Clustering and Invariant Measures for Spatial Branching Models with Infinite Variance

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Abstract

We consider two spatial branching models on $\mathbb{R}^d$: branching Brownian motion with a branching law in the domain of normal attraction of a $(1+\beta)$ stable law, $0 < \beta \leq 1$, and the corresponding high density limit measure valued diffusion.

The longtime behaviour of both models depends highly on β and d.

We show that for $d \leq 2/\beta$ the only invariant measure is $\delta_{\mathbf{0}}$, the unit mass on the empty configuration. Furthermore we give a precise condition for convergence towards $\delta_{\mathbf{0}}$.

For $d > 2/\beta$ it is known that there exists a family $(\nu_\theta,\ \theta \in [0,\infty))$ of non-trivial invariant measures. We show that every invariant measure is a convex combination of the ν_θ. Both results have been known before only under an additional finite mean assumption.

For the critical dimension $d = 2/\beta$ we show that both models display the phenomenon of diffusive clustering. This means that clusters grow spatially on a random scale. We give a precise description of the clusters via multiple scale analysis.

Our methods rely mainly on studying sub- and supersolutions of the reaction diffusion equation $\partial u/\partial t - \frac{1}{2}\Delta u + u^{1+\beta} = 0$.

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1 Introduction and main results

1.1 Survey

Branching Brownian motion (BBM) is an (infinite) particle system in which particles perform independent Brownian motions and split at random times into a random number of offspring particles. We consider the process $(\psi_t)_{t\geq 0}$ which will be either branching Brownian motion on $\mathbb{R}^d$ with offspring probability generating function $f(z) = z + \frac{1}{2}(1-z)^{1+\beta}$, $0 < \beta \leq 1$, or its high density limit measure valued diffusion, the so-called super Brownian motion (SBM). (Note that the probability distribution generated by f is in the normal domain of attraction of a stable law with index $1+\beta$. In particular, for $\beta < 1$ this law does not have variance.) In the SBM the transport of mass is governed by the (deterministic) heat flow while the local "intensity of matter" fluctuates randomly.

A path-wise construction of these processes in terms of excursions of certain random walks and Lévy processes respectively can be found in a recent paper by Le Gall and Le Jan (1998). See also Gorostiza et al. (1992) for a corresponding multi-type model.

It is well known that $\delta_{\mathbf{0}}$, the unit mass on the empty configuration, is the only invariant measure with finite intensity if $d \leq 2/\beta$. One aim of this paper is to show that the finite intensity assumption can be dropped, hence $\delta_{\mathbf{0}}$ is the *only* invariant measure for (ψ_t) if $d \leq 2/\beta$. In the case of finite variance branching ($\beta = 1$) this has been done before by Bramson, Cox and Greven (1993). Their approach (as ours) is based on the study of sub- and supersolutions $u(t,x;f)$, $x \in \mathbb{R}^d$, $t \geq 0$, to the reaction diffusion equation

$$(\partial_t - \frac{1}{2}\Delta)u + u^{1+\beta} = 0, \tag{1.1}$$

where $\partial_t = d/dt$ and Δ denotes the Laplacian in $\mathbb{R}^d$. While most of their techniques work also in our setting, part of the argument of Bramson, Cox and Greven (1993) relies on a second moment estimate and had to be replaced to cope with $\beta < 1$.

In the high dimensional case $d > 2/\beta$ it is known that there exists a family $(\nu_\theta,\ \theta \in [0,\infty))$ of extremal invariant (and translation invariant) measures for (ψ_t). All invariant measures with finite intensity can be represented as a convex combination of these ν_θ. We show in this paper that the finite intensity assumption can be dropped. For the case $\beta = 1$ this has been shown before by Bramson, Cox and Greven (1997).

The other main point of this paper is to investigate closer the clustering in the critical dimension $d = 2/\beta$. We show that the so-called *diffusive clustering* occurs. This phenomenon has first been investigated for the voter model by Cox and Griffeath (1986). Roughly speaking diffusive clustering means that clusters grow spatially at a random order of magnitude. This phenomenon has been observed for a lot of interacting particle systems and related models such as the voter model, linearly interacting diffusions, critical binary branching Brownian motion, etc. A detailed treatment can be found in Klenke (1996) and Klenke (1997). All these models have in common that the local random fluctuations (given, e.g., by the branching law or the resampling mechanism) have finite variances and that the critical dimension (in which diffusive clustering occurs) is $d = 2$.

This is however the first case in which diffusive clustering is observed in absence of a second moment. In the finite variance models it turned out that the growth of the cluster height is dominated by the Green function

$$G(t) = \int_1^t p_s(0,0)\, ds,$$

where $p_t(\cdot,\cdot)$ is the (symmetric) *interaction kernel* of the model. Of course, here p_t is the heat kernel. In the absence of a second moment we show that the cluster growth is now governed by the quantity

$$G_\beta(t) = \left(\int_1^t p_s(0,0)^\beta \, ds \right)^{1/\beta} .$$

This object naturally arises in the investigation of Kallenberg's backwards tree of the clusters (see Gorostiza and Wakolbinger (1991)). In this paper we do not make explicit use of the backwards *tree* but rely on pde methods, the connection being that (1.1) is Kolmogorov's backwards equation for the Laplace functionals of (ψ_t).

1.2 The Models

We give a short description of the models considered in this paper. For more details we refer the reader to Dawson (1993). Unfortunately we have to introduce a lot of notation first.

Basic Definitions for Random Measures

Let E be a locally compact polish space. By $\mathcal{B}(E)$ we denote the Borel σ-field on E. By $C_b(E)$ and $C_c(E)$ we denote the spaces of continuous real valued functions on E that are bounded resp. have compact support. Further let $C_c^+ = \{f \in C_c : \ f \geq 0\}$ and $C_c^{++} = \{f \in C_c^+ : \ f \not\equiv 0\}$ and define C_b^+ and C_b^{++} analogously.

A measure μ on $\mathcal{B}(E)$ is called *locally finite* if $\mu(K) < \infty$ for all compact sets $K \subset E$. Let

$$\mathcal{M}(E) = \{\text{locally finite measures on } E\} \tag{1.2}$$

and $\mathcal{M}_\mathrm{f}(E) = \{\mu \in \mathcal{M}(E) : \mu(E) < \infty\}$.

For $\mu \in \mathcal{M}(E)$ and $f : E \to \mathbb{R}$ measurable and μ-integrable we define $\langle \mu, f \rangle := \int f \, d\mu$. $\mathcal{M}(E)$ is a polish space with the vague topology, defined by $\mu_n \to \mu$ iff $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$ for all $f \in C_c(E)$. The space $\mathcal{M}_1(\mathcal{M}(E))$ of probability measures on $\mathcal{M}(E)$, equipped with the weak topology, is also polish (see, e.g., Kallenberg (1983)). For weak convergence of probability measures we use the symbol "$\Longrightarrow$".

The space of (non-negative) integer valued measures μ on $\mathcal{B}(E)$ will be denoted by

$$\mathcal{N}(E) = \{\mu \in \mathcal{M}(E) : \mu(A) \in \{0,1,2\ldots,\infty\} \quad \forall A \in \mathcal{B}(E)\}. \tag{1.3}$$

For $m \in \mathcal{M}(\mathbb{R}^d)$ we denote by $\mathcal{H}(m) \in \mathcal{M}_1(\mathcal{N}(\mathbb{R}^d))$ the distribution of the Poisson point process on $\mathbb{R}^d$ with intensity measure m, i.e., for $f \in C_c^+(\mathbb{R}^d)$,

$$\int \mathcal{H}(m)(d\mu) e^{-\langle \mu, f \rangle} = \exp(-\langle m, 1 - e^{-f} \rangle). \tag{1.4}$$

We use the notation $\mathcal{L}[X]$ for the distribution of a random variable X. Let $(X_t)_{t \geq 0}$ be a Markov process with values in E and $x \in E$ or $Q \in \mathcal{M}_1(E)$. By $\mathcal{L}^x[(X_t)_{t\geq 0}]$ and $\mathcal{L}^Q[(X_t)_{t\geq 0}]$ we denote the distributions of $(X_t)_{t\geq 0}$ with $\mathcal{L}^x[X_0] = \delta_x$ and $\mathcal{L}^Q[X_0] = Q$. If (X_t) is càdlàg, convergence of paths will be understood in the Skorohod topology. Convergence of finite dimensional marginals will be indicated by "fdd".

$(1+\beta)$-Branching Brownian Motion

Let $0 < \beta \leq 1$ and let $(p_k)_{k=0,1,\ldots}$ be the probability distribution on $\mathbb{N}_0$ with p.g.f. $f(z) = z + \frac{1}{2}(1-z)^{1+\beta}$, $z \in [0,1]$, i.e.,

$$p_k = \begin{cases} 1/2 & \text{if } k = 0, \\ (1-\beta)/2 & \text{if } k = 1, \\ \dfrac{1}{2}(-1)^k \dbinom{1+\beta}{k} & \text{if } k = 2, 3, \ldots \end{cases} \tag{1.5}$$

Note that (p_k) is critical,that is $\sum kp_k = 1$, and is in the normal domain of attraction of a stable law on $[0,\infty)$ with index $(1+\beta)$. In particular, for $\beta < 1$ the law (p_k) has infinite variance.

We will consider a particle performing a Brownian motion on $\mathbb{R}^d$ and with an exponential lifetime with mean $1/2b > 0$. At the time of death the particle produces an offspring of k particles with probability p_k. The offspring behave as k independent copies of the one-particle system started at the parent particle's final position. If we start the process with more than one particle at time 0, we assume that all particles are independent.

The process

$$\eta_t(A) = \#\{\text{particles in } A\}, \qquad A \in \mathcal{B}(\mathbb{R}^d),\ t \geq 0, \tag{1.6}$$

will be called the *branching Brownian motion on* $\mathbb{R}^d$ *with parameters* $1+\beta$ *and* b, abbreviated $\mathrm{BBM}(d, 1+\beta, b)$.

$(1+\beta)$-Super Brownian Motion

Next we consider the short lifetime high density limit of $\mathrm{BBM}(d, 1+\beta, b)$. Let $\mu \in \mathcal{M}_\mathrm{f}(\mathbb{R}^d)$ and $\mu^N \in \mathcal{N}_\mathrm{f}(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that $N^{-1}\mu^N \to \mu$, as $N \to \infty$. For $N \in \mathbb{N}$ let $(\eta_t^N)_{t\geq 0}$ be $\mathrm{BBM}(d, 1+\beta, bN^\beta)$ with initial state $\eta_0^N = \mu^N$. It is well known that there exists a càdlàg Markov process $(\zeta_t)_{t\geq 0}$ with values in $\mathcal{M}_\mathrm{f}(\mathbb{R}^d)$ such that

$$\mathcal{L}^\mu[(\zeta_t)_{t\geq 0}] = \mathrm{w} - \lim_{N\to\infty} \mathcal{L}^{\mu^N}\left[(\frac{1}{N}\eta_t^N)_{t\geq 0}\right] \tag{1.7}$$

(see Dawson (1993), Section 4.4ff).

The process $(\zeta_t)_{t\geq 0}$ will be called *super Brownian motion on* $\mathbb{R}^d$ *with parameters* $1+\beta$ *and* b, abbreviated $\mathrm{SBM}(d, 1+\beta, b)$.

For $\mu \in \mathcal{M}(\mathbb{R}^d)$ we can define $(\zeta_t)_{t\geq 0}$ with initial configuration $\zeta_0 = \mu$ as the increasing limit of $(\zeta_t^n)_{t\geq 0}$ with initial configurations $\mu^n \in \mathcal{M}_\mathrm{f}(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $\mu^n \uparrow \mu$. It is known that $\mathrm{SBM}(d, 1+\beta, b)$ takes values in $\mathcal{M}(\mathbb{R}^d)$ if we impose a regularity condition on the initial state μ, e.g., assume $\langle \mu, (1+\|\cdot\|^2)^{-p})\rangle < \infty$ for some $p > d/2$. The same condition also assures that $\eta_t \in \mathcal{N}(\mathbb{R}^d)$ a.s. for all $t \geq 0$.

