LONG-TIME ASYMPTOTICS FOR KDV AND TODA EQUATIONS WITH STEPLIKE INITIAL PROFILE

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The Cauchy problem

We are interested in the long-time asymptotic behaviour of the solution of the Korteweg - de Vries equation

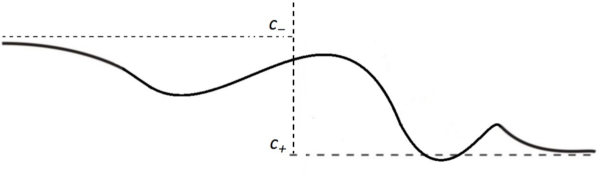
$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

with steplike initial profile $q(x,0) = q_0(x) \in \mathcal{C}^n(\mathbb{R})$:

$$\begin{cases} q_0(x) \to c_+, & \text{as} \quad x \to +\infty, \\ q_0(x) \to c_-, & \text{as} \quad x \to -\infty, \end{cases}$$

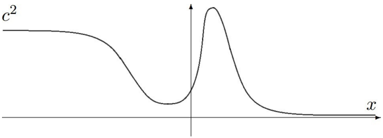
with

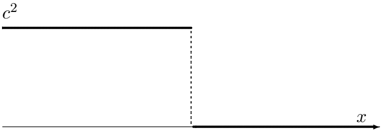
$$\int_{\mathbb{R}_+} (1 + |x|^m)|q_0(\pm x) - c_\pm| dx < \infty, \quad m \geq 1.$$



Shock and rarefaction problems

Without loss generality one can put $c_+ = 0$. If $c_- = -c^2$, $c \in \mathbb{R}$, we say about the shock problem for the KdV equation. The case $c_- = c^2$ is known as the rarefaction problem.





Initial data

$$q_0(x) = \begin{cases} 0, & x \geq 0, \\ \pm c^2, & x < 0, \end{cases}$$

are called pure step ones.

Both shock and rarefaction problem are studied in the regime:

$x \to \infty$; $t \to +\infty$; $\frac{x}{t} = O(1)$ as $t \to \infty$.

Shock problem

Well understood on a physical level of rigor for pure step initial data (Gurevich/Pitaevskii '73, Bikbaev '89, Sharipov/Novokshenov '78, Leach/Needham '08):

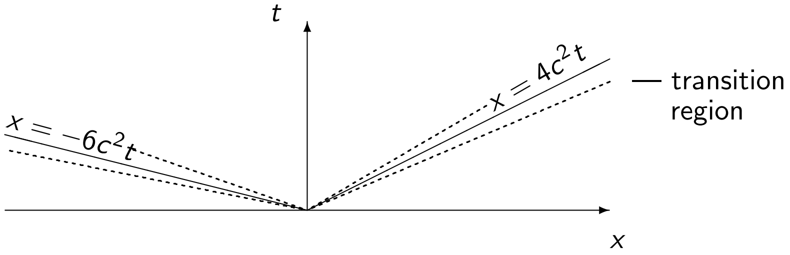
- in the domain $x < -6c^2 t$ the solution is asymptotically close to the background constant $-c^2$;
- in the domain $-6c^2 t < x < 4c^2 t$ the solution is described by the modulated elliptic wave (Whitham's method);

Inverse scattering transform (IST), steplike initial data: Ermakova '81:

- in the domain $4c^2 t < x$ the solution is asymptotically close to the sum of solitons:

$$q(x,t) = -2\sum_{j=1}^{N} \frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)} + O(t^{-1}).$$

Asymptotical solitons



Khruslov, ' 76

In the domain $4c^2 t \geq x > 4c^2 t - (2c)^{-1} \ln t^{M+1}$ as $t \to +\infty$:

$$q(x,t) = \sum_{k=1}^{[\frac{M+1}{2}]} \frac{-2c^2}{\cosh^2\{c\,x - 4c^3 t + \frac{1}{2}\ln t^{2n-1/2} + \phi_k\}} + O(t^{-1/2+\varepsilon}),$$

where $\varepsilon > 0$ is arbitrary small and the phases ϕ_k are determined by the initial scattering data.

Shock problem, numerics

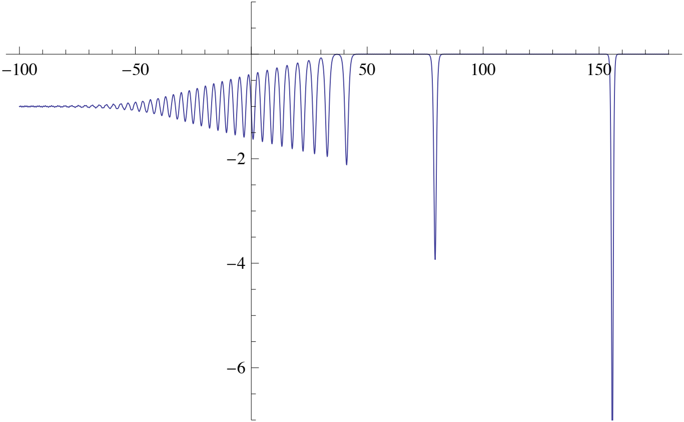


Figure: $t = 10$, $q(x,0) = \frac{1}{2}(\mathrm{erf}(x) - 1) - 5\mathrm{sech}(x-1)$.

Rarefaction problem

Well understood on the physical level of rigor for pure step initial data (Leach/Needham '14, Zakharov/Manakov/Novikov/Pitaevskii '80):

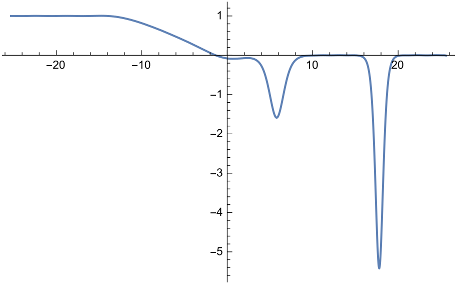


Figure: $t = 1.5$, $q_0(x) = \frac{1}{2}(1 - \mathrm{erf}(x)) - 4\,\mathrm{sech}(x-1)$.

- $q(x,t) = c^2 + o(1)$ in the domain $x < -6c^2 t$;
- $q(x,t) = -\frac{x}{6t} + o(1)$ in the domain $-6c^2 t < x < 0$;
- For $x > 0$ the solution is asymptotically close to the sum of solitons.

