On norm resolvent convergence in theory of boundary homogenization

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Boundary homogenization in bounded domains

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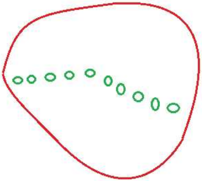
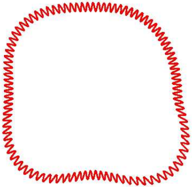
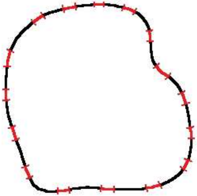


Perforation along a curve



Fast oscillating boundary



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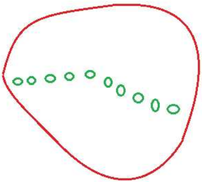

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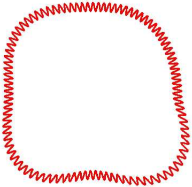

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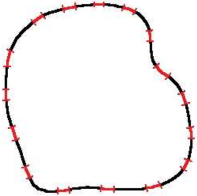

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The usual result is a strong resolvent convergence: $f \in L_2$
$(\mathcal{H}^\varepsilon - \lambda)^{-1} f \to (\mathcal{H}^0 - \lambda)^{-1} f$ strongly in L_2 and weakly in W_2^1.

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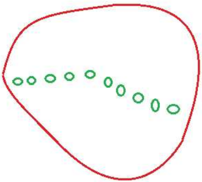

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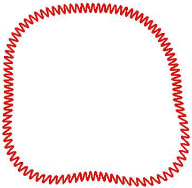

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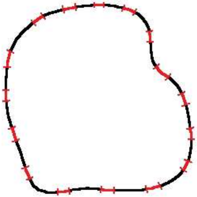

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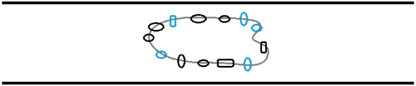
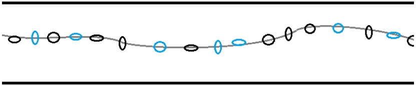
Main questions (motivated by works by M.Sh. Birman, T.A. Suslina & V.V. Zhikov, S.E. Pastukhova)

- Is there a norm resolvent convergence?
- If yes, what is the rate of the convergence?

Formulation of the problem





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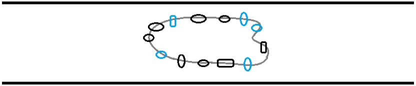

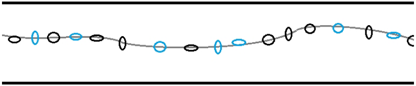

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The operator is

$$\mathcal{H}^\varepsilon := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \left(A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} \right) + A_0 \quad \text{in} \quad L_2(\Omega^\varepsilon)$$

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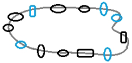

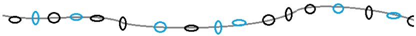

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subject to the *Dirichlet condition* on some of the holes and to the *Robin condition*

$$\left(\frac{\partial}{\partial N^\varepsilon} + a \right) u = 0, \quad \frac{\partial}{\partial N^\varepsilon} := \sum_{i,j=1}^{2} A_{ij} \nu_i^\varepsilon \frac{\partial}{\partial x_j} + \sum_{j=1}^{2} \overline{A}_j \nu_j^\varepsilon,$$

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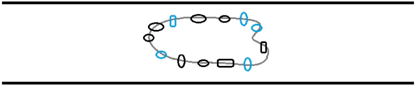

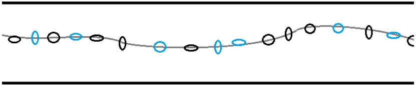

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The main aim: *to study the norm resolvent convergence for $\mathcal{H}_{\varepsilon}$ as $\varepsilon \to +0$.*

Notations

The domain and curve: $\Omega := \{x : 0 < x_2 < d\}$, γ is a C^2-curve in Ω separated from $\partial\Omega$, with a bounded curvature, with no self-intersections and is either infinite or finite and closed, s is the arc length of γ

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Position of holes: $\mathbb{M}^\varepsilon \subseteq \mathbb{Z}$ is an arbitrary set, and s_k, $k \in \mathbb{M}^\varepsilon$ are some points, $s_k < s_{k+1}$, $y_k^\varepsilon \in \gamma$ are associated with $s = s_k\varepsilon$.