Log-Laplace equation

Let $f \in C_b^+(\mathbb{R}^d)$. A prominent role in this paper is played by the solution $u(t,x;f)$, $x \in \mathbb{R}^d$, $t \geq 0$, of the Cauchy problem

$$\begin{aligned} L_\beta u(t,x;f) &= 0, && x \in \mathbb{R}^d,\ t \geq 0, \\ u(0,x;f) &= f(x), && x \in \mathbb{R}^d, \end{aligned} \tag{1.8}$$

where

$$L_\beta u(t,x;f) = (\partial_t - \frac{1}{2}\Delta)u(t,x;f) + bu(t,x;f)^{1+\beta}. \tag{1.9}$$

Since (1.8) is time-homogeneous, u has the (non-linear) semigroup property

$$u(t+s,x;f) = u(t,x;u(s,\cdot;f)), \qquad x \in \mathbb{R}^d,\ s,t \geq 0. \tag{1.10}$$

Note that for $\rho > 0$ the following scaling relation holds,

$$u(t,x;f) = \rho^{1/\beta}u(\rho t, \rho^{1/2}x; \rho^{-1/\beta}f(\rho^{-1/2}\cdot)), \qquad x \in \mathbb{R}^d,\ t > 0. \tag{1.11}$$

The reaction-diffusion equation (1.9) is linked to our branching processes by the equations (see, e.g., Dawson (1993))

$$\mathbf{E}^{\delta_x}[\exp(-\langle \eta_t, f\rangle)] = 1 - u(t,x;1-e^{-f}), \tag{1.12}$$

$$\mathbf{E}^{\delta_x}[\exp(-\langle \zeta_t, f\rangle)] = \exp(-u(t,x;f)). \tag{1.13}$$

1.3 Invariant Laws

Recall that $p_t(x) = (2\pi t)^{-d/2} \exp(-\|x\|^2/2t)$ is the heat kernel on $\mathbb{R}^d$. Define $G_\beta(t)$ by

$$G_\beta(t) = \left(\int_1^t p_s(0)^\beta \, ds \right)^{1/\beta}. \tag{1.14}$$

For $d \in \mathbb{N}$ define $\phi : (1, \infty) \times \mathbb{R}^d \to [0, \infty)$ by

$$\phi(t, x) = p_t(x)/G_\beta(t). \tag{1.15}$$

It will turn out that (in the particle language) $G_\beta(t)$ measures the concentration of particles around a certain point (say the origin), given that there is a particle. It is the typical concentration of particles or average "cluster height". On the other hand $p_t(x)$ is the expected intensity of particles at the origin if we start in δ_x. Consequently, the function $\phi(t,x)$ measures the probability of seeing a particle at the origin at time $t > 1$ if we start with one "particle" at time 0 at site $x \in \mathbb{R}^d$.

Low dimension

Recall that (ψ_t) is either BBM$(d, 1+\beta, b)$ or SBM$(d, 1+\beta, b)$.

Theorem 1 *Assume $d \leq 2/\beta$. Then the following hold.*

(i) $\mathcal{L}[\psi_t] \overset{t\to\infty}{\Longrightarrow} \delta_{\mathbf{0}}$ *if and only if*

$$\mathcal{L}[\langle \psi_0, \phi(r, \cdot)\rangle] \Longrightarrow \delta_0, \ \ r \to \infty. \tag{1.16}$$

(ii) *If condition* (1.16) *does not hold, then ψ_t is unstable, i.e. for any $f \in C_c^{++}(\mathbb{R}^d)$ the sequence $\langle \psi_t, f\rangle$ is stochastically unbounded.*

(iii) *If $\mathcal{L}[\langle \psi_0, \phi(r, \cdot)\rangle] \Longrightarrow \delta_\infty$, $r \to \infty$, then ψ_t explodes, i.e. for any $f \in C_c^{++}(\mathbb{R}^d)$ almost surely $\langle \psi_t, f\rangle \to \infty$, as $t \to \infty$.*

Corollary 1.1 *If $d \leq 2/\beta$, then the only invariant measure for ψ_t is $\delta_{\mathbf{0}}$.* □

In order to check the conditions of Theorem 1 it is useful to note that $\phi(t,x)$ can be bounded from above and below by the function $\Phi : (1, \infty) \times \mathbb{R}^d \to [0, \infty)$, defined by

$$\Phi(t,x) = \begin{cases} t^{d/2 - 1/\beta} p_t(x) & \text{if } \beta < 2/d \\ (\log t)^{-1/\beta} p_t(x) & \text{if } \beta = 2/d \\ p_t(x) & \text{if } \beta > 2/d. \end{cases} \tag{1.17}$$

More precisely, there exist $c, C > 0$ (depending only on d and β) such that for $t \geq 2$ and $x \in \mathbb{R}^d$

$$c\Phi(t,x) \leq \phi(t,x) \leq C\Phi(t,x). \tag{1.18}$$

(This is immediate from the fact that $p_t(0) = (2\pi t)^{-d/2}$.) Hence it suffices to verify the conditions of Theorem 1 for Φ instead of ϕ.

High dimension

Let $\mathcal{I}$ be the set of invariant measures for (ψ_t), by $\mathcal{I}_e \subset \mathcal{I}$ we denote its extremal elements. It is well known (see Gorostiza and Wakolbinger (1992), Theorem 1) that there exists a one parameter family $\{\nu_\theta,\ \theta \in [0,\infty)\} \subset \mathcal{I}_e$ with the following properties. Each ν_θ is translation invariant, ergodic and has intensity θ, i.e., $\int \nu_\theta(dm)\langle m, f\rangle = \theta\langle \lambda, f\rangle$ for $f \in C_c^+(\mathbb{R}^d)$. Further for $\mathcal{L}[\psi_0]$ translation invariant and ergodic with $\mathbf{E}[\langle \psi_0, f\rangle] = \theta\langle\lambda, f\rangle$,

$$\mathcal{L}[\psi_t] \overset{t\to\infty}{\Longrightarrow} \nu_\theta.$$

For any $\mu \in \mathcal{I}$ with σ-finite intensity measure there exists a unique probability measure F_μ on $[0,\infty)$ such that

$$\mu = \int \nu_\theta F_\mu(d\theta). \tag{1.19}$$

Our point is to drop the assumption of the σ-finiteness on μ to allow for a representation as in (1.19).

Theorem 2 *Let $d > 2/\beta$ and let (ψ_t) either BBM$(d, 1+\beta, b)$ or SBM$(d, 1+\beta, b)$. Then the following holds.*

$\mathcal{I}_e = \{\nu_\theta,\ \theta \in [0,\infty)\}$ and for any $\mu \in \mathcal{I}$ there exists a unique probability distribution F_μ on $[0,\infty)$ such that $\mu = \int \nu_\theta F_\mu(d\theta)$.

The crucial step to prove Theorem 2 is the following proposition.

Proposition 1.2 *Any invariant measure is translation invariant.*

1.4 Critical dimension: Diffusive clustering

Our aim is to give a precise description of the clustering in the critical dimension $d = 2/\beta$. Hence we will assume $\beta = 2/d$. For simplicity of notation we will also assume $b = 1$ in the following discussion.

Proceeding as in Klenke (1997) we introduce the following concepts for the description of the heights of the clusters and their expansions in space.

(1) High density rescaling

For time $t > 1$ we define

$$\widetilde{\psi}_t = \widetilde{\psi}_t^0 := (\log t)^{-1/\beta}\psi_t \tag{1.20}$$

with (recall that λ is the d-dimensional Lebesgue measure)

$$\mathcal{L}[\psi_0] = M_t := \begin{cases} \mathcal{H}((\log t)^{1/\beta}\lambda) & \text{if } \psi_t \text{ is BBM,} \\ \delta_{(\log t)^{1/\beta}\cdot\lambda} & \text{if } \psi_t \text{ is SBM.} \end{cases} \tag{1.21}$$

(2) Spatial rescaling

For (ψ_t) BBM$(d, 1+\beta, 1)$ or SBM$(d, 1+\beta, 1)$ let $I = [0,1]$ respectively $I = (-\infty, 1]$. We fix $\alpha \in I$ and define $(\widetilde{\psi}_t^\alpha)$ by

$$\widetilde{\psi}_t^\alpha := \mathcal{S}_{\alpha,t}\widetilde{\psi}_t, \quad \alpha \in I, \tag{1.22}$$

where

$$\mathcal{S}_{\alpha,t} : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d), \quad \mu(\cdot) \mapsto t^{-\alpha d/2}\mu(t^{\alpha/2}\,\cdot).$$

That is, for $B \in \mathcal{B}(\mathbb{R}^d)$ we set $\widetilde{\psi}_t^\alpha(B) = t^{-\alpha d/2}\widetilde{\psi}_t(t^{\alpha/2}B)$. As above we let $\widetilde{\psi}_t = \widetilde{\psi}_t^0$.

Remark: Since we intend to take the limit $t \to \infty$, it would not make sense to allow $\alpha < 0$ for BBM. Due to the particle structure in this case we would get $\mathbf{P}^{M_t}[\widetilde{\eta}_t^\alpha(B) = 0] \stackrel{t\to\infty}{\longrightarrow} 1$ for all bounded sets $B \in \mathcal{B}(\mathbb{R}^d)$. This leads us to the different choices of I.

We introduce the *total mass process* $(Z_t)_{t\geq 0}$ of SBM$(d, 1+\beta, 1)$ which is the "diffusion limit" of Galton-Watson processes with offspring probabilities defined in (1.5) above. (Z_t) is a process with independent increments which can be characterized by its log-Laplace transform

$$v(t, K, \theta) = -\log \mathbf{E}^K[\exp(-\theta Z_t)] \tag{1.23}$$

that is the unique solution of

$$\begin{aligned} v(0, K, \theta) &= \theta K, \\ \partial_t v(t, K, \theta) &= -v(t, K, \theta)^{1+\beta}. \end{aligned} \tag{1.24}$$

The solution can be given explicitly:

$$v(t, K, \theta) = (\beta t + (K\theta)^{-\beta})^{-1/\beta}. \tag{1.25}$$

Let $c_\beta = (2\pi(1+\beta)^{1/\beta})^{-1}$ and recall that λ is the Lebesgue measure.

Theorem 3 *For (ψ_t) BBM$(d, 1+2/d, 1)$ or SBM$(d, 1+2/d, 1)$ and $\alpha \in I$ the following holds,*

$$\mathcal{L}^{M_t}[\widetilde{\psi}_t^\alpha] \stackrel{t\to\infty}{\Longrightarrow} \mathcal{L}^1[Z_{c_\beta(1-\alpha)} \cdot \lambda]. \tag{1.26}$$

Multiple Scale Analysis

So far we have considered our rescaled process $\widetilde{\psi}^\alpha$ at *one* scale α. A natural task is to investigate the limit behaviour of $(\widetilde{\psi}_t^{\alpha_1}, \ldots, \widetilde{\psi}_t^{\alpha_n})$ for $\alpha_1, \ldots, \alpha_n \in I$. In order to learn more about the spatial structure of the clusters, we might also wish to choose different *points of observation* $x_t^1, \ldots, x_t^n \in \mathbb{R}^d$. Theorem 3 indicates that the distances $\|x_t^e - x_t^f\|$, $e \neq f$, of these points should grow in t on an algebraic scale $\alpha_{e,f} \in I$. Note that a consistent choice of the $\alpha_{e,f}$ implies that $2^{\alpha_{e,f}}$ is an ultra-metric on $\{1, \ldots, n\}$. Hence we may assume w.l.o.g. that the points of observation are indexed by a finite (rooted) tree $\mathbb{T}$ and that $\alpha_{e,f} = A(e \wedge f)$, where

$$A : \mathbb{T} \longrightarrow I$$

is a strictly decreasing map.

To explain this a bit, note that $\mathbb{T}$ carries a natural partial ordering $\leq$, where $e \leq f$ iff e is an ancestor of f, i.e., if e is closer to the root (denoted by $\emptyset$) than f. Hence $e \wedge f$ is the greatest common ancestor of e and f.

The pair $\mathbb{L} = (\mathbb{T}, A)$ will be called a *multiple space scale*. We will assume that $X = (x_t^e,\, e \in \mathbb{T},\, t \geq 0)$ is a family of points $x_t^e \in \mathbb{R}^d$ such that

$$\|x_t^e - x_t^f\| \approx t^{A(e\wedge f)/2}, \quad \text{as } t \to \infty.$$

As usual, $a_t \approx b_t$ means $(\log a_t)/(\log b_t) \stackrel{t\to\infty}{\longrightarrow} 1$. We refer to X as to be $\mathbb{L}$-*spaced*. Our aim is to study the asymptotics of the common distribution of (recall $\mathcal{S}$ from (1.22))

$$(\mathcal{S}_{A(e),t} \mathcal{T}_{x_t^e} \widetilde{\psi}_t)_{e\in\mathbb{T}} \quad \text{as } t \to \infty,$$

where $\mathcal{T}_z : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ is the translation by z, $(\mathcal{T}_z\mu)(\cdot) = \mu(z + \cdot)$.

We give a heuristic motivation for the next definition. Consider the simplest case $\mathbb{T} = \{\emptyset, e, f\}$, $e \wedge f = \emptyset$. Since Brownian motion has range $t^{1/2}$, the common history of the space-time points (t, x_t^e), and (t, x_t^f) ends at time $t - t^{A(e\wedge f)}$. After that time the histories of these points develop independently. In fact, asymptotically the common history of (t, x_t^e) and (t, x_t^f) is contained in $\widetilde{\psi}_{t-t^{A(e\wedge f)/2}}([-t^{A(e\wedge f)/2}, t^{A(e\wedge f)/2}]^d)$. This, together with Theorem 3, suggests that the intensity of $\mathcal{S}_{A(e),t}\mathcal{T}_{x_t^e}\widetilde{\psi}_t$ and $\mathcal{S}_{A(f),t}\mathcal{T}_{x_t^f}\widetilde{\psi}_t$ should consist of $Z_{c_\beta(1-A(e\wedge f))}$ plus two independent increments.

To be precise, let $(Z_t^e,\, e \in \mathbb{T})_{t\geq 0}$ be the following Markov process on $[0,\infty)^{\mathbb{T}}$. Each $(Z_t^e)_{t\geq 0}$ is a $(1+\beta)$ continuous state branching "diffusion" introduced in (1.23). For $e, f \in \mathbb{T}$ with $e \neq f$ we let $Z_t^e = Z_t^f$ for $t \in [0, 1 - A(e \wedge f)]$. For $t > 1 - A(e \wedge f)$ the evolutions of Z_t^e and Z_t^f shall be independent.