Dispersion: rapidly decaying initial condition

Let us take the linear part of the KdV equation

$$q_t = -q_{xxx} + 6qq_x, \qquad q(x, t = 0) = q_0(x) \to 0, \ x \to \pm\infty,$$

then the equation can be solved using the Fourier transform

$$q(x,t) = \int_{-\infty}^{\infty} \hat{q}_0(k) \mathrm{e}^{\mathrm{i}(kx + k^3 t)} dk = \int_{-\infty}^{\infty} \hat{q}_0(k) \mathrm{e}^{t\Phi(k)} dk,$$
$$\hat{q}_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(x) \mathrm{e}^{-\mathrm{i}kx} dx.$$

where the phase is given by

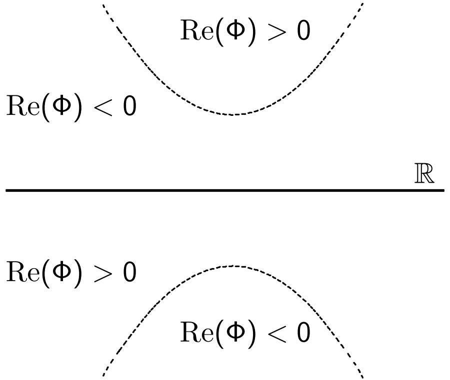
$$\Phi(k) = \mathrm{i}(ku + k^3), \qquad u = \frac{x}{t}.$$

We are interested in the asymptotics as $t \to \infty$ keeping the velocity u fixed. To this end we need to look at the stationary phase points

$$\Phi'(k) = 0 \quad \Rightarrow \quad k = \pm k_0, \ k_0 = \pm\sqrt{-\frac{u}{3}}.$$

Steepest descent

Assuming $\hat{q}_0(k)$ admits an analytic continuation to some strip $|\operatorname{Im}(k)| < \epsilon$ we can deform the integration contour $\mathbb{R} \to \Sigma$ into regions of the complex plane where the exponent $\mathrm{e}^{t\Phi(k)}$ decays exponentially as $t \to \infty$. For $u = \frac{x}{t} > 0$ there are no stationary phase points on $\mathbb{R}$ and we just shift the path of integration a bit up to obtain exponential decay:



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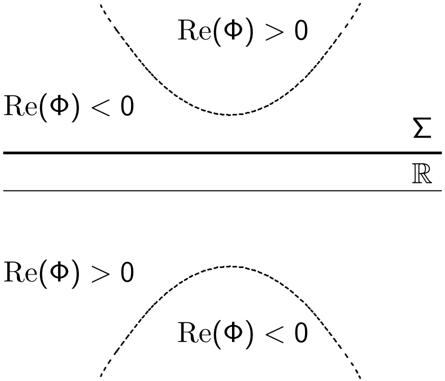
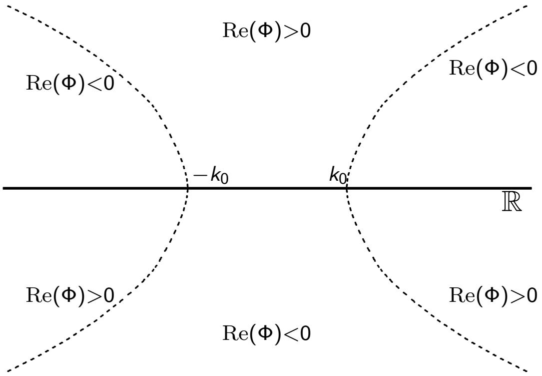


Steepest descent

For $u = \frac{x}{t} < 0$ there are two stationary phase points $\pm k_0 \in \mathbb{R}$ and we need to pass through them:



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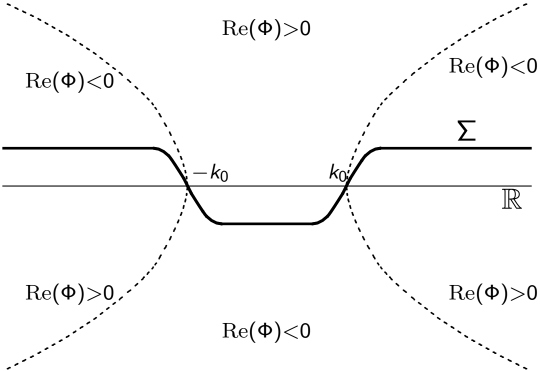
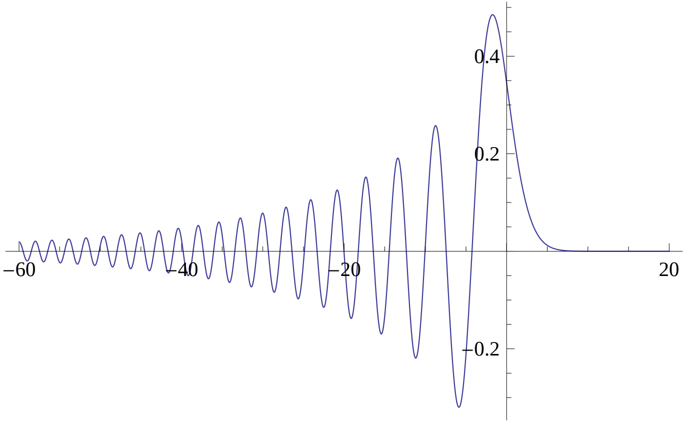


There is now a contribution of $O(t^{-1/2})$ from the two stationary phase points which can be computed by expanding around these points and explicitly evaluating the resulting integrals.

Solutions of the linearized KdV equation

Hence a typical solution of the linearized KdV equation with decaying initial condition splits into a slowly decaying dispersive tail to the left and a fast decaying part to the right. Below a numerically computed solution for $q_0(x) = \mathrm{e}^{-x^2}$ is shown:

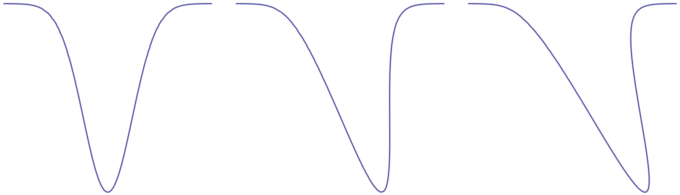


Steepening and Breaking

Now let us look at the nonlinear part of the KdV equation

$$q_t = -q_{xxx} + 6qq_x, \qquad q(x, t = 0) = q_0(x).$$

This is inviscid Burgers' equation which can be solved using the method of characteristics. The wave packets will steepen until they break:



It turns out that for KdV both effects balance each other and give rise to traveling wave solutions - solitons:

Rapidly decaying initial condition, numerics

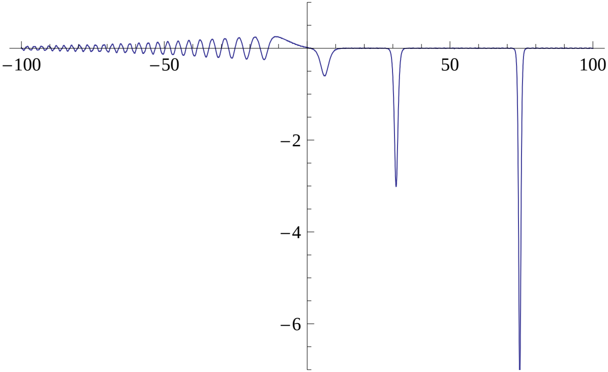


Figure: $t = 5$, $q(x, 0) = \mathrm{sech}(x + 3) - 5\mathrm{sech}(x - 1)$.