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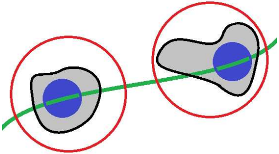
Holes with Dirichlet and Robin conditions: $\theta_D^\varepsilon := \bigcup\limits_{k\in\mathbb{M}_D} \omega_k^\varepsilon$, $\theta_R^\varepsilon := \bigcup\limits_{k\in\mathbb{M}_R} \omega_k^\varepsilon$,

$\mathbb{M}_D^\varepsilon \cup \mathbb{M}_R^\varepsilon = \mathbb{M}^\varepsilon$.

Assumptions

Sizes and position of the holes:

$\exists$ fixed numbers $0 < R_1 < R_2$, $b > 1$, $L > 0$, and points $x^k \in \mathbb{R}^2$, $k \in \mathbb{M}^\varepsilon$, s.t. $B_{R_1}(x^k) \subset \omega_k \subset B_{R_2}(0)$, $|\partial\omega_k| \leqslant L$, $B_{bR_2\varepsilon}(y_k^\varepsilon) \cap B_{bR_2\varepsilon}(y_i^\varepsilon) = \emptyset$, $i \neq k$.



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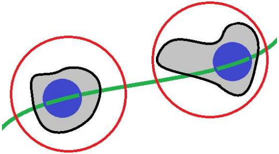

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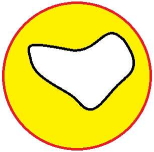
Uniform regularity of the holes:
For $R_3 := R_2(b+1)/2$ and $k \in \mathbb{M}^\varepsilon$ the b.v.p.



$$\operatorname{div} X_k = 0 \text{ in } B_{R_3}(0) \setminus \omega_k,$$
$$X_k \cdot \nu = -1 \text{ on } \partial\omega_k, \quad X_k \cdot \nu = \varphi_k \text{ on } \partial B_{R_3}(0),$$

is solvable in $L_\infty(B_{R_3}(0) \setminus \omega_k)$ and bounded in this space uniformly in $k \in \mathbb{M}^\varepsilon$. Here ν is the outward normal, $\varphi_k \in L_\infty(\partial B_{R_3}(0))$ satisfies $\int\limits_{\partial B_{R_3}(0)} \varphi_k \, ds = |\partial\omega_k|$.

Main results: Homogenized Dirichlet condition on γ

Theorem

Let $\boldsymbol{\varepsilon \ln \eta(\varepsilon) \to 0}$, $\varepsilon \to +0$. *Suppose there exists a constant* $R_4 > bR_2$ *such that*

$$\{x : \mathrm{dist}(x,\gamma) < \varepsilon b R_2\} \subset \bigcup_{k\in\mathbb{M}_D^\varepsilon} B_{R_4\varepsilon}(y_k^\varepsilon).$$

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Then

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_\mathrm{D}^0 - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{1/2}\big(|\ln \eta(\varepsilon)|^{1/2} + 1\big),$$

where $\mathcal{H}_\mathrm{D}^0$ *is the operator with the same differential expression and subject to the Dirichlet condition of* γ *and* $\partial\Omega$, C *is a positive constant independent of* ε.
The estimate is order sharp.

δ-interaction on γ

For $\beta \in W^1_\infty(\gamma)$, $\mathcal{H}^0_\beta$ denotes the operator with the above differential expression subject to the boundary conditions

$$[u]_\gamma = 0, \quad \left[\frac{\partial u}{\partial N^0}\right]_\gamma + \beta u\big|_\gamma = 0, \qquad \frac{\partial}{\partial N^0} := \sum_{i,j=1}^{2} A_{ij}\nu_i^0 \frac{\partial}{\partial x_j}, \tag{1}$$

where $\nu^0 = (\nu_1^0, \nu_2^0)$ and $[u]_\gamma = u\big|_{\tau=+0} - u\big|_{\tau=-0}$.