Theorem 4 (Multiple Scale)
Let (ψ_t) be BBM$(d, 1 + 2/d, 1)$ or SBM$(d, 1 + 2/d, 1)$ and $I = [0, 1]$, respectively $I = (-\infty, 1]$. Then the following holds.

$$\mathcal{L}^{M_t}\left[(\mathcal{S}_{A(e),t}\mathcal{T}_{x_t^e}\widetilde{\psi}_t)_{e\in\mathbb{T}}\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}\left[\left(Z^e_{c_\beta(1-A(e))}\cdot\lambda\right)_{e\in\mathbb{T}}\right].$$

By taking a linear tree $\mathbb{T}$ we obtain the following corollary.

Corollary 1.3

$$\mathcal{L}^{M_t}\left[(\widetilde{\psi}_t^\alpha(B))_{\alpha\in I}\right] \underset{\text{fdd}}{\overset{t\to\infty}{\Longrightarrow}} \mathcal{L}^1\left[\lambda(B)\cdot(Z_{c_\beta(1-\alpha)})_{\alpha\in I}\right], \qquad B \in \mathcal{B}(\mathbb{R}^d). \qquad \square$$

1.5 Outline

The rest of the paper is organized as follows. In Section 2 we give upper and lower bounds of $u(t, x; f)$ in terms of the function $\phi(t, x)$ in Proposition 2.6. This is the key for the proof of Theorem 1 and 2 in Section 2 and 3. In Section 4 we give better bounds for the special situation $\beta = 2/d$ that serve to prove Proposition 4.1. A coupling technique will be employed to infer Theorem 4.

2 Proof for the low dimensions

In this section we give some lemmas dealing with sub- and supersolutions to the equation $L_\beta u = 0$. With the aid of these lemmas we prove Theorem 1. Some of the lemmas will be used in Section 3 to prove the high-dimensional results.

The main tool for the investigation is a maximum principle for the non-linear parabolic differential operator L_β. We state the following lemma without proof and refer the reader to Protter and Weinberger (1967), Chapter 3.7. (In fact, Protter and Weinberger only deal with the case of a bounded domain. Our lemma follows by approximation arguments.)

Lemma 2.1 (Maximum principle) *Let $L = L_{t,H}$ be the semi-parabolic operator on $\mathbb{R}^d$ defined by*

$$Lu(t, x) = \partial_t u(t, x) - \frac{1}{2}\Delta u(t, x) + H(t, u(t, x)),$$

where $H : [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing in u. Let $f, g \in C_c^+(\mathbb{R}^d)$, $f \leq g$, and $T > 0$. Let $\underline{u}(t, x; f)$ and $\overline{u}(t, x; g)$ be sub- and supersolutions to $Lu = 0$ with initial conditions f, respectively g, that is,

$$\underline{u}(0, x; f) = f(x), \qquad \overline{u}(0, x; g) = g(x), \qquad x \in \mathbb{R}^d,$$

and

$$L\underline{u}(t,x;f) \le 0, \qquad L\overline{u}(t,x;g) \ge 0, \qquad x \in \mathbb{R}^d,\ t \in [0,T].$$

We also assume that $\underline{u}$ *and* $\overline{u}$ *are bounded and vanishing at infinity (as functions of* x *for any* t*). Then*

$$\underline{u}(t,x;f) \le \overline{u}(t,x;g), \qquad x \in \mathbb{R}^d,\ t \in [0,T].$$

□

To warm up we give two simple applications of the maximum principle. Recall that $u(t,x;f)$ denotes the solution of the Cauchy problem (1.8).

Lemma 2.2 *Let* $f,g \in C_c^+(\mathbb{R}^d)$ *and let* $c > 1$*. Then*

$$u(t,x;f+g) \le u(t,x;f) + u(t,x;g), \qquad x \in \mathbb{R}^d,\ t \ge 0 \tag{2.1}$$

and

$$u(t,x;cf) \le c\,u(t,x;f), \qquad x \in \mathbb{R}^d,\ t \ge 0. \tag{2.2}$$

Furthermore, if $f \le g$ *then*

$$u(t,x;f) \le u(t,x;g), \qquad x \in \mathbb{R}^d,\ t \ge 0. \tag{2.3}$$

Proof Check that $L_\beta(u(t,x;f) + u(t,x;g)) \ge 0$ and $L_\beta(c\,u(t,x;f)) \ge 0$. The last inequality is trivial. □

Lemma 2.3 *Let* $f \in C_c^+(\mathbb{R}^d)$ *and let* $v(t,x;f) = (p_t * f)(x)$ *the solution of the heat equation* $(\partial_t - \frac{1}{2}\Delta)v = 0$, $v(0,\cdot\,;f) = f$*. Then the following inequality holds,*

$$(1 + b\beta\,\|f\|_\infty^\beta\, t)^{-1/\beta} v(t,x;f) \le u(t,x;f) \le v(t,x;f), \quad x \in \mathbb{R}^d,\ t \ge 0. \tag{2.4}$$

Proof Let $U(t,x;K) = U(t;K)$, $K > 0$, the solution of $L_\beta U = 0$, $U(0,\cdot\,;K) \equiv K$. The explicit solution of this equation is

$$U(t;K) = K(1 + bK^\beta \beta t)^{-1/\beta}.$$

By the maximum principle, for $K \ge \|f\|_\infty$,

$$\begin{aligned} L_\beta u(t,x;f) &\le \partial_t u - \frac{1}{2}\Delta u + bU^\beta u \\ &= \partial_t u - \frac{1}{2}\Delta u + (bK^\beta)(1 + bK^\beta \beta t)^{-1} u. \end{aligned} \tag{2.5}$$

Hence $\widetilde{L}_\beta u \ge 0$, where $\widetilde{L}_\beta = \widetilde{L}_{\beta,t}$ is defined by

$$\widetilde{L}_{\beta,t}\tilde{u} = \partial_t \tilde{u} - \frac{1}{2}\Delta\tilde{u} + (bK^\beta)(1 + bK^\beta\beta t)^{-1}\tilde{u}.$$

Let $\tilde{u}$ be the solution the $\widetilde{L}_{\beta,t}\tilde{u} = 0$. The maximum principle implies $u(t,x;f) \ge \tilde{u}(t,x;f)$, $x \in \mathbb{R}^d$, $t \ge 0$. Note that $\tilde{u}$ can be represented as

$$\begin{aligned} \tilde{u}(t,x;f) &= v(t,x;f) \cdot \exp\left(-\int_0^t bK^\beta(1 + bK^\beta\beta s)^{-1}\,ds\right) \\ &= v(t,x;f)(1 + bK^\beta\beta t)^{-1/\beta}. \end{aligned} \tag{2.6}$$

Since the maximum principle implies $u(t,x;f) \le v(t,x;f)$, $x \in \mathbb{R}^d$, $t \ge 0$, the assertion is proved. □

Let $A > 0$, recall $G_\beta(t)$ from (1.14), and define $\overline{u}(t,x)$ by

$$\overline{u}(t,x) = \begin{cases} At^{-d/2}G_\beta(t)^{-1}\exp\left(-\frac{1}{2}\frac{\|x\|^2}{2t}\right), & \beta < 2/d \\ At^{-d/2}G_\beta(t)^{-1}\exp\left(-(1-G_\beta(t)^{-\beta})\frac{\|x\|^2}{2t}\right), & \beta = 2/d \\ At^{-d/2}\exp\left(-\frac{1}{2}\frac{\|x\|^2}{2t}\right), & \beta > 2/d. \end{cases} \tag{2.7}$$

The definition of $\overline{u}$ for the case $\beta = 2/d$ might look a little strange at first glance. Clearly for t large enough, $\overline{u}(t,x) \le At^{-d/2}G_\beta(t)^{-1}\exp\left(-\frac{1}{2}\frac{\|x\|^2}{2t}\right)$. However we will be able to show that $\overline{u}$ is a supersolution only for the definition given in (2.7).

Lemma 2.4 (Super-solution) *Assume that $\beta \le 2/d$. There exists $t_0 = t_0(\beta,d) \ge 0$ such that for A large enough and $t \ge t_0$,*

$$L_\beta\overline{u}(t,x) \ge 0, \qquad x \in \mathbb{R}^d. \tag{2.8}$$

Proof We do the proof separately for critical and low dimension.

Critical dimension ($\beta = 2/d$). Note that

$$G_\beta(t) = (2\pi)^{-d/2}(\log t)^{1/\beta} \tag{2.9}$$

and that $G'_\beta(t) = \frac{d}{dt}G_\beta(t) = (\beta t\log t)^{-1}G_\beta(t) = \frac{1}{2\pi}\frac{d}{2t}G_\beta(t)^{1-\beta}$. Hence a short calculation shows that for $t \ge \exp(2+4\pi)$,

$$\begin{aligned} L_\beta\overline{u}(t,x) &= \frac{\overline{u}(t,x)}{tG_\beta(t)^\beta}\left[-\frac{1}{\beta}\Big(1+\frac{1}{2\pi}\Big) + \Big(1-\Big(1+\frac{1}{2\pi}\Big)G_\beta(t)^{-\beta}\Big)\frac{\|x\|^2}{2t}\right. \\ &\qquad \left. +bA^\beta\exp\left(-\beta\left(1-G_\beta(t)^{-\beta}\right)\frac{\|x\|^2}{2t}\right)\right] \\ &\ge \frac{\overline{u}(t,x)}{tG_\beta(t)^\beta}\left[-\frac{1}{\beta}\Big(1+\frac{1}{2\pi}\Big) + \frac{1}{2}\frac{\|x\|^2}{2t} + bA^\beta\exp\left(-\beta\frac{\|x\|^2}{2t}\right)\right]. \end{aligned} \tag{2.10}$$

Check separately that this is nonnegative for $\|x\|^2/4t \ge (1+1/2\pi)/\beta$ and for $\|x\|^2/4t < (1+1/2\pi)/\beta$ and $A \ge \exp(4/\beta)/b^{1/\beta}$.

Low dimension ($\beta < 2/d$). Note that

$$G_\beta(t) = (2\pi)^{-d/2}\left(1-\frac{\beta d}{2}\right)^{-1/\beta}\cdot\left(t^{1-\beta d/2}-1\right)^{1/\beta} \tag{2.11}$$

and that

$$\begin{aligned} G'_\beta(t) = \frac{d}{dt}G_\beta(t) &= (2\pi)^{-d/2}\left(1-\frac{\beta d}{2}\right)^{-1/\beta}\cdot\frac{1}{\beta}\left(1-\frac{\beta d}{2}\right)\left(t^{1-\beta d/2}-1\right)^{-1+1/\beta}t^{-\beta d/2} \\ &= \left(\frac{1}{\beta}-\frac{d}{2}\right)(1-t^{1-\beta d/2})^{-1}\cdot G_\beta(t)/t \\ &\le \left(\frac{2}{\beta}-d\right)G_\beta(t)/t \qquad\qquad \text{for } t \ge 2^{1/(1-\beta d/2)}. \end{aligned}$$

Hence for $t \ge 2^{1/(1-\beta d/2)}$,

$$\begin{aligned} L_\beta\overline{u}(t,x) &= u(t,x)\left[-\frac{d}{4t}-\frac{G'_\beta(t)}{G_\beta(t)}+\frac{\|x\|^2}{8t^2}+bA^\beta t^{-\beta d/2}G_\beta(t)^{-\beta}\exp\left(-\beta\frac{1}{2}\frac{\|x\|^2}{2t}\right)\right] \\ &\ge u(t,x)\frac{1}{t}\left[-\frac{2}{\beta}+\frac{\|x\|^2}{8t}+bA^\beta(2\pi)^{\beta d/2}(1-\beta d/2)\exp\left(-\beta\frac{1}{2}\frac{\|x\|^2}{2t}\right)\right]. \end{aligned} \tag{2.12}$$

Check separately that this expression is nonnegative for $\|x\|^2/8t \geq 2/\beta$ and for $\|x\|^2/8t < 2/\beta$ and

$$A \geq \frac{2^{1/\beta}\exp(4/\beta)}{(2\pi)^{d/2}(1-\beta d/2)^{1/\beta}b^{1/\beta}\beta^{1/\beta}}.$$

□

We continue by giving sub-solutions $\underline{u}$. Recall that $p_t(x)$ is the heat kernel. We define for $a \in (0, (\beta b)^{-1/\beta})$,

$$\underline{u}(t,x) = a\, p_t(x)\, G_\beta(t)^{-1}. \tag{2.13}$$

Lemma 2.5 (Sub-solution) *In any dimension for $t > 1$,*

$$L_\beta \underline{u}(t,x) \leq 0, \qquad x \in \mathbb{R}^d. \tag{2.14}$$

Proof We calculate

$$L_\beta \underline{u}(t,x) = \underline{u}(t,x) G_\beta(t)^{-\beta}\left(ba^\beta p_t(x)^\beta - \frac{1}{\beta}p_t(0)^\beta\right) \leq 0, \tag{2.15}$$

where the inequality holds since $a \leq (\beta b)^{-1/\beta}$. □

Recall ϕ from (1.15).

