4.43), (4.46) and (4.49) we see that we can choose

$$C_n = D_n = E_n = n!\Gamma^n \tag{4.50}$$

for some $\Gamma < \infty$ (depending on Δ). □

Next we give a lemma that provides some uniformity in different spatial scalings that are $\approx t^{\alpha/2}$ (recall that $a_t \approx b_t$ means $(\log a_t)/(\log b_t) \stackrel{t\to\infty}{\longrightarrow} 1$).

Lemma 4.2 *Let (ψ_t) be BBM$(\mathbb{R}^2)$ or SBM$(\mathbb{R}^2)$ and $I = [0,1]$ resp. $]-\infty, 1]$. Fix $\alpha \in I$ and $v(t) \ll u(t)$ with $u(t), v(t) \approx t^\alpha$. Then uniformly in all sequences $w(t)$ such that $u(t) \le w(t) \le v(t)\, \forall\, t \ge 0$ the following holds*

$$h(t) := \mathbf{E}^{\widetilde{M}(t)} \left[\left(\frac{1}{u(t)} \widetilde{\psi}_t([0, \sqrt{u(t)}[^2) - \frac{1}{w(t)} \widetilde{\psi}_t([0, \sqrt{w(t)}[^2) \right)^2 \right] \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.51}$$

Proof Let

$$\phi_t = \frac{1}{u(t)} \mathbb{1}_{[0,u(t)^{1/2}[^2} - \frac{1}{w(t)} \mathbb{1}_{[0,w(t)^{1/2}[^2}.$$

Recall that (S_s) is the semigroup of Brownian motion on $\mathbb{R}^2$. By the second moment formulas (3.5) and (3.12),

$$h(t) \le a_t + b_t + c_t, \tag{4.52}$$

(with equality in the case of BBM) where

$$\begin{aligned} a_t &= \left(\frac{8\pi}{\log t} \right)^2 \int (\langle \mu, S_t \phi_t \rangle)^2 \, \widetilde{M}(t)(d\mu) \\ b_t &= \left(\frac{8\pi}{\log t} \right)^2 \int \langle \mu, S_t(\phi_t^2) - (S_t\phi_t)^2 \rangle \, \widetilde{M}(t)(d\mu) \\ c_t &= \left(\frac{8\pi}{\log t} \right)^2 \int \left\langle \mu, \int_0^T S_{t-s}((S_s\phi_t)^2)\, ds \right\rangle \widetilde{M}(t)(d\mu). \end{aligned} \tag{4.53}$$

Clearly, $a_t \stackrel{t\to\infty}{\longrightarrow} 0$ and $b_t \stackrel{t\to\infty}{\longrightarrow} 0$. To show $c_t \stackrel{t\to\infty}{\longrightarrow} 0$ we have to be more careful. By translation invariance we get (recall that λ is the Lebesgue measure)

$$c_t = \frac{8\pi}{\log t} \left\langle \lambda, \int_0^t (S_s\phi_t)^2 \, ds \right\rangle. \tag{4.54}$$

Note that by Hölder's inequality

$$\begin{aligned} \langle \lambda, (S_s\phi_t)^2 \rangle &\le \|S_s\phi_t\|_\infty = \sup_{x\in\mathbb{R}^2} |S_s\phi_t(x)| \\ &\le \min\left(\frac{1}{2\pi s}, \frac{1}{u(t)} + \frac{1}{w(t)} \right) \le \min\left(\frac{1}{2\pi s}, \frac{2}{u(t)} \right). \end{aligned} \tag{4.55}$$

Thus

$$\frac{8\pi}{\log t}\int_0^{v(t)\log t}\langle\lambda,(S_s\phi_t)^2\rangle\,ds \le \frac{8\pi}{\log t}\left[\frac{2}{\log t}+(\log(v(t)\log t)-\log(u(t)/\log t))\right] \overset{t\to\infty}{\longrightarrow} 0. \tag{4.56}$$

On the other hand

$$\begin{aligned}\|S_s\phi_t\|_\infty &\le \sup_{x\in\mathbb{R}^2}\ \sup_{y\in[0,u(t)^{1/2}[^2}\ \sup_{z\in[0,v(t)^{1/2}[^2}|p_s(x,y)-p_s(x,z)| \qquad (4.57)\\ &= \frac{1}{2\pi s}\sup_{r\in\mathbb{R}}\ \sup_{\zeta\in[-(2v(t))^{1/2},(2v(t))^{1/2}]}\left|\exp\{-r^2/2s\}-\exp\{-(r-\zeta)^2/2s\}\right|\\ &\le \frac{e^{-1}}{2\pi s}\sqrt{2v(t)/s}\,.\end{aligned}$$