Long-time asymptotical analysis :Ablowitz/Newell '73, Shabat '73, Tanaka '73, Manakov '74, Ablowitz/Segur '77, Buslaev '81, Buslaev/Sukhanov '86, Deift/ Venakides/Zhou '94, Grunert/Teschl '10.

Lax pair

The KdV equation is equivalent to the Lax equation

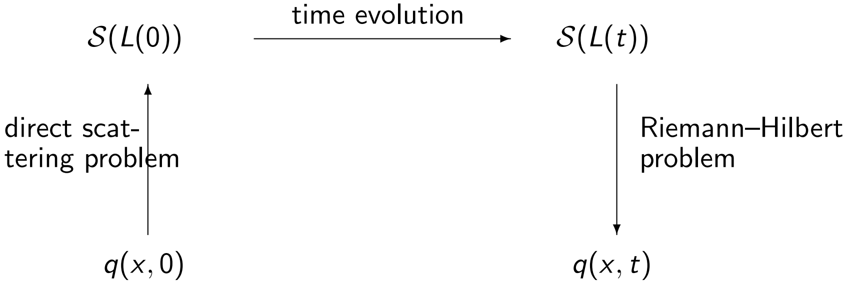
$$\frac{dL(t)}{dt} = L(t)A(t) - A(t)L(t),$$

where

$$L(t) = -\frac{\partial^2}{\partial x^2} + q(x,t),$$

$$A(t) = -4\frac{\partial^3}{\partial x^3} + 3\left(q(x,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}q(x,t)\right)$$

The initial value problem for the KdV equation can be solved via the inverse scattering transform:



The long-time asymptotics can then be found via a nonlinear steepest descent analysis.

Direct scattering problem

Consider the KdV equation with steplike initial data $q_0(x)$, and let the solution of the initial value problem satisfies

$$\int_0^\infty |x|(|q(x,t)| + |q(-x,t) \pm c^2|)dx < \infty, \qquad \forall t \in \mathbb{R}.$$

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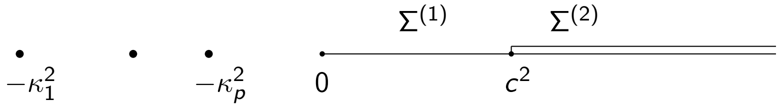
Existence and uniqueness of the Cauchy problem solution: E/ Teschl '11.

Rarefaction problem: $q_0(x) \to c^2$ as $x \to -\infty$.

Spectral picture for the Schrödinger operator

$$L(t) = \partial_x^2 + q(x,t):$$



Jost solutions

The Schrödinger equation $L(t)\phi = k^2\phi$ has the Jost solutions :

$$\lim_{x\to+\infty} \mathrm{e}^{-\mathrm{i}kx}\phi(k,x,t) = \lim_{x\to-\infty} \mathrm{e}^{\mathrm{i}k_1 x}\phi_1(k,x,t) = 1, \quad \text{as} \;\; k \in \overline{\mathbb{C}^+}.$$

Here $k_1 = \sqrt{k^2 - c^2}$. They can be represented via the transformation operators:

$$\phi(k,x,t) = \mathrm{e}^{\mathrm{i}kx} + \int_x^{+\infty} K(x,y,t)\mathrm{e}^{\mathrm{i}ky}dy,$$

$$\phi_1(k,x,t) = \mathrm{e}^{-\mathrm{i}k_1 x} + \int_{-\infty}^{x} K_1(x,y,t)\mathrm{e}^{-\mathrm{i}k_1 y}dy,$$

$$|K(x,y,t)| \leq C\int_{\frac{x+y}{2}}^{\infty} |q(z,t)|dz, \quad K_1(x,y,t)| \leq C\int_{-\infty}^{\frac{x+y}{2}} |q(z,t) - c^2|dz.$$

In particular, $\phi_1(k,x,t) \in \mathbb{R}$ as $k \in [-c,c]$.

Scattering matrix

- The Jost solutions are connected by the scattering relations

$$T(k,t)\phi_1(k,x,t) = \overline{\phi(k,x,t)} + R(k,t)\phi(k,x,t), \qquad k \in \mathbb{R},$$

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- $T(k,t)$ and $R(k,t)$ are the right transmission and reflection coefficients.
- The following representations are valid:

$$T(k,t) = \frac{2\mathrm{i}k}{\langle \phi_1, \phi \rangle}, \quad R(k,t) = -\frac{\langle \phi_1, \overline{\phi} \rangle}{\langle \phi_1, \phi \rangle},$$

where $\langle f, g \rangle = fg_x - gf_x$ is the Wronskian.

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where $\langle f, g \rangle = fg_x - gf_x$ is the Wronskian.
In particular, $|R(k,t)| = 1$ as $k \in [-c,c]$. $R(0,t) = -1$ corresponds to the non-resonant case, $R(0,t) = 1$ is the resonant case.

Evolution of scattering data

- The solution $\phi(\mathrm{i}\varkappa_j, x, t) \in \mathbb{R}$ is an eigenfunction of $L(t)$. The value

$$\gamma_j(t) = \left(\int_{\mathbb{R}} \phi^2(\mathrm{i}\varkappa_j, x, t) dx \right)^{-1},$$

is called the right normalizing constant.

- The solution $q(x,t)$ can be uniquely restored from the right scattering data

$$\mathcal{S}(L(t)) := \{R(k,t), \quad k \in \mathbb{R}; \quad -\varkappa_j^2, \ \gamma_j(t) > 0, \ j = 1, \dots, N\}.$$

- The time evolution of the scattering data can be computed explicitly from the Lax equation:

$$R(k,t) = R(k)\mathrm{e}^{8\mathrm{i}k^3 t}, \quad T(k,t) = T(k)\mathrm{e}^{4\mathrm{i}k^3 t - 4\mathrm{i}k_1^3 t + 6\mathrm{i}c\, k_1 t};$$

$$\gamma_j(t) = \gamma_j \mathrm{e}^{8\kappa_j^3 t}.$$

Here $R(k) = R(k,0)$, $\gamma_j = \gamma_j(0)$.

The Riemann-Hilbert problem

In $\overline{\mathbb{C}^+}$ introduce the vector function

$$m(k,x,t) = \left(T(k,t)\phi_1(k,x,t)\mathrm{e}^{\mathrm{i}kx}, \quad \phi(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \right).$$

Then $q(x,t)$ can be read off from m via

$$m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2\mathrm{i}k} \int_x^{+\infty} q(y,t)dy \begin{pmatrix} -1 & 1 \end{pmatrix} + O\left(\frac{1}{k^2}\right).$$

Define $m(k)$ in $\mathbb{C}^-$ by use of the symmetry condition:

$$m(k) = m(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem (no discrete spectrum)

The function $m(k)$ is the unique solution of the following Riemann-Hilbert problem: to find a holomorphic in $\mathbb{C} \setminus \mathbb{R}$ function $m(k)$, which has continuous limits $m_\pm(k) = m(k \pm \mathrm{i}0)$, $k \in \mathbb{R}$, and satisfies:

I. *the jump condition $m_+(k) = m_-(k)v(k)$, where*

$$v(k) := v(k,x,t) = \begin{pmatrix} 1 - |R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix}, \quad k \in \mathbb{R};$$

II. *the symmetry condition;*

III. *the normalizing condition*

$$m(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} + O(k^{-1}), \quad k \to \infty.$$

Here the phase function is given by $\Phi(k) = 4\mathrm{i}k^3 + 12\mathrm{i}k\xi$, where $\xi = \frac{x}{12t}$.