Proposition 2.6 *Let $d \in \mathbb{N}$ and $0 < \beta \leq 1$. Let $f \in C_c^{++}(\mathbb{R}^d)$, and let $u(t,x;f)$ the solution of $L_\beta u = 0$ with $u(0,\cdot\,;f) = f$. Then there exist constants a, A, $t_0 > 0$ such that*

$$a\phi(t/2,x) \leq u(t,x;f) \leq A\phi(4t,x), \qquad x \in \mathbb{R}^d,\ t \geq t_0. \tag{2.16}$$

Proof *Upper bound, low and critical dimension.* Assume $\beta \leq 2/d$. Let $\overline{u}$ and t_0 as in Lemma 2.4 with A' ($= A$ in the notation of (2.7)) large enough such that $f(x) \leq \overline{u}(t_0,x)$, $x \in \mathbb{R}^d$. By the maximum principle,

$$u(t,x;f) \leq \overline{u}(t+t_0,x), \qquad x \in \mathbb{R}^d,\ t \geq 0.$$

Note that there exist constants $A, C > 0$ such that for $t \geq t_0$ (note that $G_\beta(t)^{-\beta} \leq \frac{1}{2}$ if $\beta = 2/d$),

$$A\phi(4t,x) \geq C\phi(2(t+t_0),x) \geq \overline{u}(t+t_0,x).$$

Upper bound, high dimension. Assume $\beta > 2/d$. Note that by (1.18) there exist constants $c, c' > 0$ such that

$$A\phi(4t,x) \geq Ac'p_{4t}(x) \geq Acp_{t+1}(x), \qquad \text{for } t \geq 2.$$

Choose A large enough such that $Acp_1(x) \geq f(x)$. Note that

$$L_\beta(Acp_t(x)) = b(Acp_t(x))^{1+\beta} \geq 0.$$

Hence by the maximum principle $Acp_{t+1}(x) \geq u(t,x;f)$, $t \geq 0$.

Lower bound. Note that there exists $a' > 0$ such that for $\underline{u}$ of (2.13) with a' instead of a

$$u(4,x;f) \geq \underline{u}(2,x), \qquad x \in \mathbb{R}^d.$$

Arguing as above, another application of the maximum principle yields that for some $c > 0$, $a > 0$, $\underline{u}$ of (2.13) now again with a, and for $t \geq 4$,

$$u(t,x;f) \geq \underline{u}(t-2,x) = c\phi(t-2,x) \geq a\phi(t/2,x). \tag{2.17}$$

□

With Proposition 2.6 in hand we are able to prove the extinction and explosion in Theorem 1. It is, however, more subtle to show instability. Here we need better lower bounds for u that reflect the clustering in low dimensions. Roughly speaking the picture is as follows. Let $\varepsilon > 0$ and $B \in \mathcal{B}(\mathbb{R}^d)$ bounded. For t large with high probability there is no particle in B: $\eta_t(B) = 0$. With a small probability, $\eta_t(B) \gg 1/\varepsilon$ and there is no intermediate regime. Hence for large t with overwhelming probability, $e^{-\varepsilon\eta_t(B)}$ is either 0 or close to 1 implying that $e^{-\varepsilon\eta_t(B)} \approx e^{-\eta_t(B)}$. This suggests that for given $\varepsilon > 0$ and indicator function $f = 1\!\!1_B$ for $t > 0$ large enough we should have $u(t, x;\ \varepsilon f) \approx u(t, x;\ f)$. (Recall that $u(t, x;\ f)$ is the solution of $L_\beta u = 0$ with $u(0, x; f) = f(x)$.) This should carry over to $f \in C_c^+(\mathbb{R}^d)$ by approximation.

For our purpose of showing instability in Theorem 1 it will be sufficient to give a lower bound for $u(t, x;\ \varepsilon f)$.

Lemma 2.7 *Let $\beta \le 2/d$ and $f \in C_c^{++}(\mathbb{R}^d)$. There exists a constant $c_f > 0$ such that for $\varepsilon > 0$ and t large enough,*

$$u(t, x; \varepsilon f) \ge c_f \phi(t/2, x). \tag{2.18}$$

Proof Note that there exists a constant $c_f^1 \in (0, 1)$ such that

$$c_f^1\, p_1 \le (p_2 \ * \ f).$$

Hence by Lemma 2.3 and Lemma 2.2 (and the semigroup property of u (1.10)),

$$\begin{aligned} u(t, x; \varepsilon f) &= u\Big(t-2, x; u(2, \cdot\,; \varepsilon f)\Big) \\ &\ge u\Big(t-2, x; (1 + 2b\beta\varepsilon\, \|f\|_\infty^\beta)^{-1/\beta}\, \varepsilon\, (p_2 * f)\Big) \\ &\ge c_f^2\, u(t-2, x; \varepsilon\, p_1), \end{aligned} \tag{2.19}$$

where $c_f^2 := c_f^1\, (1 + 2b\beta\, \varepsilon\, \|f\|_\infty^\beta)^{-1/\beta} > 0$. Now we use two different arguments for low and critical dimension.

Critical dimension. Assume $\beta = 2/d$. Here we make use of the selfsimilarity of SBM$(d, 1 + 2/d, 1)$. Let $a \in (0, (b\beta)^{-1/\beta})$ and let $\underline{u}$ be the the corresponding sub-solution according to Lemma 2.5, that is,

$$\underline{u}(x, t) = (2\pi)^{-d/2} a \cdot t^{-d/2} (\log t)^{-1/\beta} \exp(-\|x\|^2/2t).$$

Abbreviating $\rho = \exp((a/\varepsilon)^\beta / 2\pi)$ we get by the scaling property (1.11) that

$$\begin{aligned} u(t, x; \varepsilon p_1(\cdot)) &= u(t, x; (2\pi)^{1/\beta} a (\log\rho)^{-1/\beta} \rho^{1/\beta} p_\rho(\rho^{1/2}\cdot)) \\ &= u(t, x; \rho^{1/\beta} \underline{u}(\rho, \rho^{1/2}\cdot)) \\ &= \rho^{1/\beta} u(\rho t, \rho^{1/2} x; \underline{u}(\rho, \cdot)). \end{aligned} \tag{2.20}$$

By time homogeneity, also $\underline{u}(s + \cdot\,, \cdot)$ is a subsolution for all $s \ge 0$. Hence $\underline{u}$ has the property

$$\underline{u}(t+s, x) \le u(t, x; \underline{u}(s, \cdot)), \quad s, t \ge 0,\ x \in \mathbb{R}^d. \tag{2.21}$$

We can thus continue (2.20) by

$$\begin{aligned} u(t, x; \varepsilon p_1(\cdot)) &\ge \rho^{1/\beta} \underline{u}(\rho(t+1), \rho^{1/2} x) \\ &= (\log(t+1) + \log\rho)^{-1/\beta}\, a\, p_{t+1}(x) \\ &\ge (\log(t+1) + \log\rho)^{-1/\beta}\, a 2^{-d/2}\, p_t(x) \quad \text{for } t \ge 1 \\ &\ge 2^{-d} \underline{u}(t, x), \quad \text{for } t > \rho + 1. \end{aligned} \tag{2.22}$$

(Note that for $t > \rho+1$, $\log(t+1)+\log(\rho) \le \log((t+1)(t-1)) \le 2\log t$.) Combining (2.19) and (2.22) we get that there exist c_f^3, c_f^4, and $c_f > 0$ such that for $\varepsilon > 0$ and t large enough

$$\begin{aligned} u(t,x;\varepsilon f) &\ge c_f^3 \underline{u}(t-2,x) \ge c_f^4 \underline{u}(t/2,x) \\ &= c_f \phi(t/2,x). \end{aligned} \tag{2.23}$$

Low dimension. Assume $\beta < 2/d$. We make implicit use of the fact that SBM$(d, 1+2/d, 1)$ has a density in the low dimensions. Recall from (1.13) that u is the Laplace transform of SBM. The existence of a density of SBM is known to be equivalent to the existence of a smooth solution u to $L_\beta u = 0$ with a delta distribution δ_0 as initial condition (see Fleischmann (1988)). More precisely, $u(t,x;\delta_0)$, $t>0$, $x \in \mathbb{R}^d$, is the solution of $L_\beta u = 0$ such that for $h \in C_c^+(\mathbb{R}^d)$,

$$\lim_{t\to 0} \int u(t,x;\delta_0) h(x)\, dx = h(0).$$

Brezis, Peletier and Terman (1986) show in their appendix that there exists $t_0 > 0$ such that

$$u(t,x;\delta_0) \ge \frac{1}{2} p_t(x), \qquad x \in \mathbb{R}^d,\ 0 < t \le t_0.$$

Note that the maximum principle implies

$$u(t,x;\delta_0) \le p_t(x), \qquad x \in \mathbb{R}^d,\ t > 0.$$

Hence by Lemma 2.3 for $t+s > t_0$ (note that $\|\frac{1}{2}p_{t_0}\|_\infty = \frac{1}{2}(2\pi t_0)^{-d/2}$),

$$\begin{aligned} u(t,x;p_s(\cdot)) &\ge u(t+s,x;\delta_0) \\ &= u(t+s-t_0,x;u(t_0,\cdot\,;\delta_0)) \\ &\ge u\Big(t+s-t_0,x;\frac{1}{2}p_{t_0}(\cdot)\Big) \\ &\ge \frac{1}{2}\Big(1+2^{-\beta} b\beta(t+s)(2\pi t_0)^{-\beta d/2}\Big)^{-1/\beta} p_{t+s}(x). \end{aligned} \tag{2.24}$$

Let $c_f^3 = \frac{1}{2}(1+2^{-\beta}b\beta(2\pi t_0)^{-\beta d/2})^{-1/\beta}$. By the scaling property (1.11) and by (2.19) there exists $c_f > 0$ such that for $t \ge \varepsilon^{-1/(1/\beta - d/2)}$,

$$\begin{aligned} u(t,x;\varepsilon f) &\ge c_f^2 u(t-2,x;\varepsilon p_1(\cdot)) \\ &= c_f^2 t^{-1/\beta} u\left(1-\frac{2}{t}, t^{-1/2}x\ ;\ t^{1/\beta}\varepsilon t^{-d/2} p_{1/t}(\cdot)\right) \\ &\ge c_f^2 t^{-1/\beta} u\left(1-\frac{2}{t}, t^{-1/2}x;\ p_{1/t}(\cdot)\right) \\ &\ge c_f^2 c_f^3\, t^{-1/\beta} p_{1-1/t}(t^{-1/2}x) \\ &\ge c_f \phi(t/2,x). \end{aligned} \tag{2.25}$$

□

Proof of Theorem 1

We do the proof only for $(\psi_t) = (\zeta_t)$ SBM$(d, 1+\beta, b)$. An easy application of Jensen's inequality in (1.12) and (1.13) then yields the claim for BBM$(d, 1+\beta, b)$.

To prove part (i) note that $\mathcal{L}[\zeta_t] \stackrel{t\to\infty}{\Longrightarrow} \delta_0$ is equivalent to

$$\liminf_{t\to\infty} \mathbf{E}[\exp(-\langle \zeta_t, f\rangle)] = 1, \qquad f \in C_c^{++}(\mathbb{R}^d). \tag{2.26}$$

However, for such f, by Proposition 2.6, there exist $a, A > 0$ such that

$$\begin{aligned}\liminf_{t\to\infty} \mathbf{E}[\exp(-A\langle \zeta_0, \phi(t,\cdot)\rangle)] &\leq \liminf_{t\to\infty} \mathbf{E}[\exp(-\langle \zeta_0, u(t,\ \cdot\,; f)\rangle)] \\ &= \liminf_{t\to\infty} \mathbf{E}[\exp(-\langle \zeta_t, f\rangle)] \\ &\leq \liminf_{t\to\infty} \mathbf{E}[\exp(-a\langle \zeta_0, \phi(t,\cdot)\rangle)].\end{aligned} \tag{2.27}$$

These expressions are equal to 1 if (and only if)

$$\langle \zeta_0, \phi(t,\cdot)\rangle \stackrel{t\to\infty}{\longrightarrow} 0 \qquad \text{stochastically.} \tag{2.28}$$

This proves part (i).

Now assume that (2.28) does not hold. We have to show that $\langle \zeta_t, f\rangle$ is stochastically unbounded. This is the case if and only if

$$\limsup_{\varepsilon\to 0}\ \liminf_{t\to\infty} \mathbf{E}[\exp(-\langle \zeta_t, \varepsilon f\rangle)] < 1. \tag{2.29}$$

By Lemma 2.7 there exists a constant $c_f > 0$ such that the l.h.s. of (2.29) is dominated by

$$\liminf_{t\to\infty} \mathbf{E}[\exp(-c_f\langle \zeta_0, \phi(t,\cdot)\rangle)], \tag{2.30}$$

which is strictly smaller than 1 by assumption. Hence ζ_t is unstable.

Assume that the assumption of part (iii) holds, i.e., $\mathcal{L}[\langle \zeta_0, \phi(r,\cdot)\rangle] \Longrightarrow \delta_\infty$. The assertion that ζ_t explodes is equivalent to

$$\limsup_{t\to\infty} \mathbf{E}[\exp(-\langle \zeta_t, f\rangle)] = 0, \tag{2.31}$$

for all $f \in C_c^{++}(\mathbb{R}^d)$. This however can be shown with the aid of Proposition 2.6 as in the proof of part (i). □

3 Proof for the high dimensions

Bramson, Cox and Greven (1997) give a proof for Theorem 2 in the case $\beta = 1$. Most of their proof works without *any* changes for all $\beta \in (0, 1]$. We do not repeat their entire proof here but only give an outline of their strategy and proofs of the lemmas that needed (minor) modifications.