Thus

$$\frac{8\pi}{\log t}\int_{v(t)\log t}^{t}\langle\lambda(S_s\phi_t)^2\rangle\,ds \le \frac{\sqrt{8}}{e}\sqrt{1/\log t}\overset{t\to\infty}{\longrightarrow} 0. \tag{4.58}$$

We conclude $c_t \overset{t\to\infty}{\longrightarrow} 0$ and the proof is complete. □

5 Proof of the Clustering Results for the Infinite Systems

5.1 Proof of Theorem 1

The proof of Theorem 1 will be based on an asymptotic result related to the Laplace transforms of $\widetilde{\psi}_t$. This is formulated in Proposition 5.1 and 5.2 below.

Let $x\in\mathbb{R}^2$ and $(x_t)_{t\ge 0}$ be a sequence in $\mathbb{R}^2$ such that $x_t/\sqrt{t}\overset{t\to\infty}{\longrightarrow} x$.

Proposition 5.1 *Then for $B\in\mathcal{B}(\mathbb{R}^2)$ bounded and $\theta\ge 0$,*

$$\lim_{t\to\infty}\frac{t\log t}{8\pi}\left(1-\mathbf{E}^{x_t}\left[\exp\{-\theta\widetilde{\psi}_t^\alpha(B)\}\right]\right)\overset{t\to\infty}{\longrightarrow}\varphi(x)\frac{\theta|B|}{1+\theta|B|(1-\alpha)} \tag{5.1}$$

$$\lim_{t\to\infty}\frac{\log t}{8\pi}\left(1-\mathbf{E}^{M(1)}\left[\exp\{-\theta\widetilde{\psi}_t^\alpha(B)\}\right]\right)\overset{t\to\infty}{\longrightarrow}\frac{\theta|B|}{1+\theta|B|(1-\alpha)}. \tag{5.2}$$

Proof Let

$$\phi_t(\theta)=\frac{t\log t}{8\pi}\left(1-\mathbf{E}^{x_t}[\exp\{-\theta\widetilde{\psi}_t^\alpha(B)\}]\right)\qquad \theta\in\mathbb{C},\ \mathrm{Re}(\theta)>0. \tag{5.3}$$

Then

$$|\phi_t(\theta)|\le\frac{t\log t}{8\pi}|\theta|\cdot\mathbf{E}^{x_t}[\widetilde{\psi}_t^\alpha(B)]\le|\theta|. \tag{5.4}$$

Thus $\phi_t(\theta)$ is uniformly bounded for θ in compact sets. Let $\Gamma < \infty$ be as in Lemma 4.1(b). By (4.7) for $|\theta| < \frac{1}{\Gamma}$ we can express $\phi_t(\theta)$ in terms of the moments

$$\begin{aligned} \phi_t(\theta) &= -\frac{t\log t}{8\pi}\sum_{n=1}^{\infty}\frac{(-\theta)^n \mathbf{E}^{x_t}[(\widetilde{\psi}_t^{\alpha}(B))^n]}{n!} \\ &= -\sum_{n=1}^{\infty}\frac{(-\theta)^n(8\pi)^{n-1}\widetilde{m}_n(x_t,t,t,\alpha)}{n!}. \end{aligned} \tag{5.5}$$

Hence by (4.5),

$$\phi_t(\theta) \overset{t\to\infty}{\longrightarrow} \varphi(x)\frac{\theta|B|}{1+\theta|B|(1-\alpha)},\ \ |\theta| < \frac{1}{\Gamma}. \tag{5.6}$$

By Vitali's theorem (see, e.g., Remmert (1991)), equation (5.6) holds for all θ on the right half plane.

The proof of (5.2) is analogous. Here we take

$$\phi_t(\theta) = \frac{\log t}{8\pi}\left[1 - \mathbf{E}^{M(1)}\left[\exp\{-\theta\widetilde{\psi}_t^{\alpha}(B)\}\right]\right] \tag{5.7}$$

and use (4.6) and (4.8). □

For $\alpha < 1$ and $|B| > 0$, Proposition 5.1 can be reformulated in terms of distributions.