Admissible transformations for RH problem

- Let $d(k)$ be a holomorphic function on $\mathbb{C} \setminus \Sigma$, where $\Sigma \subset \mathbb{R}$ is a symmetric with respect to the map $k \mapsto -k$ contour. If $d(-k) = d^{-1}(k)$ and $d(k) \to 1$, as $k \to \infty$, then the transformation

$$\widetilde{m}(k) = m(k) \begin{pmatrix} d^{-1}(k) & 0 \\ 0 & d(k) \end{pmatrix}$$

preserves the symmetry and the normalization conditions.

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preserves the symmetry and the normalization conditions.
- Let $\Omega^U \subset \mathbb{C}^+$ be a domain and let $\Omega^L \subset \mathbb{C}^-$ be symmetric with respect to the maps $k \mapsto -k$ and $k \mapsto \overline{k}$. The transformation
$$\widetilde{m}(k) = m(k) \begin{cases} B_{U,L}(k), & k \in \Omega^{U,L}, \\ \mathbb{I}, & \text{else}, \end{cases}$$
is admissible if
$$B_U(k) = \begin{pmatrix} 1 & a(k) \\ 0 & 1 \end{pmatrix}, \quad B_L(k) = \begin{pmatrix} 1 & 0 \\ a(-k) & 1 \end{pmatrix},$$
or vice versa, is admissible.

"Upper-lower" factorization (for decaying initial data)

For $k \in \mathbb{R} \setminus [-\sqrt{-\xi}, \sqrt{-\xi}]$, we factorize the jump matrix :

$$v(k) = \begin{pmatrix} 1 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R(k)\mathrm{e}^{2t\Phi(k)} & 1 \end{pmatrix} = B_L \cdot B_U^{-1},$$

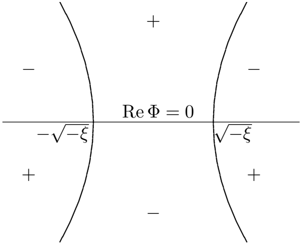
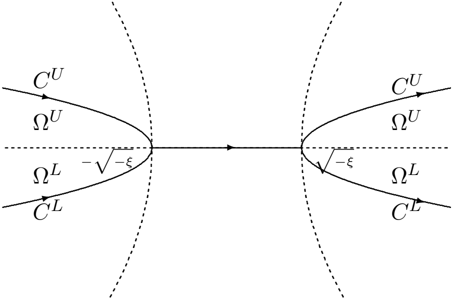


Figure: Signature table for $\operatorname{Re}\Phi(k)$ as $\xi < 0$.

The lenses mechanizm

Put

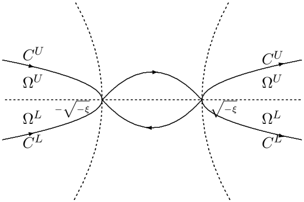
$$m^{(1)}(k) = m(k)\begin{cases} B_U(k), & k \in \Omega^U; \\ B_L(k), & k \in \Omega^L; \end{cases}$$



Then $m_+^{(1)}(k) = m_-^{(1)}(k) v^{(1)}(k)$, where for $t \to \infty$

$$v^{(1)}(k) = \begin{cases} v(k), & [-\sqrt{-\xi}, \sqrt{-\xi}], \\ \mathbb{I} + o(1), & k \in \mathcal{C}^U \cup \mathcal{C}^L. \end{cases}$$

"Lower - upper" factorization



On $k \in [-\sqrt{-\xi}, \sqrt{-\xi}]$ we solve an auxiliary scalar RH problem

$$d_+(k) = d_-(k)(1 - |R(k)|^2), \; d(-k) = d^{-1}(k), \; d(k) \to 1, \text{ as } k \to \infty,$$

and put $m^{(2)}(k) = m^{(1)}(k) d(k)^{\sigma_3}$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then

$$v^{(2)}(k) = \begin{pmatrix} 1 & 0 \\ \frac{d_-(k)^2 R(k) \mathrm{e}^{2t\Phi(k)}}{1 - |R^2(k)|} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{d_+(k)^2 R(k) \mathrm{e}^{-2t\Phi(k)}}{1 - |R^2(k)|} \\ 0 & 1 \end{pmatrix} = \tilde{B}_L \tilde{B}_U^{-1}.$$

Rarefaction problem, the "g-function" approach

In the rarefaction wave case $|R(k)| = 1$ as $k \in [-c; c]$.

We study the region between the leading and the rear wave fronts,

$$\xi \in (-\frac{c^2}{2}, 0).$$

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$$\xi \in (-\frac{c^2}{2}, 0).$$

Put $a(\xi) = \sqrt{-2\xi}$ and

$$g(k) = g(k;\xi) = 4\mathrm{i}(k^2 - a^2(\xi))\sqrt{k^2 - a^2(\xi)}.$$

Then $a(\xi) \in [0, c]$ and

$$\Phi(k) - g(k) = \frac{12\xi^2}{2\mathrm{i}k}(1 + O(k^{-1})), \quad k \to \infty.$$

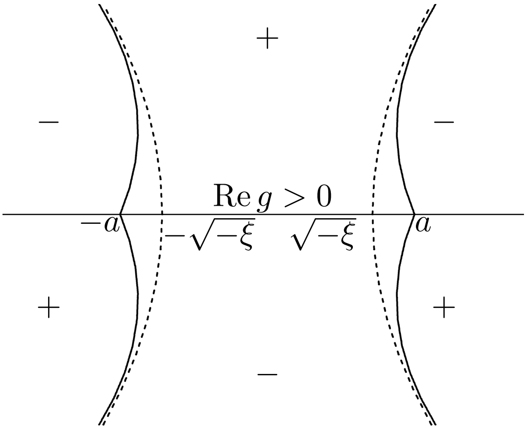

Figure: Signature tables for $\mathrm{Re}\, g$ and $\mathrm{Re}\,\Phi$ (dashed line)

STEP 1. Put

$$m^{(1)}(k) = m(k)\mathrm{e}^{-t(\Phi(k)-g(k))\sigma_3},$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ - is the Pauli matrix. Then $m_+^{(1)} = m_-^{(1)} v^{(1)}$, where

$$v^{(1)}(k) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(k)} \\ R(k) & \mathrm{e}^{-2tg_+(k)} \end{pmatrix}, & k \in [-a, a]; \\ \begin{pmatrix} 1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2tg(k)} \\ R(k)\mathrm{e}^{2tg(k)} & 1 \end{pmatrix}, & k \in \mathbb{R} \setminus [-a, a]. \end{cases}$$