Bramson, Cox and Greven (1997) use Proposition 2.6 to show that if ψ_t is stable, then $\langle \psi_0, p_t(\cdot)\rangle$ is stochastically bounded (as $t \to \infty$). They infer with the aid of Lemma 3.1 below that if $\mathcal{L}[\psi_0]$ is invariant, then for $f \in C_c^{+}(\mathbb{R}^d)$ and $z \in \mathbb{R}^d$,

$$\mathbf{E}[\exp(-\langle \psi_t, f(z+\cdot)\rangle)] - \mathbf{E}[\exp(-\langle \psi_t, f\rangle)] \stackrel{t\to\infty}{\longrightarrow} 0. \tag{3.1}$$

Since the left-hand side of (3.1) does not depend on t (by the assumption that $\mathcal{L}[\psi_0]$ is an invariant law), we know that $\mathcal{L}[\psi_t]$ is translation invariant, hence Proposition 1.2 holds.

A standard argument now yields the claim of Theorem 2.

We only give the proof of Lemma 3.1 since it is here where small changes have to be made to cover the case $\beta < 1$.

For $\rho > 0$ let $B(\rho) = \{x \in \mathbb{R}^d,\ \|x\| < \rho\}$ denote the ball with radius ρ centered at the origin.

Lemma 3.1 *Let $\varepsilon > 0$ and $0 < M < 1/4\varepsilon$. Let $f \in C_c^{++}(\mathbb{R}^d)$. For t large enough and all $x_1, x_2 \in B(\varepsilon t)$ with $\|x_1 - x_2\| \le M$,*

$$u(t, x_1; f) \le \exp\left(8\sqrt{M\varepsilon}/\beta\right) u(t, x_2; f). \tag{3.2}$$

Proof Step 1. We show that for $x \in \mathbb{R}^d$ and $t \ge 0$,

$$e^{-2\delta/\beta} v\Big(\delta t, x; u((1-\delta)t, \cdot\,; f)\Big) \le u(t, x; f) \le v\Big(\delta t, x; u((1-\delta)t, \cdot\,; f)\Big), \tag{3.3}$$

where v is defined as in Lemma 2.3. The right-hand inequality follows immediately from the maximum principle. For the other inequality we proceed as follows.

Fix $t > 0$. Define the linear operator K_β for smooth functions $w(s, x)$ by

$$K_\beta w = \partial_s w - \frac{1}{2}\Delta w + \frac{2}{\beta t} w.$$

By (2.5) for $s > t/2$,

$$K_\beta u(s, x; f) \ge L_\beta u(s, x; f).$$

Let $w = w(s, x; f)$ be the solution of the Cauchy problem $K_\beta w = 0$, $w(0, \cdot\,; f) = f$. By the maximum principle (applied to K_β), $w(\delta t, x; u((1-\delta)t, \cdot\,; f)) \le u(t, x; f)$. Note that (as in (2.6)) w can be represented as

$$\begin{aligned} w(\delta t, x; u((1-\delta)t, \cdot\,; f)) &= v(\delta t, x; u((1-\delta)t, \cdot\,; f)) \exp\left(-\int_0^{\delta t} \frac{2}{\beta}\, ds\right) \\ &= v(\delta t, x; u((1-\delta)t, \cdot\,; f)) e^{-2\delta/\beta}. \end{aligned} \tag{3.4}$$

This however yields (3.3).

Step 2. Finally we show the assertion of the lemma. Let $x \in B(\varepsilon t)$. We write

$$\begin{aligned} &v(\delta t, x; u((1-\delta)t, \cdot\,; f)) = I_1(x) + I_2(x) \\ &:= \int_{B(4\varepsilon t)} p_{\delta t}(z, x) u((1-\delta)t, z; f)\, dz + \int_{B(4\varepsilon t)^c} p_{\delta t}(z, x) u((1-\delta)t, z; f)\, dz. \end{aligned} \tag{3.5}$$

An easy estimate using $\|x - z\| \ge 3\varepsilon t$ for $x \in B(\varepsilon t)$ and $z \in B(4\varepsilon t)^c$ and Lemma 2.3 yield for t large enough,

$$\begin{aligned} I_2(x) &\le \int_{B(4\varepsilon t)^c} dz\, p_{\delta t}(x, z) \int_{\mathbb{R}^d} dy\, p_{(1-\delta)t}(z, y) f(y) \\ &\le (2\pi\delta t)^{-d/2} \exp(-9\varepsilon^2 t/2) \|f\|_1 \le \exp(-4\varepsilon^2 t)\|f\|_1. \end{aligned} \tag{3.6}$$

We obtain $u(t, x; f) \ge \exp(-3\varepsilon^2 t)\|f\|_1$ for t large by a similar estimate using the other inequality in Lemma 2.3. Hence

$$I_2(x) \le \exp(-\varepsilon^2 t) u(t, x; f) \tag{3.7}$$

Let $x_1, x_2 \in B(\varepsilon t)$ with $\|x_1 - x_2\| \le M$. For $z \in B(4\varepsilon t)$ clearly

$$\begin{aligned} \|z - x_2\|^2 - \|z - x_1\|^2 &= \|x_2 - x_1\|^2 + 2\langle z, x_2 - x_1\rangle \\ &\le \|x_2 - x_1\| \cdot \|x_2 - x_1\| + 2\|z\| \cdot \|x_2 - x_1\| \\ &\le M \cdot 2\varepsilon t + 2 \cdot 4\varepsilon t \cdot M = 10\varepsilon M. \end{aligned}$$

Hence $p_{\delta t}(z,x_1) \le \exp(5\varepsilon M/\delta) p_{\delta t}(z,x_2)$. Together with (3.3) we get (recall $\delta = \sqrt{\varepsilon M}$)

$$\begin{aligned} I_1(x_1) &\le \exp(5\varepsilon M/\delta) \int_{B(4\varepsilon t)} p_{\delta t}(z,x_2) u((1-\delta)t, z; f))\, dz && (3.8) \\ &\le \exp(5\varepsilon M/\delta) v(\delta t, x_2; u((1-\delta)t, \cdot\,; f)) && (3.9) \\ &\le \exp(2\delta/\beta + 5\varepsilon M/\delta) u(t, x_2; f) \\ &= \exp(7\delta/\beta) u(t, x_2; f). \end{aligned}$$

Together with (3.7) this implies for t large

$$u(t,x_1;f) \le \exp(8\delta/\beta) u(t,x_2;f).$$

□

4 Proof for the diffusive clustering

In order to show the weak convergence statement (1.26) of Theorem 3 we will show convergence of the Laplace transforms in the case of SBM. This is the content of Proposition 4.1 below. The case of BBM will follow by a comparison argument using the embedded particle system and the law of large numbers (here $\alpha \ge 0$ is needed).

The log-Laplace transform of the l.h.s. of (1.26) for a test function $f \in C_c^+(\mathbb{R}^d)$ is (recall (1.13) and (1.20) - (1.22))

$$\begin{aligned} -\log \mathbf{E}^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \widetilde{\zeta}_t^\alpha, f\rangle)] &= -\log \mathbf{E}^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \widetilde{\zeta}_t, t^{-\alpha d/2} f(t^{-\alpha/2}\cdot\,)\rangle)] && (4.1) \\ &= -\log \mathbf{E}^{(\log t)^{1/\beta}\lambda}[\exp(-\langle \zeta_t, (\log t)^{-1/\beta} t^{-\alpha d/2} f(t^{-\alpha/2}\cdot\,)\rangle)] \\ &= \langle (\log t)^{1/\beta}\cdot\lambda, u(t,\cdot\,; (\log t)^{-1/\beta} t^{-\alpha d/2} f(t^{-\alpha/2}\cdot\,))\rangle. \end{aligned}$$

Using the scaling relation (1.11) (with $\rho = t^\alpha$) this quantity equals (recall $\beta = 2/d$)

$$\begin{aligned} &= \langle (\log t)^{1/\beta}\cdot\lambda,\ t^{-\alpha d/2} u(t^{1-\alpha}, t^{-\alpha/2}\cdot\,; (\log t)^{-1/\beta} f)\rangle && (4.2) \\ &= \langle (\log t)^{1/\beta}\cdot\lambda,\ u(t^{1-\alpha},\ \cdot\,; (\log t)^{-1/\beta} f)\rangle \\ &= (1-\alpha)^{-1/\beta} \left\langle (\log t')^{1/\beta}\cdot\lambda, u(t',\ \cdot\,; (\log t')^{-1/\beta}(1-\alpha)^{1/\beta} f)\right\rangle, \end{aligned}$$

where we put $t' = t^{1-\alpha}$. We have to show that this expression converges as $t' \to \infty$ to (recall (1.23), (1.25) and that $c_\beta = (2\pi(1+\beta)^{1/\beta})^{-1}$)

$$\begin{aligned} -\log \mathbf{E}^1[\exp(-Z_{c_\beta(1-\alpha)}\langle\lambda, f\rangle)] &= (\beta c_\beta(1-\alpha) + \langle\lambda, f\rangle^{-\beta})^{-1/\beta} && (4.3) \\ &= (1-\alpha)^{-1/\beta}(c_\beta + \langle\lambda, (1-\alpha)^{1/\beta} f\rangle^{-\beta})^{-1/\beta}. \end{aligned}$$

Comparing this with (4.2), it is clearly enough to handle the case $\alpha = 0$ This is the content of the following proposition.

Proposition 4.1 *Assume $\beta = 2/d$ and let u be the solution of* (1.8)*. For $f \in C_c^{++}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $-\infty < \alpha \le 1$, and $s \ge 0$ the following hold.*

$$\begin{aligned} \text{(i)} \quad \lim_{t\to\infty} (t\log t)^{1/\beta} u(t-s, t^{1/2}x; (\log t)^{-1/\beta} f) &= (2\pi(\beta c_\beta + \langle\lambda, f\rangle^{-\beta}))^{-1/\beta} e^{-\|x\|^2/2}, \\ \text{(ii)} \quad \lim_{t\to\infty} \langle (\log t)^{1/\beta}\cdot\lambda, u(t,\cdot\,; (\log t)^{-1/\beta} f)\rangle &= (\beta c_\beta + \langle\lambda, f\rangle^{-\beta})^{-1/\beta} \end{aligned}$$

4.1 Proof of Proposition 4.1

The proof of Proposition 4.1 copies the proofs of Lee (1991), Theorems 2.1, 2.3, and 2.4. Lee's results cover only the case $\beta = 1$ but can easily be adapted to $\beta < 1$. For the sake of completeness we give the proof in detail.

Strategy of the Proof

Recall that $b = 1$ and that the (non-linear) operator L_β is defined by

$$L_\beta u = \partial_t u - \frac{1}{2}\Delta u + u^{1+\beta}.$$

As indicated by Proposition 2.6, a solution of $L_\beta u = 0$ should be "close" to multiple of

$$\phi(t,x) = t^{-d/2}(\log t)^{-1/\beta}\exp(-\|x\|^2/2t).$$

However, in Proposition 2.6 we had no control of the constants a and A of the upper and lower bounds of $u(t,x;f)$ in terms of $\phi(t,x)$. (The point that ϕ in the upper and lower bounds is evaluated not at time t but at $t/2$ and $4t$ respectively could be repaired rather easily.)

More precisely, we would like to find a constant θ_β such that for $\varepsilon > 0$ and t large enough

$$\theta_\beta(1-\varepsilon)(t\log t)^{-1/\beta}\exp\left(-\frac{\|x\|^2}{2t}\right) \le u(t,x;f)$$
$$\le \theta_\beta(1+\varepsilon)(t\log t)^{-1/\beta}\exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right), \qquad x \in \mathbb{R}^d.$$

We make the following ansatz to determine θ_β. For $\theta > 0$ define $\widetilde{u}$ by

$$\widetilde{u} = \theta\phi + t^{-1/\beta}(\log t)^{-(1+1/\beta)}g(t^{-1/2}x), \tag{4.4}$$

with a "smooth" function g.

Let $H = -(\frac{1}{\beta}I + \frac{1}{2}\Delta + \frac{x}{2}\nabla)$ and

$$f_\theta(t,x) = \theta^{1+\beta}\exp(-(1+\beta)\|x\|^2/2t) - \frac{\theta}{\beta}\exp(-\|x\|^2/2t).$$

Then

$$L_\beta\widetilde{u} = (t\log t)^{-(1+1/\beta)}\left[f_\theta(t,x) - (Hg)(t^{-1/2}x)\right] + o((t\log t)^{-(1+1/\beta)}).$$

Hence we search for θ such that $Hg(x) = f_\theta(1,x)$ has a solution g. In Lemma 4.2 we construct a right inverse G of H on a certain subspace $V \subset C_b(\mathbb{R}^d)$. It turns out that $f_\theta(1,\cdot) \in V$ if and only if

$$\theta = \theta_\beta = (\frac{1}{\beta}(1+\beta)^{1/\beta})^{1/\beta}. \tag{4.5}$$

The problem with this ansatz is that we can not control the tail behaviour of $g(x)$ at infinity. So now that we have determined θ_β we come back to the main idea of constructing sub- and supersolutions to $L_\beta u = 0$. We try the following ansatz:

$$\begin{aligned} \underline{u}(t,x) &= (\theta_\beta - \varepsilon)\ \phi(t,x) + t^{-1/\beta}(\log t)^{-1-1/\beta}g^-(t^{-1/2}x), \\ \overline{u}(t,x) &= (\theta_\beta + \varepsilon)\ \phi(t,x) + t^{-1/\beta}(\log t)^{-1-1/\beta}g^+(t^{-1/2}x), \end{aligned} \tag{4.6}$$

where the functions g^- and g^+ will be determined in terms of G. In Lemma 4.3 we control the tails of g^- and g^+ and in Lemma 4.4 we show that these $\underline{u}$ and $\overline{u}$ are indeed sub- and supersolutions to our problem.