Homogenized δ-interaction on γ: exponentially small Dirichlet holes

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Let $(\varepsilon \ln \eta(\varepsilon))^{-1} \to -\rho$, $\varepsilon \to +0$, *and* $\mathbb{M}_D^\varepsilon \neq \emptyset$.

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Let $\boldsymbol{(\varepsilon \ln \eta(\varepsilon))^{-1} \to -\rho}$, $\varepsilon \to +0$, *and* $\mathbb{M}_D^\varepsilon \neq \emptyset$. *Denote* $\alpha^\varepsilon(s) := \frac{\pi}{bR_2}$ *as* $|s - \varepsilon s_k| < bR_2\varepsilon$, $k \in \mathbb{M}_D^\varepsilon$, *and* $\alpha^\varepsilon(s) := 0$ *otherwise.*

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$$\sum_{q\in\mathbb{Z}} \frac{1}{|q|+1} \left| \int_n^{n+\ell} \big(\alpha^\varepsilon(s) - \alpha(s)\big) \mathrm{e}^{-\frac{\mathrm{i}q}{2\pi\ell}(s-n)}\, ds \right|^2 \leqslant \varkappa^2(\varepsilon),$$

where $n = -|\gamma|/2$, $\ell = |\gamma|$, *if* γ *is a finite curve, and* $n \in \mathbb{Z}$, $\ell = 1$, *if* γ *is infinite.*

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where $n = -|\gamma|/2$, $\ell = |\gamma|$, *if* γ *is a finite curve, and* $n \in \mathbb{Z}$, $\ell = 1$, *if* γ *is infinite. Denote* $\beta := -\alpha \frac{\rho}{A_{11}A_{22} - A_{12}^2}$, $\mu(\varepsilon) := -(\varepsilon \ln \eta(\varepsilon))^{-1} - \rho \to 0$. *Then*

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_\beta^0 - \mathrm{i})^{-1}\|_{L_2(\Omega)\to L_2(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{1/2} + \varkappa(\varepsilon) + \mu(\varepsilon)\big),$$

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$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}^0_{\alpha\mathrm{a}} - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W^1_2(\Omega^\varepsilon)} \leqslant C\big(\varepsilon^{1/2} + \varkappa(\varepsilon)\big),$$

where C *is a positive constant independent of* ε.
Term $\varepsilon^{1/2}$ *is order sharp.*

Homogenized "no condition"

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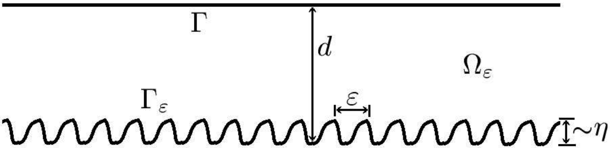
if $a \not\equiv 0$, *and*

$$\|(\mathcal{H}^\varepsilon - \mathrm{i})^{-1} f - (\mathcal{H}^0 - \mathrm{i})^{-1} f\|_{L_2(\Omega)\to W_2^1(\Omega^\varepsilon)} \leqslant C\varepsilon^{1/2}\eta(\varepsilon)(|\ln\eta(\varepsilon)|^{1/2} + 1),$$

if $a \equiv 0$, *where* C *is a positive constant independent of* ε.
The estimates are order sharp up to the absence of the term $|\ln\eta|^{1/2}$.