Proposition 5.2 *Assume $\alpha < 1$. Let (x_t) as in Proposition 5.1 and let $u > 0$. Then for $B \in \mathcal{B}(\mathbb{R}^2)$ bounded, $|B| > 0$,*

$$\lim_{t\to\infty}\frac{t\log t}{8\pi}\mathbf{P}^{x_t}\left[\widetilde{\psi}_t^{\alpha}(B) > u\right] = \frac{\varphi(x)}{1-\alpha}\exp\left\{-\frac{u}{|B|(1-\alpha)}\right\} \tag{5.8}$$

$$\lim_{t\to\infty}\frac{\log t}{8\pi}\mathbf{P}^{M(1)}\left[\widetilde{\psi}_t^{\alpha}(B) > u\right] = \frac{1}{1-\alpha}\exp\left\{-\frac{u}{|B|(1-\alpha)}\right\}. \tag{5.9}$$

Proof We show only (5.8) since the proof of the other statement ist similar. Let $F_t(u) = \frac{\log t}{8\pi}\mathbf{P}^{x_t}[\widetilde{\psi}_t^{\alpha}(B) > u]$ and $G(u) = \frac{\varphi(x)}{1-\alpha}\frac{1}{|B|(1-\alpha)}\int_0^u \exp(-s/(|B|(1-\alpha))\,ds$. Note that (5.1) states that

$$\int_0^{\infty}(1-e^{-\theta u})dF_t(u) \overset{t\to\infty}{\longrightarrow} \int_0^{\infty}(1-e^{-\theta u})dG(u). \tag{5.10}$$

Since $(u \mapsto 1 - e^{-\theta u},\ u \ge 0)$ is a separating class on $]0,\infty[$ we are done. □

Proof (of Theorem 1)

From Proposition 5.1 the proof is easy. Let $L(s,\theta) = \mathbf{E}^1[\exp\{-\theta Z_s\}]$ be the Laplace transform of Feller's diffusion (Z_s). By (1.6), $L(0,\theta) = \exp\{-\theta\}$ and

$$\frac{\partial}{\partial s}L(s,\theta) = \mathbf{E}^1[\theta^2 Z_s\exp\{-\theta Z_s\}] = -\theta^2\frac{\partial}{\partial\theta}L(s,\theta). \tag{5.11}$$

The solution of (5.11) is

$$L(s,\theta) = \exp\left\{-\frac{\theta}{1+\theta s}\right\}, \qquad \theta \geq 0,\ s \geq 0. \tag{5.12}$$

Let $\alpha \in [0,1]$ and $B \in \mathcal{B}(\mathbb{R}^2)$ bounded. Use (5.2) to obtain

$$\begin{aligned} \mathbf{E}^{\widetilde{M}(t)}\left[\exp\{-\theta\widetilde{\psi}_t^\alpha(B)\}\right] &= \left(1-\left(1-\mathbf{E}^{M(1)}\left[\exp\{-\theta\widetilde{\psi}_t^\alpha(B)\}\right]\right)\right)^{\frac{\log t}{8\pi}} \\ &\overset{t\to\infty}{\longrightarrow} \exp\left\{-\frac{\theta|B|}{1+\theta|B|(1-\alpha)}\right\}. \end{aligned} \tag{5.13}$$

Comparing this with (5.12) yields the claim.

The case $\alpha < 0$ and $\psi_t = \zeta_t$ SBM($\mathbb{R}^d$) can be done with the scaling property (1.10) as follows,

$$\begin{aligned} \mathcal{L}^{\widetilde{M}(t)}\left[\widetilde{\zeta}_t^\alpha(B)\right] &= \mathcal{L}^{\widetilde{M}(t)}\left[\frac{8\pi}{\log t}t^{-\alpha}\zeta_t(t^{\alpha/2}B)\right] \\ &= \mathcal{L}^{\widetilde{M}(t)}\left[\frac{8\pi}{\log t}\zeta_{t^{1-\alpha}}(B)\right] \\ &= \mathcal{L}^{\widetilde{M}((t^{1-\alpha})/(1-\alpha))}\left[(1-\alpha)\widetilde{\zeta}_{t^{1-\alpha}}(B)\right] \\ &\overset{t\to\infty}{\Longrightarrow} \mathcal{L}^{1/(1-\alpha)}\left[(1-\alpha)Z_1\right] = \mathcal{L}^1[Z_{1-\alpha}]. \end{aligned} \tag{5.14}$$