STEP 2. We perform the upper-lower factorization for $v^{(1)}(k)$ as $k \in \mathbb{R} \setminus [-a; a]$ as above, and apply the lenses mechanizm. We get the RH problem for $m^{(2)}(k)$ with

$$v^{(2)}(k) = \begin{cases} \begin{pmatrix} 0 & -\overline{R(k)} \\ R(k) & o(1) \end{pmatrix}, & k \in [-a, a]; \\ \mathbb{I} + o(1), & k \in \mathcal{C}^L \cup \mathcal{C}^U. \end{cases}$$

The scalar conjugation problem:

To find a holomorphic in $\mathbb{C} \setminus [-a, a]$ *function* $d(k)$*, satisfying the jump*

$$d_+(k)d_-(k) = R^{-1}(0)R(k), \quad k \in [-a, a],$$

the symmetry $d(-k) = d^{-1}(k)$*, and normalisation* $d(k) \to 1$*, as* $k \to \infty$*.*

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STEP 3. Setting $m^{(3)}(k) = m^{(2)}(k)d(k)^{-\sigma_3}$, we get

$$v^{(3)}(k) = \begin{cases} \begin{pmatrix} 0 & -R(0) \\ R(0) & \frac{d_+(k)}{d_-(k)}\mathrm{e}^{-2tg_+(k)} \end{pmatrix}, & k \in [-a, a]; \\ \begin{pmatrix} 1 & 0 \\ d(k)^{-2}R(k)\mathrm{e}^{2tg(k)} & 1 \end{pmatrix}, & k \in \mathcal{C}^U; \\ \begin{pmatrix} 1 & -d(k)^2R(-k)\mathrm{e}^{-2tg(k)} \\ 0 & 1 \end{pmatrix}, & k \in \mathcal{C}^L. \end{cases}$$

Here $R(0) = \pm 1$.

The model problem

To find a holomorphic in $\mathbb{C}\setminus[-a,a]$ *vector function* $m^{\mathrm{mod}}(k)$, *satisfying the jump*

$$m_+^{\mathrm{mod}}(k) = m_-^{\mathrm{mod}}(k)\begin{pmatrix} 0 & -R(0) \\ R(0) & 0 \end{pmatrix}, \quad k\in[-a,a],$$

and the symmetry and normalization conditions:

$$m^{mod}(k) = m^{mod}(-k)\sigma_1; \qquad m^{mod}(k) = (1,1) + O(k^{-1}), \quad k\to\infty.$$

The matrix solution of the model problem:

$$M^{\mathrm{mod}}(k) = \begin{pmatrix} \frac{\beta(k)+\beta^{-1}(k)}{2} & \frac{\beta(k)-\beta^{-1}(k)}{2\mathrm{i}} \\ -\frac{\beta(k)-\beta^{-1}(k)}{2\mathrm{i}} & \frac{\beta(k)+\beta^{-1}(k)}{2} \end{pmatrix}, \quad M^{mod}(k)\to\mathbb{I}, k\to\infty,$$

where

$$\beta(k) = \begin{cases} \sqrt[4]{\frac{k-a}{k+a}}, & \text{as } R(0)=-1 \quad \text{(non-resonant case);} \\ \sqrt[4]{\frac{k+a}{k-a}}, & \text{as } R(0)=1 \quad \text{(resonant case).} \end{cases}$$

The solution of the model problem

The vector solution: $m^{\mathrm{mod}}(k) = (1,1)M^{\mathrm{mod}}(k) =$

$$= \frac{1}{2\mathrm{i}}\left(\beta(k)(\mathrm{i}-1)+\beta^{-1}(k)(\mathrm{i}+1),\ \ \beta(k)(\mathrm{i}+1)+\beta^{-1}(k)(\mathrm{i}-1)\right).$$

We prove that this solution approximates properly the solution of the initial RH problem for sufficiently large k:

$$m_1(k) = m_1^{(3)}(k)d(k)\mathrm{e}^{t(\Phi(k)-g(k))} \sim m_1^{\mathrm{mod}}(k)d(k)\mathrm{e}^{t(\Phi(k)-g(k))}.$$

Then from formula

$$m(k) = \begin{pmatrix}1 & 1\end{pmatrix} - \frac{1}{2\mathrm{i}k}\int_x^{+\infty} q(y,t)dy\begin{pmatrix}-1 & 1\end{pmatrix} + O\left(\frac{1}{k^2}\right)$$

we compute

$$q(x,t) = -\frac{\partial}{\partial x}\lim_{k\to\infty} 2\mathrm{i}k\left(m_1(k,\xi,t)-1\right).$$

The rarefaction problem, results

Theorem (Andreiev/E/Lange/Teschl '16)

For arbitrary small $\epsilon_j > 0$, $j = 1, 2, 3$, and for $\xi = \frac{x}{12t}$, the following asymptotics are valid as $t \to \infty$ uniformly with respect to ξ:

A. In the domain $(-6c^2 + \epsilon_1)t < x < -\epsilon_1 t$:

$$q(x,t) = -\frac{x + Q(\xi)}{6t}(1 + O(t^{-1/3})), \quad \text{as } t \to +\infty, \quad \text{where}$$

$$Q(\xi) = \frac{2}{\pi}\int_{-\sqrt{-2\xi}}^{\sqrt{-2\xi}} \left(\frac{d}{ds}\log R(s) - 4\mathrm{i}\sum_{j=1}^{N} \frac{\varkappa_j}{s^2 + \varkappa_j^2} \right) \frac{ds}{\sqrt{s^2 + 2\xi}} \mp \frac{1}{2\sqrt{-2\xi}},$$

and $\pm$ corresponds to resonant/nonresonant cases.

B. In the domain $x < (-6c^2 - \epsilon_2)t$ in the nonresonant case:

$$q(x,t) = c^2 + \sqrt{\frac{4\nu y}{3t}} \sin(16ty^3 - \nu \log(192ty^3) + \delta)(1 + o(1)),$$

where $y = y(\xi) = \sqrt{\frac{c^2}{2} - \xi}$, $\nu = \nu(\xi) = -\frac{1}{2\pi} \log\left(1 - |R(y)|^2\right)$ and

$$\delta(\xi) = -\frac{3\pi}{4} + \arg(R(y) - 2T(y) + \Gamma(\mathrm{i}\nu)) \\ - \frac{1}{\pi} \int_{\mathbb{R}\setminus[-y,y]} \log \frac{1 - |R(s)|^2}{1 - |R(y)|^2} \frac{s\,ds}{s^2 - c^2 - (\frac{c^2}{2} + \xi)^{1/2}(c^2 - s^2)^{1/2}}.$$

Here Γ is the Gamma-function.