The second step in the proof of Proposition 4.1 is to deal with the fact that we rescale the initial value of our Cauchy problem, i.e. that we start u in $(\log t)^{-1/\beta} f$. We use our new estimates on the (non-rescaled) behaviour of u to give a refinement of Lemma 2.7 that also contains an upper bound (Lemma 4.7). First we reduce the problem to initial conditions of the form $a\, p_1$, $a > 0$, (Lemma 4.6). Next we use a similar scaling and comparison argument as in the proof of Lemma 2.7 to deal with general initial data.

The conclusion of part (i) of Proposition 4.1 is then easy. Statement (ii) follows by a simple dominated convergence argument.

The Details

We start by giving a right inverse G of $H = -(\frac{1}{\beta} I + \frac{1}{2}\Delta + \frac{x}{2}\nabla)$ on the linear space V generated by functions $\mathbb{R}^d \to \mathbb{R}$ of the type

$$q_{a,b}(x) = a^{-d/2} \exp\left(-\frac{\|x\|^2}{2a}\right) - b^{-d/2} \exp\left(-\frac{\|x\|^2}{2b}\right), \qquad a, b > 0.$$

Define for $s > 0$ and $y, z \in \mathbb{R}^d$,

$$G(s,y,z) = s^{-1} (2\pi(1-s))^{-d/2} \exp\left(-\frac{\|y - s^{1/2} z\|^2}{2(1-s)}\right).$$

Lemma 4.2 *We can define a linear operator $G : V \to C_b(\mathbb{R}^d)$ by*

$$(G q_{a,b})(y) = \int_0^1 \int_{\mathbb{R}^d} G(s,y,z) q_{a,b}(z)\, dz\, ds.$$

G is the right inverse of H on V, that is, $HG = \mathrm{id}_V$.

Proof It is easily verified that the integral converges. Hence G is well defined. Now let

$$v(t,x) = \int_0^t \int_{\mathbb{R}^d} (2\pi)^{-d/2} (t-s)^{-d/2} \exp\left(-\frac{\|x-z\|^2}{2(t-s)}\right) s^{-1-d/2} q_{a,b}(s^{-1/2} z)\, dz\, ds.$$

Note that the integral converges, that $v(1,\cdot) = G q_{a,b}$ and that

$$\left(\partial_t - \frac{1}{2}\Delta\right) v(t,x) = t^{-1-d/2} q(t^{-1/2} x).$$

By the substitution $s' = ts$ we obtain $v(t,x) = t^{-d/2} v(1, t^{-1/2} x)$. Thus $(\partial_t - \frac{1}{2}\Delta) v(t,x) = t^{-1-d/2} (Hv(1,\cdot))(t^{-1/2} x)$. Hence $q_{a,b}(y) = Hv(1,y) = H(G q_{a,b})(y)$ as desired. □

Note that we can find a function g such that $Hg = f_\theta(1,\cdot)$ if $f_\theta(1,\cdot) \in V$. This, however, is equivalent to $\theta = \theta_\beta$ with θ_β defined in (4.5)

The next aim is to construct the functions g^- and g^+ of (4.6) and to give upper and lower bounds for g^- and g^+.

For $0 < \varepsilon < \min(\frac{\beta}{1-\beta}, \theta_\beta/2)$ (where $\frac{\beta}{1-\beta} = \infty$ if $\beta = 1$) define

$$q^-(x) = \rho^- q_{1,1/(1+\beta)}$$

and

$$q^+(x) = \rho^+ q_{1+k\varepsilon, 1/(1+\beta)},$$

where

$$\begin{aligned} \rho^- = \rho^-(\varepsilon) &= \frac{(\theta_\beta - \varepsilon)^{1+\beta}}{(1+\beta)^{1/\beta}}, \\ \rho^+ = \rho^+(\varepsilon) &= \frac{(\theta_\beta + \varepsilon)^{1+\beta}}{(1+\beta)^{1/\beta}}, \end{aligned}$$

and $k = \beta^2/(2\theta_\beta)$. Now let $g^- = Gq^-$ and $g^+ = Gq^+$.

Lemma 4.3 *There exist constants $0 < m, M < \infty$ such that for all $x \in \mathbb{R}^d$*

$$-m \exp\left(-\frac{\|x\|^2}{2}\right) \le g^-(x) \le M \exp\left(-\frac{\|x\|^2}{2(1+\varepsilon)}\right), \tag{4.7}$$

$$-m \exp\left(-\frac{\|x\|^2}{2}\right) \le g^+(x) \le M \exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right). \tag{4.8}$$

Proof We only give the proof of (4.8). The proof of (4.7) is similar but easier.

By the substitution $y = s^{1/2}z$ and the Chapman-Kolmogorov equation we get

$$\begin{aligned} g^+(x) &= \rho^+ \int_0^1 ds \int_{\mathbb{R}^d} dz\; s^{-1}(2\pi(1-s))^{-d/2} \exp\left(-\frac{\|x - s^{1/2}z\|^2}{2(1-s)}\right) \qquad (4.9) \\ &\quad \times \left[(1+k\varepsilon)^{-d/2} \exp\left(-\frac{\|z\|^2}{2(1+k\varepsilon)}\right) - (1+\beta)^{d/2} \exp\left(-(1+\beta)\frac{\|z\|^2}{2}\right)\right] \\ &= \rho^+ \int_0^1 ds\; s^{-1} \left[(1+sk\varepsilon)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1+sk\varepsilon)}\right)\right. \\ &\qquad \left. - \left(1 - \frac{\beta s}{1+\beta}\right)^{-d/2} \exp\left(-\frac{\|x\|^2}{2\left(1-\frac{\beta s}{1+\beta}\right)}\right)\right]. \end{aligned}$$

Choose $a \in (0,1)$. Then clearly $\int_a^1 \cdots\; ds \le (\int_a^1 s^{-1}(1+sk\varepsilon)^{-d/2}\; ds) \exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right)$. By partial integration we see that the $\int_0^a$ term equals

$$\begin{aligned} &\int_0^a ds\; \log s \left[k\varepsilon \frac{d(1+sk\varepsilon) + \|x\|^2}{2(1+sk\varepsilon)^{2+d/2}} \exp\left(-\frac{\|x\|^2}{2(1+sk\varepsilon)}\right)\right. \qquad (4.10) \\ &\qquad \left. + \frac{\beta}{1+\beta} \frac{d(1-\frac{\beta s}{1+\beta}) + \|x\|^2}{2\left(1-\frac{\beta s}{1+\beta}\right)^{2+d/2}} \exp\left(-\frac{\|x\|^2}{2\left(1-\frac{\beta s}{1+\beta}\right)}\right)\right] \\ &\qquad + (\log a) \left[(1+ak\varepsilon)^{-d/2} \exp\left(-\frac{\|x\|^2}{2(1+ak\varepsilon)}\right)\right. \\ &\qquad \left. - \left(1 - \frac{\beta a}{1+\beta}\right)^{-d/2} \exp\left(-\frac{\|x\|^2}{2\left(1-\frac{\beta a}{1+\beta}\right)}\right)\right] \\ &= O\left((1+\|x\|^2) \exp\left(-\frac{\|x\|^2}{2(1+ak\varepsilon)}\right)\right) = O\left(\exp\left(-\frac{\|x\|^2}{2(1+k\varepsilon)}\right)\right). \end{aligned}$$

□

Now we are able to construct our sub- and supersolutions $\underline{u}$ and $\overline{u}$.

Lemma 4.4 *Let $0 < \varepsilon < \min(\frac{\beta}{1-\beta}, \theta_\beta/2)$ and g^- and g^+ as in Lemma 4.3. Define $\underline{u}$ and $\overline{u}$ by*

$$\begin{aligned}\underline{u}(t,x) &= (\theta_\beta - \varepsilon)\ \phi(t,x) + t^{-1/\beta}(\log t)^{-1-1/\beta} g^-(t^{-1/2}x),\\ \overline{u}(t,x) &= (\theta_\beta + \varepsilon)\ \phi(t,x) + t^{-1/\beta}(\log t)^{-1-1/\beta} g^+(t^{-1/2}x).\end{aligned}$$

Then $L_\beta \underline{u} \leq 0$ and $L_\beta \overline{u} \geq 0$. Furthermore for t large enough and all $x \in \mathbb{R}^d$

$$0 \leq (\theta_\beta - 2\varepsilon)\ \phi(t,x) \quad \leq \underline{u}(t,x) \leq \quad \theta_\beta\ (t\log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right), \tag{4.11}$$

$$0 \leq \theta_\beta\ \phi(t,x) \quad \leq \overline{u}(t,x) \leq \quad (\theta_\beta + 2\varepsilon)\ (t\log t)^{-1/\beta} \exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right). \tag{4.12}$$

Proof From (4.7) and (4.8) it is clear that (4.11) and (4.12) hold. In order to show that $\underline{u}$ and $\overline{u}$ are sub- and supersolutions we have to give upper and lower bounds for $L_\beta \underline{u}$ and $L_\beta \overline{u}$. Note that for $a, b \in \mathbb{R}$, $a > 0$ and $|b|/a$ small enough $|(a+b)^{1+\beta} - a^{1+\beta}| < 4a^\beta b$. We use this estimate to show that

$$\begin{aligned}L_\beta \underline{u}(t,x) &\leq L_\beta((\theta_\beta - \varepsilon)\phi(t,x)) + (t\log t)^{-1-1/\beta}(Hg^-)(t^{-1/2}x) \qquad (4.13)\\ &\quad +4M((\theta_\beta-\varepsilon)\phi(t,x))^\beta t^{-1/\beta}(\log t)^{-1-1/\beta}\exp\left(-\frac{\|x\|^2}{2t(1+\varepsilon)}\right)\\ &\quad +m(1+1/\beta)t^{-1-1/\beta}(\log t)^{-2-1/\beta}\exp(-\|x\|^2/2t)\\ &= (t\log t)^{-1-1/\beta}\Big[(\theta_\beta-\varepsilon)^{1+\beta}e^{-(1+\beta)\|x\|^2/2t} - \frac{\theta_\beta-\varepsilon}{\beta}e^{-\|x\|^2/2t} + \rho^- e^{-\|x\|^2/2t}\\ &\quad -\rho^-(1+\beta)^{1/\beta}e^{-(1+\beta)\|x\|^2/2t}\Big] + \mathrm{O}(t^{-1-1/\beta}(\log t)^{-2-1/\beta}e^{-\|x\|^2/2t})\\ &= (t\log t)^{-1-1/\beta}\left(\rho^- - \frac{\theta_\beta-\varepsilon}{\beta}\right)e^{-\|x\|^2/2t} + \mathrm{O}(t^{-1-1/\beta}(\log t)^{-2-1/\beta}e^{-\|x\|^2/2t})\\ &\leq 0,\end{aligned}$$

for t large enough, since $\rho^- - \frac{\theta_\beta - \varepsilon}{\beta} < 0$ for $0 < \varepsilon < \theta_\beta$.

Similarly, for t large,

$$\begin{aligned}L_\beta \overline{u} &\geq L_\beta((\theta_\beta+\varepsilon)\phi(t,x)) + (t\log t)^{-1-1/\beta}(Hg^+)(t^{-1/2}x) \qquad (4.14)\\ &\quad -4m((\theta_\beta+\varepsilon)\phi(t,x))^\beta t^{-1/\beta}(\log t)^{-1-1/\beta}e^{-\|x\|^2/2t}\\ &\quad -M(1+1/\beta)(t^{-1-1/\beta}(\log t)^{-2-1/\beta}e^{-\|x\|^2/2t(1+k\varepsilon)})\\ &= (t\log t)^{-1-1/\beta}\Big[\rho^+(1+k\varepsilon)^{-d/2}e^{-\|x\|^2/2t(1+k\varepsilon)} - \frac{\theta_\beta+\varepsilon}{\beta}e^{-\|x\|^2/2t}\Big]\\ &\quad +O((t^{-1-1/\beta}(\log t)^{-2-1/\beta}e^{-\|x\|^2/2t(1+k\varepsilon)})\\ &\geq 0,\end{aligned}$$

since $\rho^+(1+k\varepsilon)^{-d/2} - \frac{\theta_\beta+\varepsilon}{\beta} > 0$. To see the latter inequality, note that

$$1 + k\varepsilon = 1 + \frac{1}{2}\beta^2\varepsilon/\theta_\beta = \frac{\beta^\beta}{1+\beta}\left[\theta_\beta^{\beta^2} + \frac{1}{2}\beta^2\theta_\beta^{\beta^2-1}\varepsilon\right] < \frac{\beta^\beta}{1+\beta}[\theta_\beta+\varepsilon]^{\beta^2}.$$

This inequality holds since $\varepsilon < \theta_\beta$. Thus

$$(\theta_\beta + \varepsilon)\left[\frac{(\theta_\beta+\varepsilon)^\beta}{(1+\beta)^{1/\beta}}(1+k\varepsilon)^{-1/\beta} - \frac{1}{\beta}\right] > 0,$$

as claimed. □

A simple consequence of Lemma 4.4 is the following result on the asymptotic behaviour of u.