Model

Domain:



$\Omega_\varepsilon := \{x \in \mathbb{R}^2 : \eta b(x_1\varepsilon^{-1}) < x_2 < d\}$, $b \in C^2(\mathbb{R})$, b is 1-periodic, $b \geqslant 0$, $\eta = \eta(\varepsilon) > 0$, $\varepsilon \to +0$

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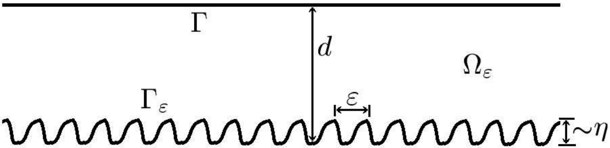

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Operator: $\mathcal{H}_\varepsilon := -\sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum\limits_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{A_j} + A_0$ in $L_2(\Omega_\varepsilon)$

$u = 0$ on Γ

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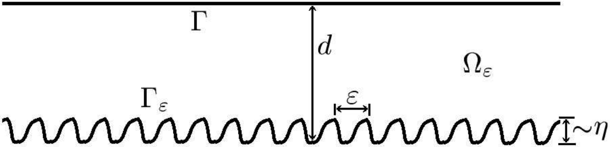

$\Omega_\varepsilon := \{x \in \mathbb{R}^2 : \eta b(x_1\varepsilon^{-1}) < x_2 < d\}$, $b \in C^2(\mathbb{R})$, b is 1-periodic, $b \geqslant 0$,
$\eta = \eta(\varepsilon) > 0$, $\varepsilon \to +0$

Operator: $\mathcal{H}_\varepsilon := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{A_j} + A_0$ in $L_2(\Omega_\varepsilon)$

$u = 0$ on Γ

Boundary condition on Γ_ε is the Dirichlet, Neumann or Robin one

Dirichlet condition on Γ_ε

Suppose we have the Dirichlet condition on Γ_ε.

Dirichlet condition on Γ_ε

Suppose we have the Dirichlet condition on Γ_ε.

Homogenized operator: $\Omega_0 := \{x \in \mathbb{R}^2 : 0 < x_2 < d\}$

$\mathcal{H}_0 := -\sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum\limits_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0$ in $L_2(\Omega_0)$ subject to the Dirichlet condition on Γ and on $\Gamma_0 := \{x : x_2 = 0\}$.

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Main result: *The estimate holds:*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega_0) \to W_2^1(\Omega_\varepsilon)} \leqslant C\eta^{1/2}$$

This estimate is order sharp.

Neumann condition on Γ_ε

Suppose we have Neumann condition on Γ_ε.

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Homogenized operator: $\Omega_0 := \{x \in \mathbb{R}^2 : 0 < x_2 < d\}$

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$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega_0)\to W_2^1(\Omega_\varepsilon)} \leqslant C\eta^{1/2}$$

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Robin condition on Γ_ε

Boundary condition on Γ_ε: $\left(\dfrac{\partial}{\partial\nu_\varepsilon} + a\right) u_\varepsilon = 0$, $a \geqslant 0$.

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Cases

- Weakly oscillating boundary: $\lim\limits_{\varepsilon \to +0} \varepsilon^{-1}\eta(\varepsilon) = \alpha \geqslant 0$
- Highly oscillating boundary: $\lim\limits_{\varepsilon \to +0} \varepsilon^{-1}\eta(\varepsilon) = +\infty$

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$$\mathcal{H}_0 := -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_j}A_{ij}\frac{\partial}{\partial x_i}+\sum_{j=1}^{2}A_j\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_j}\overline{A_j}+A_0 \text{ in } L_2(\Omega_0)$$

Boundary condition on Γ_0: $\left(\frac{\partial}{\partial\nu_0}+a_0\right)u=0.$

$a_0 := a(x_1,0)\int_0^1\sqrt{1+\alpha^2 b'(t)^2}\,dt.$

Robin condition on Γ_ε:

Weakly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon) = \alpha \geqslant 0$

Homogenized operator:

$$\mathcal{H}_0 := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{A_j} + A_0 \text{ in } L_2(\Omega_0)$$

Boundary condition on Γ_0: $\left(\frac{\partial}{\partial \nu_0} + a_0\right) u = 0.$

$a_0 := a(x_1, 0) \int_0^1 \sqrt{1 + \alpha^2 b'(t)^2}\, dt.$

Main result: *The estimate holds*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega_0)\to W_2^1(\Omega_\varepsilon)} \leqslant C(\eta^{1/2} + |\varepsilon^{-1}\eta - \alpha|)$$

The estimate is order sharp.