In the last step we have used the scaling property of Feller's diffusion, $\mathcal{L}^{\rho/\gamma}[\gamma Z_\beta] = \mathcal{L}^\rho[Z_{\gamma\beta}]$, $\beta,\gamma,\rho > 0$. □

5.2 Proof of Theorem 2

In order to understand why Theorem 2 should be true, we draw a time-space picture (see Figure 4). Consider a point $(x,t) \in \mathbb{R}^2 \times [0,\infty[$. We want to investigate the events $C(x,t)$ that form the history of (x,t). Since Brownian motion at time s has range $\sim \sqrt{s}$, we may roughly set

$$C(x,t) = \{(u,s), \|u-x\| \leq (t-s)^{1/2},\ u \in \mathbb{R}^2,\ s \in [0,t]\}.$$

Now let for $\alpha \in [0,1]$,

$$C_\alpha(x,t) = C(x,t) \cap (\mathbb{R}^2 \times \{t-t^\alpha\})$$

be the events at time $t-t^\alpha$ that may influence (x,t). Fix $\alpha \in [0,1]$ and let (x_t), $(y_t) \in \mathbb{R}^2$ be such that $\|x_t - y_t\| \sim t^{\alpha/2}$. Then for $\gamma < \alpha$ we have that $C_\gamma(x_t,t)$ and $C_\gamma(y_t,t)$ are (asymptotically) completely disjoint. For $\beta > \alpha$ we have that $C_\beta(x_t,t)$ and $C_\beta(y_t,t)$ (asymptotically) overlap completely. By the Markov property the common history is contained in $C_\alpha(x_t,t) \approx C_\alpha(y_t,t)$. After time $t-t^\alpha$ the evolutions leading to (x_t,t) and (y_t,t) are independent.

We have to justify that the information contained in $C_\alpha(x_t,t) \approx C_\alpha(y_t,t)$ is sufficiently well described by the common value of $Z_{1-\alpha}$. This will be done by showing that the distribution of mass is not "too inhomogeneous".

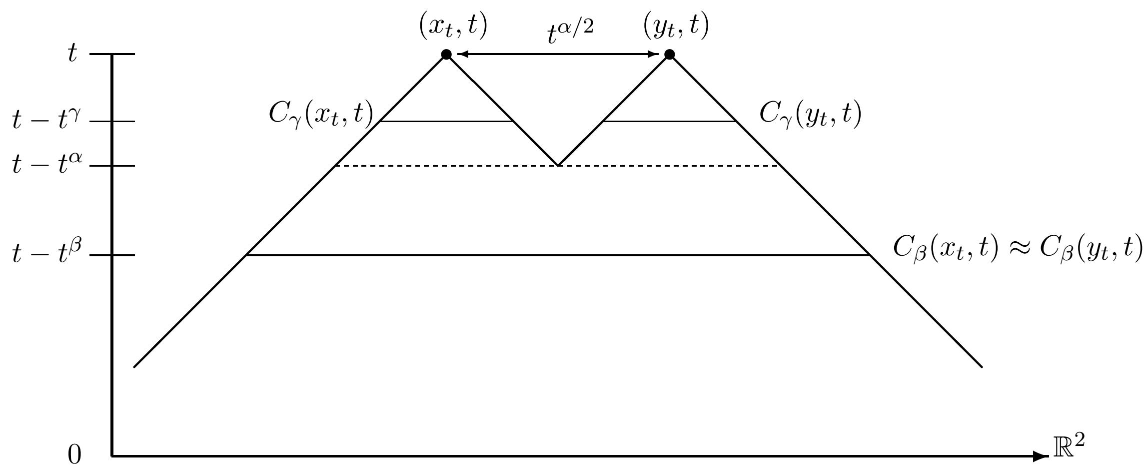

Figure 4. Historical cones for $\|x_t - y_t\| \sim t^{\alpha/2}$

We make the preceding idea precise. It is sufficient to check that

$$\mathcal{L}^{\widetilde{M}(t)}\left[(\mathcal{S}_{A(e),t}\mathcal{T}_{x_t^e}\widetilde{\psi}_t(B^e))_{e\in\mathbb{T}}\right] \overset{t\to\infty}{\longrightarrow} \mathcal{L}\left[\left(|B^e|Z^e_{1-A(e)}\right)_{e\in\mathbb{T}}\right], \tag{5.15}$$

for $B^e \in \mathcal{B}(\mathbb{R}^2)$ bounded for all $e \in \mathbb{T}$.