Lemma 4.5 *For $f \in C_c^{++}(\mathbb{R}^d)$ the following holds,*

$$\lim_{t\to\infty} (t\log t)^{1/\beta} u(t, t^{1/2}x, f) = \theta_\beta\, e^{-\|x\|^2/2}, \qquad x \in \mathbb{R}^d.$$

Proof The proof of this statement is simple with Lemma 4.4 at hand. Since we will not make use of this Lemma we omit the details and refer the reader to Lee (1993), pages 304/5. □

So far we have considered only a fixed initial condition $u(0,\cdot;f) = f$ of the Cauchy problem $L_\beta u = 0$. The next aim is to change the initial condition to $(\log t)^{-1/\beta} f$.

Define

$$b(x;f) = \inf_{s\ge 0} \liminf_{t\to\infty} (t\log t)^{1/\beta} u(t-s, t^{1/2}x; (\log t)^{-1/\beta} f)$$

and

$$B(x;f) = \sup_{s\ge 0} \limsup_{t\to\infty} (t\log t)^{1/\beta} u(t-s, t^{1/2}x; (\log t)^{-1/\beta} f).$$

In order to proof Proposition 4.1 we will give upper and lower bounds for B and b respectively.

The first step is to reduce the situation to $f(x)$ replaced by $\langle\lambda, f\rangle(2\pi)^{-d/2}\exp(-\|x\|^2/2)$. Recall that p_t is the heat kernel.

Lemma 4.6 *Let $f \in C_c^{++}(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ the following inequalities hold:*

$$\begin{aligned} b(x;f) &\ge b(x; \langle\lambda, f\rangle\, p_1), \\ B(x;f) &\le B(x; \langle\lambda, f\rangle\, p_1). \end{aligned}$$

Proof Note that for $x, y \in \mathbb{R}^d$ and $t, \delta > 0$,

$$\frac{\|x\|^2}{2t(1+\delta)} - \frac{\|x-y\|^2}{2t} \le \frac{1}{t}\cdot\frac{1+\delta}{2\delta}\|y\|^2.$$

Hence for $K \subset \mathbb{R}^d$ compact and $\varepsilon > 0$ we can find $\delta > 0$ and $t_0 > 0$ such that for $f \in C_c^{++}(\mathbb{R}^d)$ with $\mathrm{supp}(f) \subset K$,

$$p_t * f \le (1+\varepsilon)\,\langle\lambda, f\rangle\, p_{(1+\delta)t}, \qquad t \ge t_0.$$

Similarly we get $\delta > 0$ and $t_0 > 0$ such that

$$p_t * f \ge (1-\varepsilon)\,\langle\lambda, f\rangle\, p_{(1-\delta)t}, \qquad t \ge t_0.$$

Hence if we let $d_t = \varepsilon\,\|f\|_\infty^{-\beta}\cdot\log t$, we get with Lemma 2.3 and Lemma 2.2 that for $d_t > t_0$,

$$\begin{aligned} u(t-s, t^{1/2}x; (\log t)^{-1/\beta}\, f) &= u\Big(t-s-d_t, t^{1/2}x; u(d_t, \cdot\,; (\log t)^{-1/\beta}\, f)\Big) \\ &\ge u\Big(t-s-d_t, t^{1/2}x; (1-\varepsilon)\,(\log t)^{-1/\beta}\,(p_{d_t} * f)\Big) \\ &\ge u\Big(t-s-d_t, t^{1/2}x; (1-\varepsilon)^2\,(\log t)^{-1/\beta}\,\langle\lambda, f\rangle p_{(1-\delta)d_t}\Big) \\ &\ge (1-\varepsilon)^2\, u\Big(t-s-d_t, t^{1/2}x; (\log t)^{-1/\beta}\,\langle\lambda, f\rangle p_{(1-\delta)d_t}\Big) \\ &\ge (1-\varepsilon)^2\, u\Big(t-s-\delta d_t - 1, t^{1/2}x; (\log t)^{-1/\beta}\,\langle\lambda, f\rangle p_1\Big). \end{aligned}$$

Since $d_t \ll t$, we obtain

$$b(x;f) \geq (1-\varepsilon)^2 \; b(x; \langle \lambda, f\rangle \, p_1).$$

Now let $\varepsilon \to 0$ to obtain the claim. The claim for B follows analogously using the opposite inequality in Lemma 2.3. □

Now we give the bounds for $b(x; a\,p_1)$ and $B(x; a\,p_1)$. The result is a refinement of Lemma 2.7 and will be obtained by a similar scaling argument.

Lemma 4.7 *For $a > 0$ and $x \in \mathbb{R}^d$ the following inequalities hold*

$$b\left(x; a\exp(-\|\;\cdot\;\|^2/2)\right) \geq \left(\frac{(\theta_\beta a)^\beta}{a^\beta + (\theta_\beta)^\beta}\right)^{1/\beta} \cdot \exp(-\|x\|^2/2)$$

$$B\left(x; a\exp(-\|\;\cdot\;\|^2/2)\right) \leq \left(\frac{(\theta_\beta a)^\beta}{a^\beta + (\theta_\beta)^\beta}\right)^{1/\beta} \cdot \exp(-\|x\|^2/2).$$

Proof

We do the proof only for the second inequality. The proof of the other inequality is quite similar.

Recall from (1.11) that

$$u(t,x;\rho^{1/\beta} f(\rho^{1/2}\cdot)) = \rho^{1/\beta} u(\rho t, \rho^{1/2}x; f), \quad \rho > 0. \tag{4.15}$$

Thus for $s > 0$, $t - s > 0$ large enough and $\rho = \rho(t) = t^{(\theta_\beta)^\beta/a^\beta}$ we have by (4.12)

$$\begin{aligned} & u(t-s,x;(\log t)^{-1/\beta} a\exp(-\|\cdot\|^2/2)) && (4.16)\\ &\qquad = u(t-s,x;\theta_\beta(\log\rho)^{-1/\beta}\exp(-\|\cdot\|^2/2)) \\ &\qquad \leq u(t-s,x;\rho^{1/\beta}\overline{u}(\rho,\rho^{1/2}\cdot)) \\ &\qquad = \rho^{1/\beta} u(\rho(t-s),\rho^{1/2}x;\overline{u}(\rho,\cdot)) \\ &\qquad \leq \rho^{1/\beta}\overline{u}(\rho(t-s+1),\rho^{1/2}x). \end{aligned}$$

We infer that for $\varepsilon > 0$ uniformly in $x \in \mathbb{R}^d$,

$$\begin{aligned} & \sup_{s\geq 0}\limsup_{t\to\infty}(t\log t)^{1/\beta} u(t-s,t^{1/2}x;(\log t)^{-1/\beta} a\exp(-\|\cdot\|^2/2)) && (4.17)\\ & \leq \sup_{s\geq 0}\limsup_{t\to\infty}(t\log t)^{1/\beta}\rho^{1/\beta}\overline{u}(\rho(t-s+1),(\rho t)^{1/2}x) \\ & \leq \sup_{s\geq 0}\limsup_{t\to\infty}(t\log t)^{1/\beta}(t-s+1)^{-1/\beta}(\log(\rho(t-s+1)))^{-1/\beta}\theta_\beta \\ & \qquad \times(1+\varepsilon)\exp\left(-\frac{\|x\|^2 t}{2(1+\varepsilon)(t-s+1)}\right) \\ & = \sup_{s\geq 0}\limsup_{t\to\infty}\left(\frac{\log t}{\log\rho + \log t}\right)^{1/\beta}\theta_\beta(1+\varepsilon)\exp(-\|x\|^2/2(1+\varepsilon)) \\ & = \theta_\beta(1+\varepsilon)\exp(-\|x\|^2/2(1+\varepsilon))\left(\frac{(\theta_\beta a)^\beta}{a^\beta+(\theta_\beta)^\beta}\right)^{1/\beta}. \end{aligned}$$

Now let $\varepsilon \to 0$. □

Proof of Proposition 4.1

The proof of Proposition 4.1 is now easy. Combine Lemma 4.6 and Lemma 4.7 to obtain part (i).
In order to prove part (ii) note that by Lemma 2.3

$$\begin{aligned}(t\log t)^{1/\beta}u(t,t^{1/2}x;(\log t)^{-1/\beta}\,f) &\le t^{1/\beta}(p_t*f)(t^{1/2}x)\\ &= (p_1*(t^{1/\beta}f(t^{1/2}\cdot)))(x)\\ &\le c_f\,p_2(x)\qquad\text{for } t\ge 1\end{aligned}$$

for some $c_f>0$. Hence dominated convergence yields the claim. □

4.2 Proof of Theorem 3

First consider the case where (ψ_t) is SBM$(d,1+2/d,1)$. Note that here the assertion is immediate from Proposition 4.1, (ii), by (1.13) and the scaling relation (see (4.2))

$$\mathcal{L}^{\rho\cdot\lambda}[t^{-\alpha/\beta}\zeta_t(t^{\alpha/2}\cdot)]=\mathcal{L}^{\rho\cdot\lambda}[\zeta_{t^{1-\alpha}}(\cdot)]. \tag{4.18}$$

A more detailed discussion precedes Proposition 4.1.

Now assume that $(\psi_t)=(\eta_t)$ is BBM$(d,1+2/d,1)$. The link to the SBM is the embedded particle system (an idea that goes back to Gorostiza et al., Lemme 1):

> "For fixed time horizon t, poissonizing the initial state m first and then running a BBM (η_s) is the same as running SBM (ζ_s) with initial state m and then poissonizing the random population ζ_t." (4.19)

To make this precise we define a new random measure X_t such that $\mathcal{L}[X_t|\zeta_t]=\mathcal{H}(\zeta_t)$ (recall that $\mathcal{H}(m)$ is the law of a Possion point process with intensity measure m) . Then (4.19) says that for $m\in\mathcal{M}(\mathbb{R}^d)$,

$$\mathcal{L}^{\mathcal{H}(m)}[\eta_t]=\mathcal{L}^m[X_t]. \tag{4.20}$$

To check this let $f\in C_c^+(\mathbb{R}^d)$. Then (recall (1.12) and (1.13)),

$$\begin{aligned}\mathbf{E}^{\mathcal{H}(m)}[\exp(-\langle\eta_t,f\rangle)] &= \exp\left(-\int m(dx)\left(1-\mathbf{E}^{\delta_x}[\exp\left(-\langle\eta_t,f\rangle\right)]\right)\right)\\ &= \exp(-\langle m,u(t,\cdot,1-e^{-f})\rangle)\\ &= \mathbf{E}^m[\exp(-\langle\zeta_t,1-e^{-f}\rangle)]\\ &= \mathbf{E}^m[\exp(-\langle X_t,f\rangle)].\end{aligned} \tag{4.21}$$

Now for $A\in\mathcal{B}(\mathbb{R}^d)$ bounded and $\alpha\ge0$ by the law of large numbers,

$$\mathbf{E}^{(\log t)^{1/\beta}\cdot\lambda}[t^{-\alpha d/2}(\log t)^{-1/\beta}|X_t(t^{\alpha/2}A)-\zeta_t(t^{\alpha/2}A)|]\overset{t\to\infty}{\longrightarrow}0. \tag{4.22}$$

□

4.3 Proof of Theorem 4

Note that as above, the case of (ψ_t) BBM can be derived from the case (ψ_t) SBM. Hence we will now assume that $(\psi_t)=(\zeta_t)$ is SBM$(d,1+2/d,1)$.

The idea of the proof is an induction over the length of the tree $\mathbb{T}$. Recall the heuristics given in the discussion preceding Theorem 4. The key point in the induction is to show that the "important" information about $\widetilde{\zeta}_{t-t^\alpha}\Big|_{[-t^{\alpha/2},t^{\alpha/2}]^d}$ is already contained in $\widetilde{\zeta}_{t-t^\alpha}([-t^{\alpha/2},t^{\alpha/2}]^d)$. We do so by constructing a coupling of $(\zeta_{t-t^\alpha+s})_{s\geq 0}$ with a SBM$(d,1+2/d,1)$ $(\zeta_s^2)_{s\geq 0}$ started in $t^{-\alpha d/2}\zeta_{t-t^\alpha}([-t^{\alpha/2},t^{\alpha/2}]^d)\cdot\lambda$. We show that the coupling is successful in a certain sense within time $s=t^\alpha$.

We prepare for the proof of Theorem 4 by stating a coupling lemma and a comparison lemma both taken from Klenke (1997) (stated there for the case $\beta=1$ only). Note that we give a new proof of the comparison lemma since the second moment used in Klenke (1997) is not available here.

Lemma 4.8 (Coupling) *Let $S>R>0$. Consider $(\zeta_s^1)_{s\geq 0}$ SBM$(d,1+2/d,1)$. Assume that $\mathcal{L}[\zeta_0^1]$ is translation invariant and that $\varepsilon>0$ and $0<\rho<\infty$ are chosen such that*

$$\begin{aligned}
\mathbf{E}\Big[\,\zeta_0^1\left([0,1)^d\right)\Big] &= \rho\\
\mathbf{E}\Big[\,\big|R^{-d}\zeta_0^1([0,R)^d)-S^{-d}\zeta_0^1([0,S)^d)\big|\,\Big] &< \varepsilon\\
\mathbf{E}\Big[\,\big|\zeta_0^1([0,S)^d)-\zeta_0^1\left(S(z+[0,1)^d)\right)\big|\,\Big] &< \varepsilon S^d \qquad \forall\, z\in[-1,1]^d.
\end{aligned}$$

Then there exists a coupling $(\zeta_s^1,\zeta_s^2)_{s\geq 0}$ (i.e., (ζ_s^2) is also SBM$(d,1+2/d,1)$ and both processes are defined on the same probability space) such that

$$\mathcal{L}[\zeta_0^2\,|\,\zeta_0^1]=S^{-d}\zeta_0^1([0,S)^d)\cdot\lambda \tag{4.23}$$

and

$$\mathbf{E}[\|(\zeta_s^1-\zeta_s^2)\Big|_B\|]\leq\lambda(B)\cdot\left[4\varepsilon+d\,e^{-D^2/2s}+2\sqrt{\frac{d}{\pi}}\,\rho R s^{-1/2}\right], \tag{4.24}$$

where $B\in\mathcal{B}(\mathbb{R}^d)$, $B\subset[0,S)^d$ and $D=\operatorname{dist}(B,\mathbb{R}^d\setminus[0,S)^d)$.