Robin condition on Γ_ε: highly oscillating boundary and positive coefficient

Highly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon) = +\infty$

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Boundary condition on Γ_0: Dirichlet condition.

Robin condition on Γ_ε: highly oscillating boundary and positive coefficient

Highly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon) = +\infty$

Homogenized operator:

$$\mathcal{H}_0 := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{A_j} + A_0 \text{ in } L_2(\Omega_0)$$

Boundary condition on Γ_0: Dirichlet condition.

Main result: *Let $a \geqslant c > 0$. Then the estimate holds*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega_0)\to W_2^1(\Omega_\varepsilon)} \leqslant C(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2})$$

The term $\varepsilon^{1/2}\eta^{-1/2}$ is order sharp.

Robin condition on Γ_ε: highly oscillating boundary and non-negative coefficient

Highly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon) = +\infty$

Homogenized operator: as above

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Homogenized operator: as above

Main result: *Let* $a \geqslant 0$ *and for small* δ *the set* $\{x : a(x) \leqslant \delta,\ 0 < x_2 < (\sup b + 1)\eta\}$ *is contained in at most countably many rectangles* $\{x : |x_1 - X_n| < \mu(\delta),\ 0 < x_2 < (\sup b + 1)\eta\}$*, where* $\mu(\delta) \to +0$ *as* $\delta \to +0$*, numbers* X_n*,* $n \in \mathbb{Z}$ *are independent of* δ *and satisfy the estimate* $|X_n - X_m| \geqslant c > 0$*,* $n \neq m$*. Then the estimate holds*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_0 - \mathrm{i})^{-1}\|_{L_2(\Omega_0)\to W_2^1(\Omega_\varepsilon)} \leqslant C\big(\eta^{1/2} + \varepsilon^{1/2}\eta^{-1/2}\delta^{-1/2} + \mu^{1/2}(\delta)|\ln\mu(\delta)|^{1/2}\big),$$

where $\delta = \delta(\varepsilon) \to +0$*,* $\varepsilon \to +0$ *is an arbitrary function.*

Robin condition on Γ_ε: highly oscillating boundary and non-negative coefficient

Highly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon) = +\infty$

The most general case: $\mu(\delta) \sim \delta^{1/2}$

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Highly oscillating boundary: $\lim\limits_{\varepsilon\to+0} \varepsilon^{-1}\eta(\varepsilon)=+\infty$

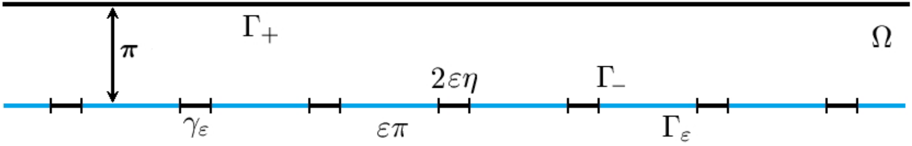
The most general case: $\mu(\delta)\sim\delta^{1/2}$

Main estimate:

$$\|(\mathcal{H}_\varepsilon-\mathrm{i})^{-1}-(\mathcal{H}_0-\mathrm{i})^{-1}\|_{L_2(\Omega_0)\to L_2(\Omega_\varepsilon)} \leqslant C\big(\eta^{1/2}+\varepsilon^{1/6}\eta^{-1/6}|\ln\varepsilon\eta^{-1}|^{-2/3}\big).$$

Formulation of the problem

Consider the waveguide



$\varepsilon \to +0$, $\eta = \eta(\varepsilon)$, $0 < \eta < \frac{\pi}{2}$

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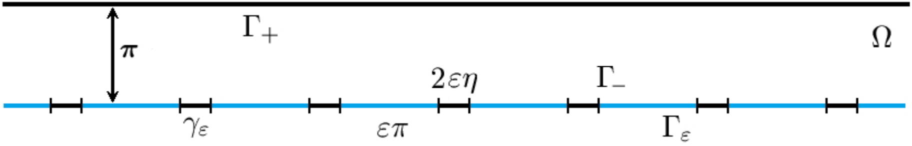