We do the proof by induction over the length of the tree $\mathbb{T}$. For $\mathbb{T} = \{\emptyset\}$ this is the assertion of Theorem 1. Now assume that the claim has been shown for all trees shorter than $\mathbb{T}$.

The idea of the proof is the following. We introduce a time scale $L(t) \approx t^{A(\emptyset)}$ and couple (ψ_s) for $s \geq t - L(t)$ with another process (ψ_s^2). This process shall have initial configuration $M(\rho)$, where ρ is the empirical population density of $\psi^1_{t-L(t)}$ in a box of length $\approx t^{A(\emptyset)/2}$. $L(t)$ will be chosen small enough that the evolutions of the subtrees (resulting from eliminating the root $\emptyset$ from $\mathbb{T}$) are asymptotically independent. On the other hand, $L(t)$ has to be chosen large enough so that the local coupling with local size $R(t) \approx t^{A(\emptyset/2)}$ is successful. Here are the details.

Let $b = \max\{\mathrm{diam}(B^e),\, e \in \mathbb{T}\}$. Let $d_t \downarrow 0$, $t \to \infty$, such that

$$\begin{aligned} t^{(A(e\wedge f)-d_t)/2} &\leq \|x_t^e - x_t^f\| - b(t^{A(e)/2} + t^{A(f)/2}) \\ &\leq \|x_t^e - x_t^f\| + b(t^{A(e)/2} + t^{A(f)/2}) \leq \frac{1}{2}t^{(A(e\wedge f)+d_t)/2} \end{aligned} \tag{5.16}$$

for all $e, f \in \mathbb{T}$. We may and will assume that $t^{d_t} \overset{t\to\infty}{\longrightarrow} \infty$. Let $\alpha := A(\emptyset)$. Let

$$\begin{aligned} S = S(t) &= t^{(\alpha+d_t)/2}, \\ R = R(t) &= t^{(\alpha-3d_t)/2}, \\ L = L(t) &= t^{\alpha-2d_t}. \end{aligned}$$

Let $B_t^e = x_t^e + t^{A(e)/2}B^e$ and $B_t = \bigcup_{e\in\mathbb{T}} B_t^e$. By shifting $X = (x_t^e,\, e \in \mathbb{T})$, if necessary, we can assume that $B_t \subset [0, S[^2$ for all $t > 0$ and

$$L^{-1/2} \cdot \mathrm{dist}(B_t, \mathbb{R}^2 \setminus [0, S[^2[) \overset{t\to\infty}{\longrightarrow} \infty. \tag{5.17}$$

Apply Corollary 3.7 with $\psi_0^1 = \psi_{t-L(t)}$, $L(t)$ instead of t, $\rho = \log t/8\pi$, and with $\varepsilon = \delta = \frac{\log t}{8\pi}\varepsilon_t$, where $\varepsilon_t \overset{t\to\infty}{\longrightarrow} 0$. This last choice is possible due to Lemma 4.2. Thus we obtain a coupling $(\psi_s^1, \psi_s^2)_{s\geq 0}$ with $\mathcal{L}[\psi_0^2|\psi_0^1] = M(S^{-2}\psi_0^1([0,S[^2))$ such that there exists a sequence $\delta_t \downarrow 0$ with

$$\mathbf{E}^{\widetilde{M}(t)}\left[\left|(\widetilde{\psi}^1_{L(t)} - \widetilde{\psi}^2_{L(t)})(C)\right|\right] \leq \delta_t \cdot |C| \quad \forall\, C \in \mathcal{B}(\mathbb{R}^2) \text{ bounded.} \tag{5.18}$$

So all we have to show is

$$\mathcal{L}^{\widetilde{M}(t)}\left[\frac{8\pi}{\log t}\left(t^{-A(e)}\psi^2_{L(t)}(B^e_t)\right)_{e\in\mathbb{T}}\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}^1\left[(|B^e|Z^e_{1-A(e)})_{e\in\mathbb{T}}\right]. \tag{5.19}$$