C. In the domain $x > \epsilon_3 t$:

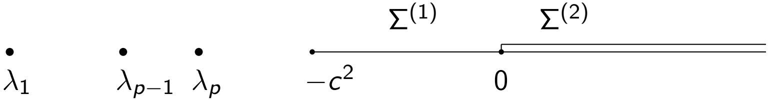
$$q(x,t) = -\sum_{j=1}^{N} \frac{2\varkappa_j^2}{\cosh^2\left(\varkappa_j x - 4\varkappa_j^3 t - \frac{1}{2}\log\frac{\gamma_j(0)}{2\varkappa_j} - \sum_{i=j+1}^{N} \log\frac{\varkappa_i - \varkappa_j}{\varkappa_i + \varkappa_j}\right)} + O(\mathrm{e}^{-\epsilon_3 t/2}).$$

The shock initial profile

$$q_t(x,t) = 6q(x,t)q_x(x,t) - q_{xxx}(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

$$q_0(x) \to \begin{cases} 0, & x \to +\infty, \\ -c^2, & x \to -\infty, \end{cases}$$

$$\int_0^{+\infty} \mathrm{e}^{C_0 x}(|q_0(x)| + |q_0(-x) + c^2|)dx < \infty, \quad C_0 > c > 0.$$



The right scattering data, associated with the initial data:

$$\{R(k),\ k \in \mathbb{R};\ \ |T(k)|,\ k \in [0, \mathrm{i}c];\ \ -\kappa_1^2, ..., -\kappa_p^2,\ \ \gamma_1, ..., \gamma_p.\}$$

Statement of the RH problem

We consider the same regime: $x \to \infty$, $t \to \infty$, $\frac{x}{12t} = \xi$, the nonresonant case only.

Introduce

$$m(k) = \left(T(k,t)\phi_1(k,x,t)\mathrm{e}^{\mathrm{i}kx}, \quad \phi(k,x,t)\mathrm{e}^{-\mathrm{i}kx} \right), k \in \overline{\mathbb{C}^+ \setminus (0, \mathrm{i}c])}.$$

Define it in $\mathbb{C}^-$ by the symmetry condition

$$m(k) = m(-k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Asymptotic behavior as $k \to +\mathrm{i}\infty$

$$m(k) = (1,1) - \frac{1}{2\mathrm{i}k} \left(\int_x^{+\infty} q(y,t) dy \right) (-1,1) + O\left(\frac{1}{k^2} \right).$$

Theorem

Function $m(k)$ is the unique solution of the following RH problem: To find a vector function $m(k)$, meromorphic away from $\mathbb{R}\cup[-\mathrm{i}c,\mathrm{i}c]$, with simple poles at points $\pm\mathrm{i}\kappa_j$, which satisfies:

- The jump condition $m_+(k)=m_-(k)v(k)$, where

$$v(k)=\begin{cases}\begin{pmatrix}1-|R(k)|^2 & -\overline{R(k)}\mathrm{e}^{-2t\Phi(k)}\\ R(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}, & k\in\mathbb{R},\\ \begin{pmatrix}1 & 0\\ \chi(k)\mathrm{e}^{2t\Phi(k)} & 1\end{pmatrix}, & k\in[\mathrm{i}c,0),\\ \begin{pmatrix}1 & \chi(k)\mathrm{e}^{-2t\Phi(k)}\\ 0 & 1\end{pmatrix}, & k\in[0,-\mathrm{i}c],\end{cases}$$

  where $\chi(k):=-\frac{\sqrt{k^2+c^2}}{k}|T(k)|^2$;

- the pole condition

$$\operatorname{Res}_{\mathrm{i}\kappa_j} m(k) = \lim_{k\to \mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & 0 \\ \mathrm{i}\gamma_j^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} & 0 \end{pmatrix},$$

$$\operatorname{Res}_{-\mathrm{i}\kappa_j} m(k) = \lim_{k\to -\mathrm{i}\kappa_j} m(k) \begin{pmatrix} 0 & -\mathrm{i}\gamma_j^2 \mathrm{e}^{2t\Phi(\mathrm{i}\kappa_j)} \\ 0 & 0 \end{pmatrix},$$

- the symmetry condition

$$m(-k) = m(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

- the normalizing condition

$$\lim_{\kappa\to\infty} m(\mathrm{i}\kappa) = (1 \quad 1).$$

Phase function: $\Phi(k) = \Phi(k,x,t) = 4\mathrm{i}k^3 + \mathrm{i}k\frac{x}{t} = 4\mathrm{i}k^3 + 12\mathrm{i}k\xi.$

Region $-c^2/2 < \xi < c^2/3$, $\xi = \frac{x}{12t}$

Point $a = a(\xi) \in (0, c)$ is given implicitly by the equation

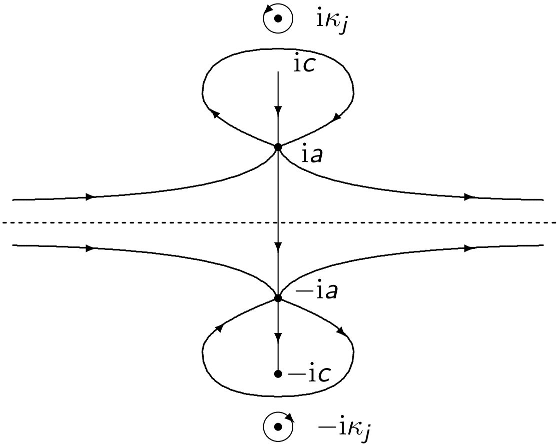
$$\int_a^0 \left(\xi + \frac{c^2 - a^2}{2} - s^2\right) \sqrt{\frac{s^2 - a^2}{c^2 - s^2}}\, ds = 0.$$

We conjugate with the "g - function":

$$g(k) = 24\mathrm{i} \int_{\mathrm{i}c}^{k} \left(k^2 + \xi + \frac{c^2 - a^2}{2}\right) \sqrt{\frac{k^2 + a^2}{k^2 + c^2}}\, dk,$$

$m^{(1)}(k) = m(k)\mathrm{e}^{-t(\Phi(k) - g(k))\sigma_3}$, and then make upper-lower and lower-upper factorizations.

Deformation of the contour and the model problem.

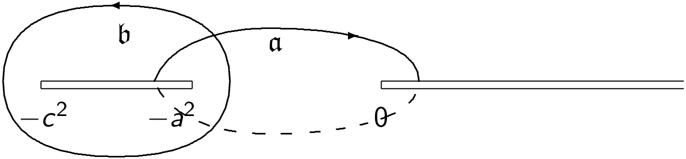


Model problem

$$m_+^{mod}(k) = m_-^{mod}(k) v^{mod}(k),$$

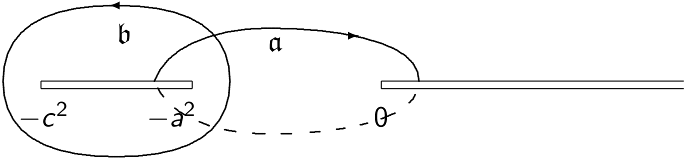
$$v^{mod}(k) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, & k \in [\mathrm{i}c, \mathrm{i}a], \\ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}tB(\xi)+\Delta(\xi)} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}tB(\xi)-\Delta(\xi)} \end{pmatrix}, & k \in [\mathrm{i}a, -\mathrm{i}a], \\ \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, & k \in [-\mathrm{i}a, -\mathrm{i}c] \end{cases}$$

Here $B(\xi) = 2g(\mathrm{i}a + 0)$, and $\Delta(\xi)$ depend of the scattering data of the initial profile.