Proof This is Corollary 3.7 in Klenke (1997). In fact the proof given there does not rely on the finite variance available there. □

With the tool of the coupling lemma we are able to give a proof of the following lemma that does not rely on second moments.

Lemma 4.9 (Comparison) *Let $\alpha\in(-\infty,1]$ and let $a(t),b(t)\approx t^\alpha$, i.e., $\lim_{t\to\infty}\frac{\log a(t)}{\log t}=\lim_{t\to\infty}\frac{\log b(t)}{\log t}=\alpha$. Then*

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}\left[(\log t)^{-1/\beta}\Big|a(t)^{-d}\zeta_t([0,a(t))^d)-b(t)^{-d}\zeta_t([0,b(t))^d)\Big|\right]\overset{t\to\infty}{\longrightarrow}0. \tag{4.25}$$

Proof The main tool for the proof is the coupling lemma, Lemma 4.8. We prepare for the use of it.

By the basic scaling relation (4.18) we may w.l.o.g. assume $\alpha=0$. Also we can restrict ourselves to the case $b(t)\equiv 1$ and $a(t)\downarrow 0$, $\log a(t)\ll\log t$. Since

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}|S^{-d}\zeta_t([0,S)^d)-R^{-d}\zeta_t([0,R)^d)|]\overset{t\to\infty}{\longrightarrow}0 \tag{4.26}$$

for any $R, S > 0$, we may find $R = R(t) \downarrow 0$, $S = S(t) \uparrow \infty$ such that

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}|S(t)^{-d}\zeta_{t-1}([0,S(t))^d) - R(t)^{-d}\zeta_{t-1}([0,R(t))^d)|] =: \varepsilon_t \overset{t\to\infty}{\longrightarrow} 0. \qquad (4.27)$$

By a similar argument we obtain (maybe by enlarging ε_t and changing $R(t)$ and $S(t)$ a little)

$$\sup_{z\in[0,1]^d} \mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}S^{-d}|\zeta_{t-1}([0,S)^d) - \zeta_{t-1}(S(z+[0,1)^d))|] \le \varepsilon_t \overset{t\to\infty}{\longrightarrow} 0. \qquad (4.28)$$

(For example, for fixed S choose $N \in \mathbb{N}$ large and take the maximum over $z \in \{0, \frac{1}{N}, \ldots, 1\}^d$. This term clearly vanishes as $t \to \infty$. The error term of the two maxima results from the (less than) $N^{d-1}3^d$ blocks of size $[0, \frac{1}{N})^d$ at the surface of $(z + [0,1)^d)$ and is thus bounded by $3^d\frac{1}{N}$. Now let $S = S(t)$ and $N = N(t)$ increase slowly to ∞.)

We apply the coupling lemma 4.8 to obtain a SBM$(d, 1+\beta, 1)$ $(\zeta_s^2)_{s\ge t-1}$ with initial state $\mathcal{L}[\zeta_{t-1}^2|\zeta_{t-1}] = S(t)^{-d}\zeta_{t-1}([0,S(t))^d)\cdot\lambda$ and

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}\|(\zeta_t^1 - \zeta_t^2)\Big|_B\|] \le \lambda(B)[4\varepsilon_t + d\cdot e^{-D^2} + 2\sqrt{d/\pi}\,R(t)], \qquad (4.29)$$

where $B \in \mathcal{B}(\mathbb{R}^d)$ is bounded and $D = \mathrm{dist}(B, \mathbb{R}^d \setminus [0,S(t)]^d)$. In particular, for $M < \infty$ there exists $\delta_t^M \overset{t\to\infty}{\longrightarrow} 0$ such that for any Borel set $B \subset [-M,M]^d$

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}\|(\zeta_t^1 - \zeta_t^2)\Big|_B\|] \le \delta_t^M\lambda(B). \qquad (4.30)$$

Now fix a value $\rho = (\log t)^{-1/\beta}S(t)^{-d}\zeta_{t-1}([0,S(t))^d)$. According to Theorem 3 (and the basic scaling)

$$(\log t)^{-1/\beta}a(t)^{-d}\zeta_t^2([0,a(t))^d) \overset{t\to\infty}{\longrightarrow} \rho \qquad (4.31)$$

and

$$(\log t)^{-1/\beta}\zeta_t^2([0,1)^d) \overset{t\to\infty}{\longrightarrow} \rho.$$

A simple uniform integrability argument yields

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}[(\log t)^{-1/\beta}|a(t)^{-d}\zeta_t'([0,a(t))^d) - \zeta_t'([0,1)^d)| \overset{t\to\infty}{\longrightarrow} 0. \qquad (4.32)$$

Combined with (4.30) the proof is complete. □

Proof of Theorem 4

Recall that we do the proof for the case in which $(\psi_t) = (\zeta_t)$ is SBM$(d, 1+2/d, 1)$. The proof is almost identical to that given in Klenke (1997). The only "real" difference to the case $\beta = 1$ is the modified proof of Lemma 4.9 given above and some changes in the constants. However for the sake of completeness we give the proof here in detail.

We do the proof by induction over the length of the tree $\mathbb{T}$. For $\mathbb{T} = \{\emptyset\}$ this is the assertion of Theorem 3. Now assume that the claim has been shown for all trees shorter than $\mathbb{T}$.

The idea of the proof is the following. We introduce a time scale $L(t) \approx t^{A(\emptyset)}$ and couple (ζ_s) for $s \ge t - L(t)$ with another process (ζ_s^2). This process shall have initial configuration $M(\rho)$, where ρ is the empirical population density of $\zeta_{t-L(t)}^1$ in a box of length $\approx t^{A(\emptyset)/2}$. $L(t)$ will be chosen small enough that the evolutions of the subtrees (resulting from eliminating $\emptyset$ from $\mathbb{T}$) are approximately independent. On the other hand $L(t)$ has to be chosen large enough so that the coupling of Lemma 4.8 with $R(t) \approx t^{A(\emptyset)/2}$ is successful. Here a the details.

Let $b = \max\{\operatorname{diam}(B^e),\, e \in \mathbb{T}\}$. Let $d_t \downarrow 0$, $t \to \infty$ such that

$$\begin{aligned} t^{(A(e\wedge f)-d_t)/2} &\leq \|x_t^e - x_t^f\| - b(t^{A(e)/2} + t^{A(f)/2}) \\ &\leq \|x_t^e - x_t^f\| + b(t^{A(e)/2} + t^{A(f)/2}) \leq \frac{1}{2} t^{(A(e\wedge f)+d_t)/2} \end{aligned} \tag{4.33}$$

for all $e, f \in \mathbb{T}$. We may assume that $t^{d_t} \overset{t\to\infty}{\longrightarrow} \infty$. Let $\alpha := A(\emptyset)$ and define

$$\begin{aligned} S = S(t) &= t^{(\alpha+d_t)/2}, \\ R = R(t) &= t^{(\alpha-3d_t)/2}, \\ L = L(t) &= t^{\alpha-2d_t}. \end{aligned}$$

Let

$$B_t^e = x_t^e + t^{A(e)/2} B^e \tag{4.34}$$

and

$$B_t = \bigcup_{e\in\mathbb{T}} B_t^e. \tag{4.35}$$

By shifting $X = (x_t^e,\, e \in \mathbb{T})$, if necessary, we can assume that $B_t \subset [0, S)^d$ for all $t > 0$ and that

$$L^{-1/2} \cdot \operatorname{dist}(B_t, \mathbb{R}^d \setminus [0,S)^d[) \overset{t\to\infty}{\longrightarrow} \infty. \tag{4.36}$$

Apply Corollary 4.8 with $\zeta_0^1 = \zeta_{t-L(t)}$, $s = L(t)$, $\rho = (\log t)^{1/\beta}$ and with $\varepsilon = (\log t)^{1/\beta}\varepsilon_t$, where $\varepsilon_t \overset{t\to\infty}{\longrightarrow} 0$. This last choice is possible due to Lemma 4.9. Thus we obtain a coupling $(\zeta_s^1, \zeta_s^2)_{s\geq 0}$ with $\mathcal{L}[\zeta_0^1 | \zeta_s^1] = M(S^{-d}\zeta_0^1([0,S)^d))$ such that there exists a sequence $\delta_t \downarrow 0$ with

$$\mathbf{E}^{(\log t)^{1/\beta}\lambda}\left[\left|(\widetilde{\zeta}_{L(t)}^1 - \widetilde{\zeta}_{L(t)}^2)(C)\right|\right] \leq \delta_t \cdot \lambda(C) \quad \forall\, C \in \mathcal{B}(\mathbb{R}^d) \text{ bounded.} \tag{4.37}$$

So all we have to show is

$$\mathcal{L}^{(\log t)^{1/\beta}\lambda}\left[(\log t)^{-1/\beta}\left(t^{-A(e)d/2}\zeta_{L(t)}^2(B_t^e)\right)_{e\in\mathbb{T}}\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}^1\left[(\lambda(B^e) Z^e_{(1-A(e))c_\beta})_{e\in\mathbb{T}}\right]. \tag{4.38}$$

By Theorem 3 we know that

$$\mathcal{L}^{(\log t)^{1/\beta}\lambda}\left[(\log t)^{-1/\beta} S^{-d}\zeta_0^1([0,S)^d)\right] \overset{t\to\infty}{\Longrightarrow} \mathcal{L}^1[Z_{(1-\alpha)c_\beta}]. \tag{4.39}$$

Hence it suffices to show that for $\rho \geq 0$,

$$\begin{aligned} \mathcal{L}^{\rho(\log t)^{1/\beta}\lambda}\left[(\log t)^{-1/\beta}\left(t^{-A(e)d/2}\zeta_{L(t)}(B_t^e)\right)_{e\in\mathbb{T}}\right] &\overset{t\to\infty}{\Longrightarrow} \mathcal{L}^{\rho}\left[(Z^e_{(\alpha-A(e))c_\beta})_{e\in\mathbb{T}}\right] \\ &= \mathcal{L}^{\rho\alpha^{-1/\beta}}\left[(\alpha^{1/\beta} Z^e_{(1-A(e)/\alpha)c_\beta})_{e\in\mathbb{T}}\right]. \end{aligned} \tag{4.40}$$

In the second line we have used the scaling property of $(1+\beta)$ branching “diffusion”.

Let $\mathbb{T}_j = \{(j, l_2, \ldots, l_n) \in \mathbb{T}, n \in \mathbb{N}\}$, $j = 1, \ldots, J$, be the partition of $\mathbb{T}$ into subtrees $\mathbb{T}_j$ according to the offspring of the root ($\mathbb{T} = \{\emptyset\} \cup \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_J$). In order to prove (4.40) by the induction hypothesis it suffices to show that

$$\left((\log t)^{-1/\beta} t^{-A(e)d/2}\zeta_{L(t)}(B_t^e)\right)_{e\in\mathbb{T}_j}, \quad j = 1, \ldots, J, \tag{4.41}$$

are asymptotically independent random variables.

For each $j=1,\dots,J$, fix one $e_j\in\mathbb{T}_j$ and let $C_j=C_j(t)=x_t^{e_j}+[-R(t),R(t))^d$ and $C_0=\mathbb{R}^d\setminus(C_1\cup\dots\cup C_J)$. Then for t large enough we have $C_i\cap C_j=\emptyset$ for $i\neq j$. Let

$$\Delta_j=\Delta_j(t)=\inf_{e\in\mathbb{T}_j}\operatorname{dist}(B_t^e,\mathbb{R}^d\setminus C_j).$$

Since $A:\mathbb{T}\to I$ is strictly decreasing, we have $\Delta_j(t)/\sqrt{L(t)}\overset{t\to\infty}{\longrightarrow}\infty$.

Let $(\chi_s^j)_{s\geq 0}$, $j=0,1,\dots,J$, be independent SBM(d,1+2/d,1) with initial states

$$\chi_0^j=\mathbb{1}_{C_j}\rho(\log t)^{1/\beta}\lambda,\qquad j=0,1,\dots,J.$$

We can assume $\zeta_s=\chi_s^0+\dots+\chi_s^J$. Now for $j=1,\dots,J$ and $e\in\mathbb{T}_j$,

$$\mathbf{E}\left[(\log t)^{-1/\beta}t^{-A(e)d/2}\sum_{i=0,i\neq j}^{J}\chi_{L(t)}^i(B_t^e)\right] \tag{4.42}$$

$$\leq\rho\lambda(B^e)t^{-A(e)d/2}\int_{\mathbb{R}^d\setminus C_j}dx\int_{B_t^e}dy\,p_{L(t)}(x,y)\leq\rho\lambda(B^e)\exp\left(-\Delta_j^2/L(t)\right)\overset{t\to\infty}{\longrightarrow}0.$$

Thus (4.41) holds and the proof is complete.

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