By Theorem 1 (and Lemma 4.2) we know that

$$\mathcal{L}^{\widetilde{M}(t)}\left[\frac{8\pi}{\log t}S^{-2}\psi_0^1([0,S[^2)\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}[Z_{1-\alpha}]. \tag{5.20}$$

Hence (using the Chapman-Kolmogorov equation) showing (5.19) amounts to showing for $\rho \geq 0$,

$$\begin{aligned}\mathcal{L}^{M(\rho\log t/8\pi)}\left[\frac{8\pi}{\log t}\left(t^{-A(e)}\psi_{L(t)}(B^e_t)\right)_{e\in\mathbb{T}}\right] &\overset{t\to\infty}{\Longrightarrow} \mathcal{L}^\rho\left[(Z^e_{\alpha-A(e)})_{e\in\mathbb{T}}\right] \\ &= \mathcal{L}^{\rho/\alpha}\left[(\alpha Z^e_{1-A(e)/\alpha})_{e\in\mathbb{T}}\right].\end{aligned} \tag{5.21}$$

The last equality is the basic scaling property of Feller's diffusion.

Let $\mathbb{T}_j = \{(j, l_2, \ldots, l_n) \in \mathbb{T}, n \in \mathbb{N}\}$, $j = 1, \ldots, J$ be the partition of $\mathbb{T}$ into subtrees $\mathbb{T}_j$ ($\mathbb{T} = \{\emptyset\} \cup \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_J$). To prove (5.21) it suffices (by the induction hypothesis) to show that

$$\left(\tfrac{8\pi}{\log t}t^{-A(e)}\psi_{L(t)}(B^e_t)\right)_{e\in\mathbb{T}_j}, \quad j = 1, \ldots, J, \tag{5.22}$$

are J asymptotically independent random variables.

For each $j = 1, \ldots, J$, fix one $e_j \in \mathbb{T}_j$ and let $C_j = C_j(t) = x_t^{e_j} + [-R(t), R(t)[^2$ and $C_0 = \mathbb{R}^2 \setminus (C_1 \cup \cdots \cup C_J)$. Then for t large enough we have $C_i \cap C_j = \emptyset$ for $i \neq j$. Let

$$\Delta_j = \Delta_j(t) = \inf_{e\in\mathbb{T}_j} \operatorname{dist}(B^e_t, \mathbb{R}^2 \setminus C_j).$$

Since $A : \mathbb{T} \to I$ is strictly decreasing, we have $\Delta_j(t)/\sqrt{L(t)} \overset{t\to\infty}{\longrightarrow} \infty$.

Let $(\chi^j_s)_{s\geq 0}$, $j = 0, 1, \ldots, J$, be independent BBM($\mathbb{R}^2$) or SBM($\mathbb{R}^2$) with $\chi^j_0 = M(\frac{\log t}{8\pi}\rho)\Big|_{C_j}$, $j = 0, 1, \ldots, J$. We can assume $\psi_s = \chi^0_s + \cdots + \chi^J_s$. Now for $j = 1, \ldots, J$ and $e \in \mathbb{T}_j$,

$$\begin{aligned}&\mathbf{E}\left[\frac{8\pi}{\log t}t^{-A(e)}\sum_{\substack{i=0\\ i\neq j}}^J \chi^i_{L(t)}(B^e_t)\right] \\ &\quad\leq \rho|B^e|t^{-A(e)}\int_{\mathbb{R}^2\setminus C_j} dx \int_{B^e_t} dy\, p_{L(t)}(x,y) \leq \rho|B^e|\exp\{-\Delta_j^2/L(t)\} \overset{t\to\infty}{\longrightarrow} 0.\end{aligned} \tag{5.23}$$

Thus (5.22) holds and the proof is complete. □

6 Proofs for Finite Systems

6.1 Proof of Theorem 3

The idea of the proof is again to introduce a new time scale $L(\ell) \ll \ell^2$ and to let $T'(\ell) = T(\ell) - L(\ell)$. As in the previous section, we want to couple (locally) given $\ell^{-d}\psi_{T'(\ell)}(\Lambda_\ell^d) = \rho$ with a process started in $M_\ell(\rho)$. This latter process will then be compared to the infinite process started in $M(\rho)$. So as to impose the local coupling, we will have to cut Λ_ℓ^d into a growing (with ℓ) number of boxes $N(\ell)^d$. $N(\ell)$ has to be chosen such that the empirical densities of $\psi_{T'(\ell)}$ within the boxes and within Λ_ℓ^d are asymptotically close.