Let $\mathbb{M}$ — be the Riemann surface of $\mathcal{R}(\lambda) = \sqrt{\lambda(\lambda + c^2)(\lambda + a^2)}$, with the sheets glued along intervals $[-c^2, -a^2] \cup [0, \infty)$, and let $p = (\lambda, \pm)$ denote a point on this surface.

Introduce the canonical basis of cycles.



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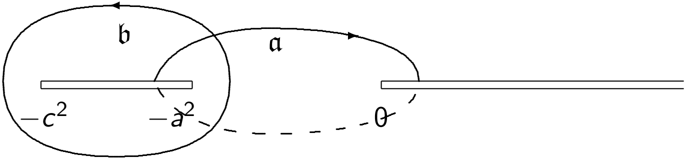
Introduce the canonical basis of cycles.

Let $d\omega$ be the holomorphic Abel differential, $\int_{\mathfrak{a}} d\omega = 2\pi \mathrm{i}$;
Denote by $\mathcal{K}(\xi) = -\frac{\tau(\xi)}{2} + \pi \mathrm{i}$ the Riemann constant, where $\tau(\xi) = \int_{\mathfrak{b}} d\omega$;

Let $A(p, \xi) = \int_{\infty}^{p} d\omega$ be the Abel map.



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$$\Delta(\xi) = \frac{\int_{\mathfrak{b}} \mathcal{R}^{-1}(\lambda) \left(\log |\chi(k)| + 2 \log \frac{k - \mathrm{i}\kappa_j}{k + \mathrm{i}\kappa_j} \right) d\lambda}{\int_{\mathfrak{a}} \mathcal{R}^{-1}(\lambda) d\lambda}.$$

The following Jacobi inversion problem

$$A(p,\xi) + \mathcal{K}(\xi) = -\Delta(\xi)$$

has the unique solution $p_0 = (\lambda_0, \pm)$, $\lambda_0 \in [-a^2, 0]$.

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$$V(\xi) = \int_{\mathfrak{b}} d\Omega_1, \quad W(\xi) = \int_{\mathfrak{b}} d\Omega_3, \quad \text{then}$$

$$B(\xi) = V(\xi) + \xi W(\xi).$$

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Let

$$\theta(v) := \theta(v,\xi) = \sum_{m\in\mathbb{Z}} \exp\left(\pi\mathrm{i}m^2\tau(\xi) + 2\pi\mathrm{i}m\, v\right)$$

be the Jacobi theta-function.

Put $h(\xi) = a^2(\xi) + c^2 + 2\int_{\mathfrak{a}} \lambda\mathcal{R}^{-1}(\lambda)d\lambda\left(\int_{\mathfrak{a}}\mathcal{R}^{-1}(\lambda)d\lambda\right)^{-1}$.

Shock problem, results

Gladka/E/Teschl '16, Gladka/E/Kotlyarov/Teschl '13

In the domain $-6c^2t < x < 4c^2t$ the solution has the following asymptotical behaviour as $t \to \infty$:

$$q(x,t) = -2\frac{d^2}{dx^2}\log\theta\left(V(\xi)x - W(\xi)t - A(p_0,\xi) - \mathcal{K}(\xi)\right) - h(\xi) + o(1).$$

In the domain $x > 4c^2t$:

$$q(x,t) = -2\sum_{j=1}^{N}\frac{\kappa_j^2}{\cosh^2(\kappa_j x - 4\kappa_j^3 t - p_j)} + O(t^{-1}),$$

where

$$p_j = \frac{1}{2}\log\left(\frac{\gamma_j^2}{2\kappa_j}\prod_{i=j+1}^{N}\left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j}\right)^2\right);$$

In the domain $x < -6c^2 t$

$$q(x,t) = -c^2 + P(x,t),$$

$$P(x,t) = \sqrt{\frac{4\nu(k_{1,0})k_{1,0}}{3t}} \sin(16tk_{1,0}^3 - \nu(k_{1,0})\log(192tk_{1,0}^3) + \delta(k_{1,0})) + O(t^{-\alpha})$$

for any $1/2 < \alpha < 1$.

Here $k_{1,0} = \sqrt{\frac{c^2}{2} - \frac{x}{12t}}$,

$$\nu(k_{1,0}) = -\frac{1}{2\pi}\log(1 - |R_1(k_{1,0})|^2),$$

$$\delta(k_{1,0}) = \frac{\pi}{4} - \arg(R_1(k_{1,0})) + \arg(\Gamma(\mathrm{i}\nu(k_{1,0}))) - \\ - \frac{1}{\pi}\int_{\mathbb{R}\setminus[-k_{1,0},k_{1,0}]} \log\left(\frac{1 - |R_1(\zeta)|^2}{1 - |R_1(k_{1,0})|^2}\right) \frac{1}{\zeta - k_{1,0}} d\zeta.$$

The Toda lattice

We consider the Cauchy problem for the Toda lattice equation

$$\begin{aligned} \dot{b}(n,t) &= 2(a^2(n,t) - a^2(n-1,t)), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \qquad (n,t) \in \mathbb{Z} \times \mathbb{R}_+,$$

with the steplike initial data

$$\begin{aligned} a(n,0) &\to a_+, \quad b(n,0) \to b_+, \quad \text{as } n \to +\infty, \\ a(n,0) &\to a_-, \quad b(n,0) \to b_-, \quad \text{as } n \to -\infty. \end{aligned}$$

The Toda lattice equation is equivalent to the Lax equation $\dot{\mathcal{H}} = [\mathcal{H}, \mathcal{A}]$:

$$\begin{aligned} (\mathcal{H}(t)y)(n) &:= a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1), \\ (\mathcal{A}(t)y) &:= -a(n-1,t)y(n-1) + a(n,t)y(n+1). \end{aligned}$$

Background spectra

The Jacobi equation:

$$a(n-1,t)y(n-1) + b(n,t)y(n) + a(n,t)y(n+1) = \lambda y(n).$$

By shifting and scaling of the spectral parameter the initial data of a general location are:

$$a(n,0) \to \frac{1}{2}, \quad b(n,0) \to 0, \quad \text{as } n \to +\infty,$$
$$a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty.$$

Background operators

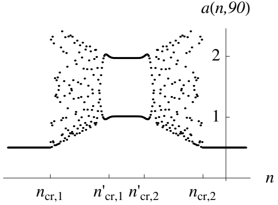
$$\begin{matrix} (H_1 y)(n) := ay(n-1) + by(n) + ay(n+1), \\ (Hy)(n) := \frac{1}{2}y(n-1) + \frac{1}{2}y(n+1). \end{matrix}, \quad n \in \mathbb{Z},$$

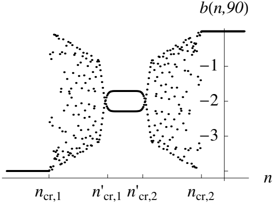
define the continuous spectrum of $\mathcal{H}(t)$.