$\varepsilon \to +0$, $\eta = \eta(\varepsilon)$, $0 < \eta < \frac{\pi}{2}$
$\mathcal{H}_\varepsilon := -\Delta$ in $L_2(\Omega)$ subject to D.b.c. on ■ and to N.b.c. on ■

Homogenized operators

If $\varepsilon \ln \eta(\varepsilon) \to 0$ as $\varepsilon \to 0$,

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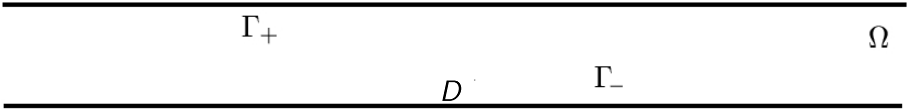

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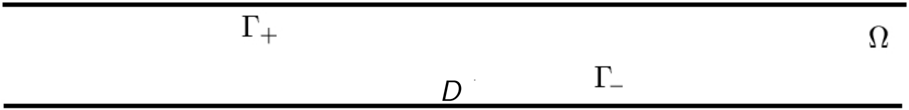

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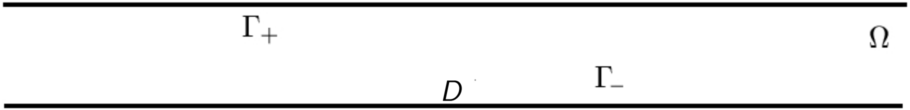
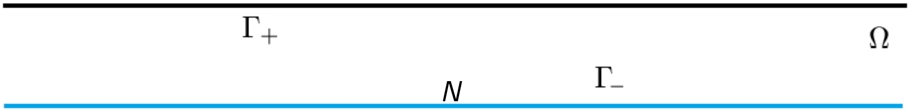


If $\varepsilon \ln \eta(\varepsilon) \to -\infty$ as $\varepsilon \to 0$, then the homogenized operator is $\mathcal{H}_N := -\Delta$ in $L_2(\Omega)$ subject to the b.c. as on the figure



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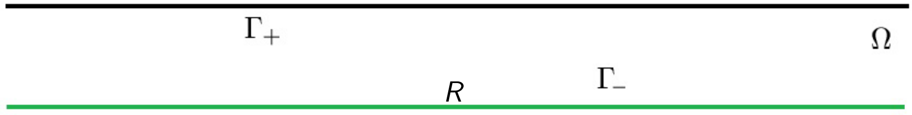
If $\boldsymbol{\varepsilon \ln \eta(\varepsilon) \to -1/K}$ as $\varepsilon \to 0$, $K \geqslant 0$ then the homogenized operator is $\mathcal{H}_R := -\Delta$ in $L_2(\Omega)$ subject to the b.c. as on the figure



R: $\left(-\frac{\partial}{\partial x_2} + K\right)u = 0$ is the condition modeling δ-potential

Resolvent convergence

Theorem (Dirichlet case). *Let* $\varepsilon \ln \eta(\varepsilon) \to 0$*. Then*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (\mathcal{H}_D - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant C\varepsilon^{1/2}|\ln\sin\eta(\varepsilon)|^{1/2}$$

holds true, where $\|\bullet\|_{X\to Y}$ *is the norm of an operator from* X *to* Y*.*

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$\mathcal{H}_N^{(\mu)} := -\Delta$ *with Dirichlet b.c. on* Γ_+ *and* $(\frac{\partial}{\partial\nu} + \mu)u = 0$ *on* Γ_-. *There exists a corrector* $W = W(x,\varepsilon,\mu)$ *defined explicitly such that*

$$\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (1+W)(\mathcal{H}_N^{(\mu)} - \mathrm{i})^{-1}\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant C\varepsilon\mu|\ln \varepsilon\mu|.$$

Resolvent convergence

Theorem (Robin case). *Let* $\boldsymbol{\varepsilon \ln \eta(\varepsilon) \to -1/K}$, $\mu := -(\varepsilon \ln \eta(\varepsilon))^{-1} - K \to +0$. *Then*

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Based on the papers:

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