Step 1. We start with showing this latter point. Let $A, B \in \mathcal{B}(\Lambda_1^d)$, $|A|, |B| > 0$, and $\phi_\ell = |\ell A|^{-1}1\!\!1_{\ell A} - |\ell B|^{-1}1\!\!1_{\ell B}$, $\ell > 0$. Then by the second moment formulas (3.5) and (3.12) (recall that (S_t) is the semigroup and $p_{\ell,t}(\cdot,\cdot)$ the transition density of Brownian motion on Λ_ℓ^d),

$$\begin{aligned}&\mathbf{E}^{M_\ell(\rho)}[(|\ell A|^{-1}\psi_{\ell,T(\ell)}(\ell A) - |\ell B|^{-1}\psi_{\ell,T(\ell)}(\ell B))^2] \qquad (6.1)\\ &\leq \int \left(\langle \mu, S_{T(\ell)}\phi_\ell\rangle\right)^2 + \langle \mu, S_{T(\ell)}(\phi_\ell^2) - (S_{T(\ell)}\phi_\ell)^2\rangle + \left\langle \mu, \int_0^T S_{T(\ell)-s}(S_s\phi_\ell)^2\, ds\right\rangle M_\ell(\rho)(d\mu),\end{aligned}$$

with equality in the case of BBM. Fix a sequence $\gamma(\ell)$ such that $\ell^2 \ll \gamma(\ell) \ll T(\ell)$. Then

$$\sup_{t\geq\gamma(\ell)} \sup_{z\in\Lambda_\ell^d} |\ell^d p_{\ell,t}(0,z) - 1| =: \varepsilon_\ell \overset{\ell\to\infty}{\longrightarrow} 0. \qquad (6.2)$$

Thus for $t \geq \gamma(\ell)$,

$$\sup_{x\in\Lambda_\ell^d} |\langle \delta_x, S_t\phi_\ell\rangle| \leq 2\varepsilon_\ell \ell^{-d} \qquad (6.3)$$

and, of course, for all $t \geq 0$

$$\sup_{x\in\Lambda_\ell^d} |\langle \delta_x, S_t\phi_\ell\rangle| \leq (|A|^{-1} + |B|^{-1})\ell^{-d}. \qquad (6.4)$$

Note that $\phi_\ell^2 \leq \ell^{-2d}(|A|^{-1} + |B|^{-1})^2$. Hence (6.1) is dominated by

$$4\varepsilon_\ell^2 \ell^{-2d}(\rho^2\ell^{2d} + \rho\ell^d) + \rho(|A|^{-1} + |B|^{-1})^2\ell^{-d} + \rho\left[\varepsilon_\ell^2 T(\ell)\ell^{-d} + (|A|^{-1} + |B|^{-1})^2\gamma(\ell)\ell^{-d}\right] \overset{\ell\to\infty}{\longrightarrow} 0. \qquad (6.5)$$

If we replace $T(\ell)$ by $T'(\ell)$, this convergence is uniform in all sequences $T'(\ell)$ such that $\frac{1}{2}T(\ell) \leq T'(\ell) \leq T(\ell)$. Thus we can find a sequence $N(\ell) \uparrow \infty$, $\frac{\log N(\ell)}{\log \ell} \overset{\ell\to\infty}{\longrightarrow} 0$, and define $L(\ell) = \frac{\ell^2}{N(\ell)}$, $T'(\ell) = T(\ell) - L(\ell)$ such that

$$\ell^{-2d}\mathbf{E}^{M_\ell(\rho)}\left[\left|\psi_{\ell,T'(\ell)}(\Lambda_\ell^d) - N(\ell)^d\psi_{\ell,T'(\ell)}([0, N(\ell)^{-1}\ell[^d)\right|\right] =: \delta_\ell \overset{\ell\to\infty}{\longrightarrow} 0. \qquad (6.6)$$

Step 2. (Coupling) We continue arguing as in the proof of Theorem 2. We let $(\chi_{\ell,t}^1, \chi_{\ell,t}^2)_{t\geq0}$ be the local coupling of BBM(Λ_ℓ^d) or SBM(Λ_ℓ^d) according to Corollary 3.6 with $R = R(\ell) = \frac{\ell}{N(\ell)}$.