The Toda shock problem

Let the constants $b, a \in \mathbb{R}$ satisfy the condition $b < -1$, $0 < 2a < -b - 1$. Then the background spectra $[b - 2a, b + 2a]$ and $[-1, 1]$ are not overlapping and the left background spectrum is located to the left of the right background spectrum, that corresponds to the Toda shock problem.

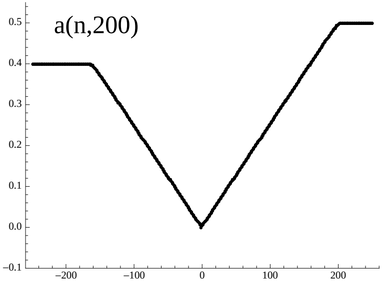
- Direct/inverse scattering - Oba '91, E/ Michor/ Teschl '08
- The Cauchy problem solution - E/Michor/Teschl '09
- Long time asymptotics (for a=1/2, in the middle interval) - Venakides/ Deift/ Oba '91
- Five regions, divided by rays $\frac{n}{t} = \xi_{cr}$: $\xi_{cr,1} < \xi'_{cr,1} < \xi'_{cr,2} < \xi_{cr,2}$.

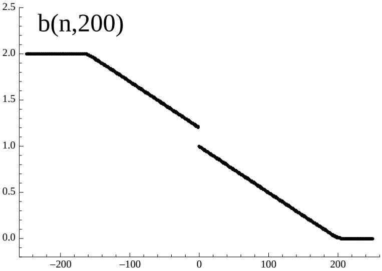




Toda rarefaction problem, numerics

Non-overlapping background: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [1.2, 2.8]$, $a = 0.4$, $b = 2$.





Toda rarefaction problem, RH problem approach

$$a(n,0) \to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty,$$
$$a(n,0) \to \frac{1}{2} \quad b(n,0) \to 0, \quad \text{as } n \to +\infty,$$

where $a > 0$, $b \in \mathbb{R}$ satisfy the condition $1 < b - 2a$.
Results (E/Michor/Teschl):

- In the region $n > t$, the solution $\{a(n,t), b(n,t)\}$ is asymptotically close to the coefficients of the right background Jacobi operator $\{\frac{1}{2}, 0\}$, plus a sum of solitons corresponding to the eigenvalues $\lambda_j < -1$.
- In the region $n < -2at$, the solution is close to the left background constants $\{a, b\}$, plus a sum of solitons corresponding to the eigenvalues $\lambda_j > b + 2a$.

Results

- In the region $-2at < n < 0$, as $t \to \infty$ we have

$$a(n,t) = -\frac{n}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = b - 2a - \frac{n}{t} + O\Big(\frac{1}{t}\Big).$$

- In the region $0 < n < t$, as $t \to \infty$ we have

$$a(n,t) = \frac{n}{2t} + O\Big(\frac{1}{t}\Big), \quad b(n,t) = 1 - \frac{n}{t} + O\Big(\frac{1}{t}\Big).$$

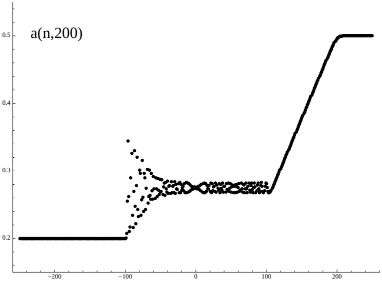
$$O\Big(\frac{1}{t}\Big) = -\frac{n}{8t^2}\Bigg(\frac{\sqrt{1-\frac{n}{t}} + 1 + \frac{2n}{t}}{\frac{n}{t}\sqrt{1-\frac{n}{t}}\,\big(1+\sqrt{1-\frac{n}{t}}\big)}$$

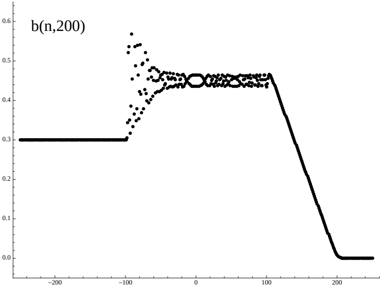
$$+\frac{1}{\sqrt{2\pi}}\int_{\theta_0}^{2\pi-\theta_0} \frac{2u'(\theta)\sin\frac{\theta}{2} - u(\theta)\cos\frac{\theta}{2}}{\sqrt{1-\frac{2n}{t}-\cos\theta}\,\sin^2\frac{\theta}{2}}\,d\theta\Bigg)(1+o(1))$$

$$\theta_0 = \arccos\Big(1-\frac{n}{2t}\Big), \quad u(\theta) = R(\cos\theta)\left(\prod_{z_k\in(-1,0)} |z_k|\frac{e^{i\theta} - z_k^{-1}}{e^{i\theta} - z_k}\right)^{-2}.$$

Mixed cases, numerics

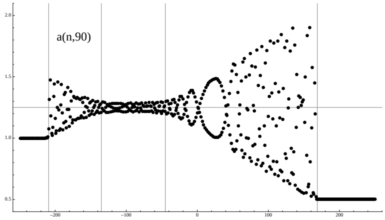
Embedding backgrounds: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [-0.1, 0.7]$.

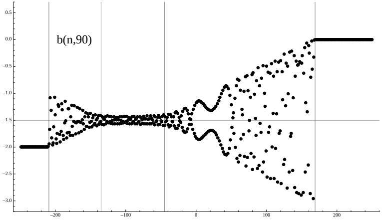




Toda shock problem, numerics

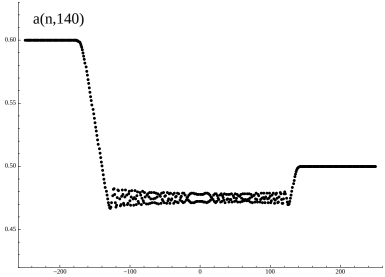
Overlapping backgrounds: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [-4, 0]$;
$a = 1, b = -2$.

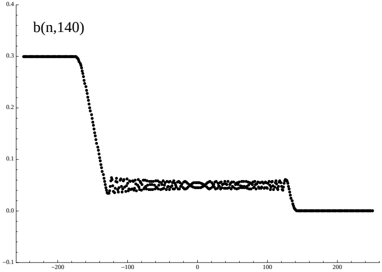




Toda rarefaction problem, numerics

Overlapping backgrounds: $\sigma(H) = [-1, 1]$, $\sigma(H_1) = [-0.9,\ 1.5]$;
$a = 0.6$, $b = 0.3$.





Thank you for your attention!