The initial configuration shall be $\chi^1_{\ell,0} = \psi_{\ell,T'(\ell)}$ and $\mathcal{L}[\chi^2_0|\chi^1_0] = M_\ell(\ell^{-2}\chi^1_0(\Lambda^d_\ell))$. By Corollary 3.6 we get for $B \in \mathcal{B}(\mathbb{R}^d)$ bounded

$$\mathbf{E}^{M_\ell(\rho)}\left[\left\|(\chi^1_{\ell,L(\ell)} - \chi^2_{\ell,L(\ell)})\Big|_B\right\|\right] \le |B|\cdot\left[\delta_\ell + 2\sqrt{\rho R(\ell)^{-d}} + 2\sqrt{d/\pi}, \rho N(\ell)^{-1/2}\right] \overset{\ell\to\infty}{\longrightarrow} 0. \tag{6.7}$$

Step 3. (Comparison) We apply the comparison lemma (Lemma 3.8) to $(\chi^3_t)_{t\ge 0}$ with $\mathcal{L}[\chi^3_0|\chi^1_0] = M(\ell^{-d}(\Lambda^d_\ell))$ and $(\chi^2_{\ell,t})$ and with $A_\ell \equiv B$ to obtain

$$\mathbf{E}\left[\left|\chi^2_{\ell,L(\ell)}(B) - \chi^3_{L(\ell)}(B)\right|\right] \overset{\ell\to\infty}{\longrightarrow} 0. \tag{6.8}$$

Thus

$$\mathbf{E}\left[\left|\chi^1_{\ell,L(\ell)}(B) - \chi^3_{L(\ell)}(B)\right|\right] \overset{\ell\to\infty}{\longrightarrow} 0. \tag{6.9}$$

Step 4. (Conclusion) Fix $f \in C_{\mathrm{c}}(\mathbb{R}^d)$ and $F \in C_{\mathrm{b}}(\mathbb{R})$. Then

$$\begin{aligned}\mathbf{E}^{M_\ell(\rho)}[F(\langle\psi_{\ell,T(\ell)}, f\rangle)] &= \mathbf{E}[F(\langle\chi^1_{\ell,L(\ell)}, f\rangle)] \\ &= \mathbf{E}[F(\langle\chi^2_{\ell,L(\ell)}, f\rangle)] + o(1) \\ &= \mathbf{E}[F(\langle\chi^3_{\ell,L(\ell)}, f\rangle)] + o(1) \\ &= \int_0^\infty \mathbf{P}^\rho[Z_{\sigma/2} \in d\rho']F(\langle\nu_{\rho'}, f\rangle) + o(1).\end{aligned} \tag{6.10}$$

The last equality holds because of (1.13) and (2.9). □

6.2 Proof of Theorem 4 and 5

The proofs are similar to that of Theorem 3. Hence we give only an outline. Recall that $\beta(\ell) = \ell^2 \log \ell$. By (2.9) we know that

$$\mathcal{L}^{\widetilde{M}_\ell(\beta(\ell))}\left[\frac{8\pi}{\log\beta(\ell)}\ell^{-2}\|\psi_{\ell,T'(\ell)}\|\right] \overset{\ell\to\infty}{\Longrightarrow} \mathcal{L}^1[Z_{2\pi\sigma}]. \tag{6.11}$$

Choose $L(\ell) \ll \ell^2$ such that $\lim\limits_{\ell\to\infty}\frac{\log L(\ell)}{\log\beta(\ell)} = \lim\limits_{\ell\to\infty}\frac{\log L(\ell)}{\log \ell^2} = 1$. Now we can proceed as in the proof of Theorem 3. We couple locally with the configuration

$$\int_0^\infty \mathbf{P}^1[Z_{2\pi\sigma} \in d\rho] M_\ell\left(\rho\,\frac{\log\beta(\ell)}{8\pi}\right) \tag{6.12}$$

and compare this with the infinite system started in

$$\int_0^\infty \mathbf{P}^1[Z_{2\pi\sigma} \in d\rho] M\left(\rho\,\frac{\log\beta(\ell)}{8\pi}\right). \tag{6.13}$$

Now we apply Theorem 1 resp. 2 to obtain the conclusions. □

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